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Quantum uncertainty: what to teach?

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Abstract

We present a new secondary school teaching method of quantum uncertainties of two-state systems. Intending to be a material teachable in schools, only two-state systems described by real numbers can be considered. An elementary argumentation based on school statistics leads to the identification of the uncertainty of a physical quantity in such systems with the standard deviation of two random variables. We provide a qualitative picture on the state-dependence of the uncertainty, leading to a pictorial representation in the form of four petals of a flower. When considering the product of uncertainty of two essentially different physical quantities we conclude that the general feature: "if the measurement of one of the quantities is certain, the other remains uncertain", cannot be faithfully expressed by means of an inequality, the product has no lower bound different from zero. The application of techniques used by school materials for teaching quantum physics leads to an exact formula for the state-dependence of the uncertainty valid in any two-state system described by real numbers, in full harmony with the qualitative picture. We compare the two-state case with the celebrated Heisenberg position-momentum uncertainty relation and show that these are both specific facets, but only the Heisenberg relation can be expressed by an inequality. The latter hardly provides any hint

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on the uncertainties of physical quantities in two-state systems. We conclude that the two-state approach is worth teaching in schools also in relation to the uncertainty relation, even if the Heisenberg relation is not part of the curriculum.

Keywords: uncertainty, uncertainty relation, quantum mechanics

1. Introduction

In the last decades two-state approaches to teaching quantum physics in secondary schools became popular because these do not need advanced mathematics, while at the same time such simple setups are sufficient to demonstrate fundamental concepts. The first published approach uses an analogy with the phenomenon of light polarisation in which the two basic states correspond to a free transmission of a photon through a polariser and to its full absorption [1]. The material is based on school experiments with polarisers leading to the verification of Malus' law which is interpreted as the probability of a photon passing through a polariser. Since then, alternative teaching materials also appeared, for instance, studying quantum phenomena related to photon-beam splitter experiments [2], or using a computer-based environment to explore the behaviour of a particle in a simplified double-well potential box [3]. A possible motivation behind these developments is the preparation of secondary school teaching for the age of quantum computers or quantum cryptography [4–6], also supported by the choice of topics of The Nobel Prize Committee in Physics 2022 [7]. Teaching experiments have also been performed showing encouraging positive outcomes [8–13]. Teaching materials [1–3] are based on Dirac's approach [14] as well as on later textbooks dealing with two-state quantum systems [15–18], although the mathematics used by them is unavoidably restricted to *real* numbers and planar vectors. We note that the International Business Machines (IBM) material [4] and a popular textbook on quantum computing [19] illustrate that one can go quite far from a conceptional point of view by the use of only real numbers.

Teaching materials [1–3] establish the probabilistic nature of individual quantum measurements: a measurement can have probabilistic outcome even if ideal instruments are used. Teaching materials apply the term *quantum uncertainty* to describe the feature that a typical quantum state is a superposition state. In such a state a measurement of a physical quantity can take on, say, two different values only. A single measurement can yield any of these values. In a sequence of repeated measurements, however, probabilities less than 1 can be associated with the values as explained in detail in [1]. In other words, the physical quantity does not possess a unique value in a general state, it is *uncertain*. Here we go one step further and address the question of how an *uncertainty of a physical quantity* in a sequence of repeated measurements should be defined, and if an *uncertainty relation* between two such uncertainties can be formulated. In addition, the uncertainty of a physical quantity will be shown to be interpretable as a *standard deviation*.

The aim of this paper is to show that these concepts can be presented by means of secondary school mathematics. Uncertainty arises in a sequence of repeated measurements carried out in the same fixed superposition state supposing individually precise measurements. We also consider how the product of the uncertainties of two different quantities behaves in different quantum states. First, in section 2, we recall the Heisenberg position-momentum uncertainty relation and speculate what features of it can remain generally valid in other cases. Next, in section 3, we consider two-state systems described by real numbers. The argumentation applied relies on elementary knowledge of secondary school statistics. The consideration, illustrated in a pictorial way, shows that a facet of the uncertainty relation different from that characterizing the Heisenberg relation appears here. The only common feature is that the two uncertainties cannot, in general, vanish simultaneously. In section 4 these arguments are supported by elementary calculations based on the techniques provided by the school materials, and exact general expressions are obtained which can be applied to the particular cases of e.g. [1–3].

2. Reviewing the Heisenberg relation

The uncertainty principle, one of the fundamental laws of quantum mechanics, is mostly expressed by the Heisenberg uncertainty relation:

$$\Delta x \Delta p_x \geqslant \hbar/2. \tag{1}$$

When taught in school [20], teachers commonly say that Δx and Δp_x are the uncertainty of position and momentum, respectively. Relation (1) implies that the uncertainty in the momentum becomes large, when the uncertainty in the position is small, or vice versa (see e.g. [21]). In fact, in the free motion of a particle described by a plane wave, the momentum can in principle be measured with certainty, but the wave extends without any limit, and hence the position is extremely uncertain. The uncertainties appearing here can be arbitrarily large.

However, the uncertainty relation is a more general law of quantum mechanics, which holds for any two essentially different quantities. In addition, teachers have no opportunity to make it clear that uncertainty relation (1) only holds since position and momentum are canonically conjugated quantities. This implies that the relation can be different for other quantity pairs, in particular, for the ones occurring in two-state systems. As we shall see, an uncertainty inequality exists (see (4) below), but its form turns out to be useless. An inequality free formulation of the relation can and will, however, be given.

When thinking about basic features of the uncertainty relation (1) (with the equality fulfilled), the following candidates emerge regarding how uncertainties of two generally chosen physical quantities are related when the quantum state is changing:

(a) when one uncertainty decreases the other must increase because of the reciprocal relation,
(b) when one is zero, the other one is as large as possible,
(c) the product of uncertainties is larger than a positive constant, not depending on the state,
(d) when one is zero, the other one is different from zero.

The surprising conclusion of this paper is that two-state approaches are only consistent with property *(d)*, all the other options do not hold. Next, we support this conclusion by means of elementary, mostly pictorial, arguments.

3. Elementary arguments: uncertainties in two-state systems

Teaching materials [1–3] all make it clear that for a given physical quantity A, certain measuring results can be obtained in special states only, and there are only two such states, say φ_+ and φ_- with associated measured values λ_+ and λ_-, respectively[1]. Without the loss of generality (apart from the degenerate case $\lambda_+ = \lambda_-$), we can assume that $\lambda_+ > \lambda_-$. In a general state, both values can be measured, and in a sequence of repeated measurements λ_+ occurs with some probability p_+, and λ_- with probability p_-. These teaching materials also point out that a measurement carried out in a general state leading to outcome $\lambda_\pm$ implies that the state is converted into $\varphi_\pm$. Because there are only two possible outcomes, $p_- + p_+ = 1$. The probabilities depend on the state. In the basic states φ_+ (φ_-) one of the permitted values can be measured with certainty that is $p_+ = 1$ ($p_- = 1$). The expected value $<A>$ of quantity A over all measurements in a given general state is

$$<A> \; = p_-\lambda_- + p_+\lambda_+ \tag{2}$$

as follows from the statistic of two random variables discussed in elementary school books (see e.g. [22]).

In a schematic diagram (figure 1) one sees that the expected value lies between λ_- and λ_+ since it is a weighted average of the two permitted values λ_- and λ_+. If p_+ is close to one, outcome λ_+ occurs with a high chance, so the expected value is close to λ_+. The expected value differs from both λ_- and λ_+, and a deviation Δ_- and Δ_+ between the measured permitted value and the expected value is present, as also illustrated by figure 1.

It is useful to consider the uncertainty as a result of measuring the deviation Δ_+ with probability p_+ and Δ_- with p_-. An advantage of the

[1] Materials [1, 3] call the permitted values λ_+ and λ_-, eigenvalues, and the basic states eigenstates/eigenvectors.

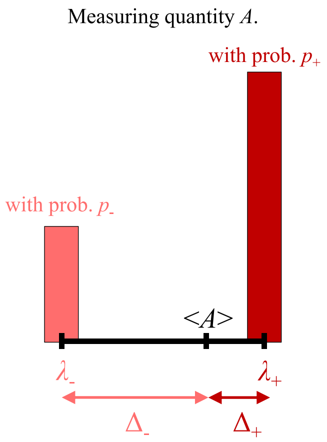

Figure 1. Schematic diagram illustrating the probabilistic outcomes of measurements in a two-state system. The permitted value λ_+ (λ_-) and its deviation Δ_+ (Δ_-) from the expected value $<A>$ are measured with probability p_+ (p_-).

two-state approach is that it enables us to illustrate that the uncertainty is a statistical characteristics. Since the deviations are signed quantities, but contribute positively to the overall uncertainty, it is worth considering the square of the deviations. In analogy with (1), the square of the uncertainty ΔA is the *expected value* of the squares of the deviations (as also suggested by figure 1):

$$\Delta A^2 = \ < \Delta^2 > \ = p_- \Delta_-{}^2 + p_+ \Delta_+{}^2. \qquad (3)$$

This relation replaces the loose concept of uncertainty widely used in secondary school teaching with a quantity uniquely following from the statistics of measurements. The square of the uncertainty is the *variance* of the statistics of the measurements, whereas the uncertainty itself is its square root, the *standard deviation* [22].

An important qualitative feature of the uncertainty can be deduced from relation (3). When p_+ is close to 1 and thus Δ_+ is relatively small (as in figure 1), this small deviation occurs several times as p_+ is relatively large. In the same set of measurements, Δ_- is relatively large, however, this larger deviation is measured rarely, so the resulting variance (3) is not large. The variance ΔA^2 vanishes if either Δ_- or Δ_+ is zero, and close to these extremes it remains small. The variance must thus have a maximum in the middle, at about $p_+ = p_- = 1/2$. Since the uncertainty

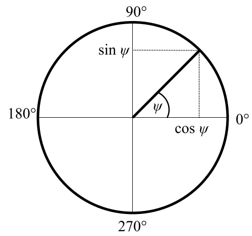

Figure 2. Quantum states of two-state systems described by real numbers can be represented by a point along a unit circle, and hence via a single angle ψ.

cannot be larger than the difference between λ_+ and λ_-, an upper limit is set by this difference (for a precise value see equation (9) in section 4). The variance is thus *bounded from above* in two-state systems (and in any finite-state systems), it can never grow without any limit in contrast to what is possible in (1). As another consequence, the graph of function ΔA^2 vs either p_+ or p_- is thus a single-humped curve between the extremal zero values.

Any state of a two-state system described by real numbers can be represented as a unit vector of the plane [19] which can be written as (cos ψ, sin ψ) where ψ is an arbitrary angle between 0° and 360°. We can and will use this angle to represent the state itself. Graphically the set of all states corresponds to the perimeter of a unit circle (see figure 2). In teaching materials [1–3] state ψ carries a particular physical meaning: in the polarisation, beam-splitting and double-well approach it corresponds to the polarisation direction of a photon, the input state of a photon, and one of the lowest energy states, respectively.

It is worth marking on this graph the basic state φ_- (or φ_+) where the permitted value λ_- (or λ_+) can be measured with certainty. We have the opportunity to illustrate these states with diagonals, since the measured value in state $\varphi_+ + 180°$ ($\varphi_- + 180°$) is the same as in φ_+ (φ_-)[2]. These diagonals form a cross since the basic states exclude each other, corresponding to orthogonal directions: $\varphi_- = \varphi_+ - 90°$. The special states φ_- and φ_+ have a physical meaning in all two-state

[2] This representation is also supported by the polarisation approach [1], where rotating the polariser by 180° does not change the physical situation.

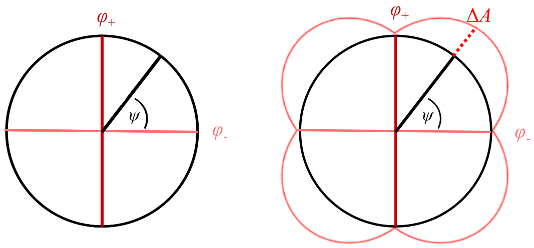

Figure 3. Left panel: The pair of basic states φ_+, φ_- (in which quantity A is certain with permitted values λ_+, λ_-) is represented by a cross inside the circle. Right panel: In these states the uncertainty is zero, it increases for states ψ lying further away. The red curve is a schematic representation of uncertainty ΔA, as given in (3), in dependence on ψ: the uncertainty is proportional to the dashed line segment.

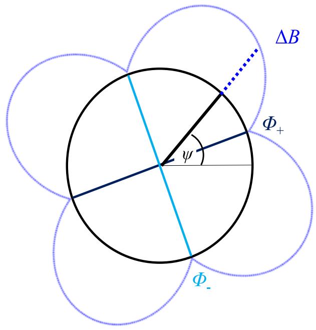

Figure 4. Cross representing the basic states Φ_+ and Φ_- for physical quantity B and the schematic graph of uncertainty ΔB (blue curve) as a function of ψ. The blue dashed line segment is proportional to the uncertainty ΔB in state ψ.

approaches, as possible final states after measurements. In the polarisation approach [1] φ_- and φ_+ correspond to full absorption and to free passing through, respectively. In the beam-splitter [2] and double-well [3] approaches these represent the transmitted and reflected paths and states being localized entirely in one of the two wells, respectively. In the left panel of figure 3, besides a general state ψ, the basic states are also marked, where for simplicity, we choose φ_- to be zero, i.e. the cross of $\varphi_\pm$ forming a coordinate frame as usual in schools.

Recall that $p_- = 1$ corresponds to state φ_- and the uniquely measured value λ_-, and $p_+ = 1$ to state φ_+ and value λ_+. In general, to any state $\psi \neq \varphi_\pm$ there belongs a unique nonzero probability p_- (or p_+) < 1 with which λ_- (λ_+) occurs. One can thus consider the standard deviation ΔA to be a function of ψ as well. When plotted as a function of state ψ, ΔA is also a single-humped curve e.g. between φ_- and φ_+. It can be represented along the perimeter of the circle as follows: the uncertainty in state ψ is proportional to the distance along the line of angle ψ between the unit circle and the red curve (dashed line segment) representing ΔA in the right panel of figure 3. The graph of the uncertainty corresponds to four 'petals' of a 'flower'.

After understanding the uncertainty of quantities as standard deviations, one can explore the content of the uncertainty relation and if a form analogous to (1) exists. To this end, a *different* physical quantity B with basic states different from those of A is needed, and the standard deviations ΔA and ΔB have to be investigated in the same quantum state. Such often occurring quantity pairs will be called essentially different[3]. The permitted values of B are denoted by Λ_- and $\Lambda_+ > \Lambda_-$ occurring in states Φ_- and Φ_+ where $\Phi_- = \Phi_+ - 90°$. In a graphical representation, Φ_- can be an arbitrary angle different from 0. The basic states of B and uncertainty ΔB can graphically be represented as above: In figure 4 we show the 'flower' corresponding to quantity B. For a cleaner visual distinction, we assume that the maximum of the uncertainty of quantity B is different from that of quantity A, the 'petals' of ΔB appear thus more elongated.

At this point we recognize that in a state where quantity A is measured with certainty, quantity B remains uncertain, since the basic states are different. In order to see how these two quantities change with ψ, figure 5 shows the combined graph containing both ΔA and ΔB as a function of ψ.

The observation of ΔA and ΔB reveals that there are intervals in ψ in which both uncertainties grow or decrease simultaneously. This excludes option *(a)* of section 2.

[3] In technical term, these are called non-commuting observables.

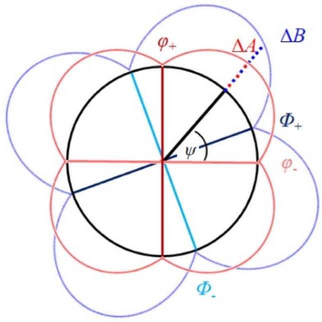

Figure 5. Simultaneous representation of both ΔA and ΔB along the perimeter of the circle.

A more transparent view is provided by figure 6(a) where angle ψ is represented by points of a straight axis. This graph also shows that option *(b)* is not valid either because when one function has a minimum the other one does not have necessarily a maximum (the minimum-maximum coincidence occurs only in the special case of $\Phi_+ - \varphi_+ = 45°$).

To find an opportunity for a straight comparison with the Heisenberg relation (1), we also plot in figure 6(b) the product of ΔA and ΔB as a function of ψ. This clearly indicates that property *(c)* does not hold either: there is no positive lower bound for the product of the uncertainties. The only inequality one can write down is

$$\Delta A \Delta B \geqslant 0. \quad (4)$$

This is, however, an empty statement since the uncertainties are nonnegative quantities by themselves. Furthermore, the mathematical form of (4) allows for the simultaneous disappearance of both uncertainties. It cannot, thus, be considered a faithful representation of the uncertainty relation in two-state systems described by real numbers. Therefore, no strict analogue of relation (1) is found for two-state (or any finite-state) systems. The situation is so much different here since an upper limit does exist for both ΔA and ΔB. We conclude that for two-state approaches of secondary school teaching materials only property *(d)* holds: when one quantity can be measured with certainty, the other one is uncertain, all the other options of section 2 are excluded.

Thus, the general formulation of the uncertainty relation valid for any quantum system can only be: *if A and B are essentially different quantities (with different basic states) and the measurement of one of the quantities is certain, the other remains uncertain:*

If $\Delta A = 0$, then $\Delta B \neq 0$ and vice versa. Uncertainties here are considered standard deviations and are evaluated in the same quantum state[4].

4. Quantitative arguments concerning uncertainties in two-state systems

When turning to quantitative arguments, it is worth expressing that probabilities p_- and p_+ are not independent. We shall use a single probability $p = p_+$ corresponding to the probability of finding λ_+ in a measurement on a general state.

Let us then express the expected values from (2) with probability p as

$$< A >= (1-p)\lambda_- + p\lambda_+. \quad (5)$$

The deviation of it from λ_- and λ_+ is thus:

$$\Delta_- = \lambda_- - < A > \; = p(\lambda_- - \lambda_+). \quad (6)$$

and

$$\Delta_+ = \lambda_+ - < A > \; = \; (1-p)\,(\lambda_+ - \lambda_-), \quad (7)$$

respectively. Substituting these into expression (3) for the variance, we find

$$\begin{aligned} \Delta A^2 = &\; (1-p)p^2(\lambda_- - \lambda_+)^2 \\ &+ p(1-p)^2(\lambda_+ - \lambda_-)^2 \\ &= p(1-p)\,(\lambda_+ - \lambda_-)^2. \end{aligned} \quad (8)$$

This is the analytic form of the $p-$dependence qualitatively deduced in section 3. Its graph is indeed of single-humped shape, with a maximum exactly at $p = 1/2$, of value $(\lambda_+ - \lambda_-)^2/4$. So, the maximum of the standard deviation is

$$\Delta A_{\max} = (\lambda_+ - \lambda_-)/2. \quad (9)$$

As mentioned in the first paragrapf of section 3, p represents the probability of measuring the

[4] Note that this formulation of the relation is consistent with the Heisenberg relation (1).

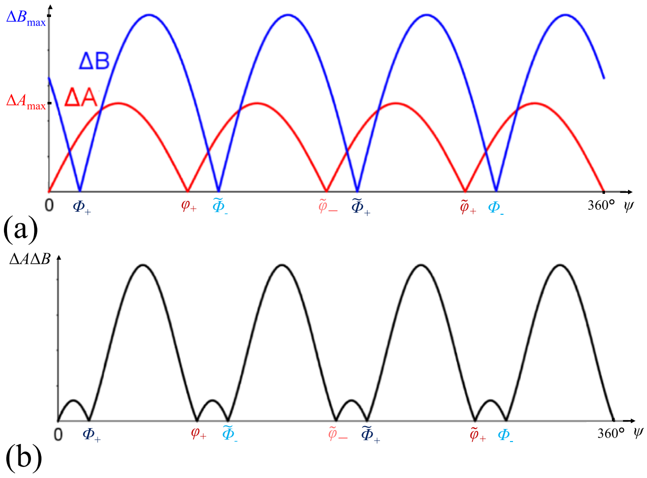

Figure 6. Graphs of the functions $\Delta A(\psi)$, $\Delta B(\psi)$ in (a) and of the product $\Delta A\Delta B(\psi)$ in (b). These are no longer schematics, rather the results of (12–13), evaluated with $\lambda_+ - \lambda_- = 1$ ($\Delta A_{\max} = 1/2$), $\varphi_+ = 90^\circ$, and $\Lambda_+ - \Lambda_- = 2$ ($\Delta B_{\max} = 1$), $\Phi_+ = 20^\circ$. The tildes above angles mark the untilded angles shifted by 180°.

larger permitted value λ_+ in a general state ψ, and the state is becoming converted into φ_+. In [1] teachers help students to recognize that this probability p can be calculated based on their experiments with light polarisation: the projection of ψ onto φ_+ corresponding to a scalar product should be taken and the probability is the square of this scalar product. The scalar product is just the cosine of the angle difference $(\psi - \varphi_+)$. Thus, probability p is

$$p = \cos^2(\psi - \varphi_+)\,. \tag{10}$$

This general expression in the particular case of polarisation is nothing but Malus' law (recall that ψ and φ_+ are the directions of the photon polarisation and of the transmission axis, respectively).

The variance ΔA^2 based on (8) is then:

$$\begin{aligned}\Delta A^2 &= p\,(1-p)\,(\lambda_+ - \lambda_-)^2\\ &= \cos^2(\psi - \varphi_+)\,\sin^2(\psi - \varphi_+)\,(\lambda_+ - \lambda_-)^2\\ &= \sin^2\psi\,\cos^2\psi\,(\lambda_+ - \lambda_-)^2.\end{aligned} \tag{11}$$

In the last line we have taken into account that $\varphi_+ = 90^\circ$ is chosen. So, the standard deviation, the uncertainty, of quantity A is

$$\Delta A(\psi) = 1/2\,|\sin(2\psi)|\,(\lambda_+ - \lambda_-)\,. \tag{12}$$

In analogy with (11) the standard deviation of quantity B is

$$\Delta B(\psi) = 1/2\,|\sin[2(\psi - \Phi_+)]|\,(\Lambda_+ - \Lambda_-)\,. \tag{13}$$

It is clear from (12 and 13) that the uncertainties of quantity A and B depend on state ψ. In fact, the graphs of figure 6 are plotted by using these expressions, i.e. they represent analytically correct results for arbitrary two-state systems. The two functions (12 and 13) are only equal for $\varphi_\pm = \Phi_\pm$. Two quantities can be measured in a given state with certainty only if their basic states are the same. If this is not the case, i.e., for essentially different quantity pairs, a simultaneous disappearance of both uncertainties cannot occur.

The quantitative description also confirms the previous results to hold: none of the properties *(a-c)* are valid, the only valid statement is *(d)*.

The teaching method presented here was part of a pilot project. We extended the polarisation material [1] with an extra class of 1 hour length in two groups of 11 students (of age 17). These were devoted solely to an elementary discussion of the problem of the uncertainty of physical quantities. The probabilistic treatment of the expected value resulting from a series of experiments, as well as the corresponding variance (equations (1) and (3)), followed the argumentation presented here. All students had an elementary background in statistics and were happy to learn that uncertainty appeared as standard deviation, i.e. a mathematically well-defined quantity. The bests of the students were able to follow the material up to the expression of $\Delta A(\psi)$ (equation (12) with $\lambda_- = 0$, $\lambda_+ = 1$, as natural in the polarisation approach). They all were shown a graph corresponding to figure 6(a), and based on this they easily accepted the general rule that two quantities cannot be certain simultaneously. Prior to the pilot project, some of the students have already had learned about the Heisenberg relation as part of the standard curriculum. These students were found to have difficulties with the comprehension of the strongly different forms of the two-state case and the Heisenberg relation. In addition, we experienced that in-service physics teachers also have difficulties interpreting the two-state form of the uncertainty relation. We hope that our paper might be of help in pointing out that the different forms are not in contradiction, just different facets of the same quantum feature.

5. Conclusions

We have illustrated that the uncertainty ΔA and ΔB of physical quantities A and B, respectively, occurring in quantum measurements of two-state systems described by real numbers can be evaluated in secondary school. Argumentations based on elementary statistics lead to precise expressions, and to the interpretation of the uncertainty as the standard deviation of a discrete random variable. The product $\Delta A\Delta B$ changes, however, with the state in a way different from what the Heisenberg relation (1) suggests. This is because uncertainties are limited from above in any finite state system. A lower limit of $\Delta A\Delta B$ other than zero cannot be found. The only general feature is that for essentially different (non-commuting) quantities both uncertainties cannot be zero simultaneously.

We emphasise that the results shown are obtained without using the standard formalism of quantum mechanics. To put the argumentation in scope for teachers using the language of university curriculum, quantities A and B correspond to 2 by 2 real symmetric matrices representing non-commuting observables. The application of the mathematical tools of quantum mechanics leads, as shown in [23], exactly to the same results (equations (12) and (13)) as found in section 3 based on secondary school arguments. The probabilistic nature of individual quantum measurements in a superposition state is what is called quantum uncertainty in [1–3]. Here we have discussed the more traditional aspect of uncertainty and determined the uncertainty of a sequence of measurements in the form of standard deviation. The most general form of the uncertainty relation (see e.g. [17]) is

$$\Delta A\Delta B \geqslant \ |< C >| \ / \ 2 \tag{14}$$

where C denotes the commutator of A and B. For symmetric real matrices, the commutator C proves to be an antisymmetric matrix. Therefore, its expected value in *any* real state turns out to be zero, in harmony with (4).

The striking feature of the Heisenberg relation (1), which is also a special case of (14), namely that one of the uncertainties can grow without any limit when the other shrinks toward zero, is due to two reasons: the system possesses continuously many states[5], and the quantities form a conjugate pair in the sense of classical mechanics.

In secondary school teaching it is impossible to really illuminate these reasons. It might therefore be sufficient to teach only the version suggested by the two-state cases when speaking about uncertainty relations, even if the position and momentum uncertainty is the subject. The teaching approach presented here might be a useful element also in a two-state-system-based introduction of quantum computation in schools.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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[5] Obviously, neither the position nor the momentum can generally be represented in the language of two-state systems.

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Tamás Tél is an emeritus professor at ELTE. In the last decades he offered a number of courses on different branches of theoretical physics, including quantum mechanics, for trainee teachers. He is particularly interested in statistical physics, chaos research and environemntal science. He served as the head of the MTA-ELTE Physics Education Research Group (2016–2021).