Physical observables of the Ising spin glass in $6-\epsilon$ dimensions: Asymptotical behavior around the critical fixed point

T. Temesvári*

MTA-ELTE Theoretical Physics Research Group, Eötvös University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary

(Received 5 May 2016; revised manuscript received 18 March 2017; published 11 July 2017)

The asymptotical behavior of physical quantities, like the order parameter, the replicon, and longitudinal masses, is studied around the zero-field spin-glass transition point when a small external magnetic field is applied. An effective field theory to model this asymptotics contains a small perturbation in its Lagrangian which breaks the zero-field symmetry. A first-order renormalization group supplemented by perturbational results provides the scaling functions. The perturbative zero of the scaling function for the replicon mass defines a generic Almeida-Thouless surface stemming from the zero-field fixed point.

DOI: 10.1103/PhysRevB.96.024411

I. INTRODUCTION

Since the invention of the renormalization group (RG) by Wilson [1], replacing a statistical system (which is close to its critical state) by an effective field theory has become the basic analytical tool to calculate the asymptotical behavior of physical quantities around a critical point. Such an effective theory is defined by its Lagrangian $\mathcal{L}$, usually called the Landau-Ginzburg-Wilson (LGW) Lagrangian, which depends on the fluctuating order parameter components, the "fields", with the statistical weight of a configuration being $\sim e^{-\mathcal{L}}$. This formalism was set up in the 1970s for the prototype spin-glass model of Edwards and Anderson (EA) [2], with an immediate application of the renormalization group [3]. The EA model for N Ising spins on a d-dimensional hypercubic lattice is defined by the Hamiltonian

$$\mathcal{H} = -\sum_{(ij)} J_{ij} s_i s_j - H \sum_i s_i, \tag{1}$$

where the J_{ij}'s are independent, Gaussian-distributed random variables with zero mean and variance J^2, and a homogeneous external magnetic field H was also included. Summations are over nearest-neighbor pairs (ij) of lattice sites in the first sum but over the N lattice sites in the second one. Averages over the quenched disorder of the EA model are managed by the replica trick, and as a result, the effective theory representing the lattice system close to criticality is a cubic *replicated* field theory with the fluctuating fields (in momentum space) $\phi_{\mathbf{p}}^{\alpha\beta} = \phi_{\mathbf{p}}^{\beta\alpha}$ and $\phi_{\mathbf{p}}^{\alpha\alpha} = 0$ for $\alpha,\beta = 1,\dots,n$, with the replica number n going to zero at the end of a calculation. Harris *et al.* [3], and later Refs. [4,5], deduced the following LGW Lagrangian for the zero-external-field case, i.e., for $H = 0$:

$$\mathcal{L}_{\text{zero-field}} = \frac{1}{2}\sum_{\mathbf{p}} \left(\frac{1}{2}p^2 + m\right) \sum_{\alpha\beta} \phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\alpha\beta} - \frac{1}{6\sqrt{N}} {\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}}' w \sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\alpha}. \tag{2}$$

Momentum conservation is indicated by the primed sum, and a continuum of $\mathbf{p}$'s, cutoff at some Λ, results in the thermodynamic limit $N \to \infty$. Replica summations above and in the following are unrestricted. For a nonzero magnetic field H which is not necessarily small, the Lagrangian $\mathcal{L}$ gets additional replica symmetric (RS) invariants (i.e., homogeneous polynomials built up of the fields $\phi_{\mathbf{p}}^{\alpha\beta}$, which are invariant under *any* permutation of the n replicas; see Ref. [6]), and the theory becomes the generic cubic RS field theory with $\mathcal{L} = \mathcal{L}_{\text{zero-field}} + \delta\mathcal{L}$, with m and w in $\mathcal{L}_{\text{zero-field}}$ replaced by $m_1 = m + \delta m_1$ and $w_1 = w + \delta w_1$, respectively, and

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{2}N^{\frac{1}{2}}h^2 \sum_{\alpha\beta}\phi_{\mathbf{p}=0}^{\alpha\beta} \\ &+\frac{1}{2}\sum_{\mathbf{p}}\left[m_2\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}}^{\alpha\gamma}\phi_{-\mathbf{p}}^{\beta\gamma} + m_3\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\gamma\delta}\right] \\ &-\frac{1}{6\sqrt{N}}{\sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}}'\left[w_2\sum_{\alpha\beta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\alpha\beta}\right. \\ &+w_3\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\alpha\gamma} + w_4\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\gamma\delta} \\ &+w_5\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\gamma}\phi_{\mathbf{p}_3}^{\beta\delta} + w_6\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\gamma}\phi_{\mathbf{p}_3}^{\alpha\delta} \\ &\left.+w_7\sum_{\alpha\beta\gamma\delta\mu}\phi_{\mathbf{p}_1}^{\alpha\gamma}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\delta\mu} + w_8\sum_{\alpha\beta\gamma\delta\mu\nu}\phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\gamma\delta}\phi_{\mathbf{p}_3}^{\mu\nu}\right].\end{aligned} \tag{3}$$

The zero-field Lagrangian of Eq. (2) contains RS invariants with all the replica indices occurring an even number of times, thus reflecting the spin-inversion symmetry of the EA model without an external magnetic field. Although the insertion of a small magnetic field breaks the spin-inversion symmetry and consequently the higher symmetry of the field theory with $\mathcal{L}_{\text{zero-field}}$, the generic RS field theory with all the coupling constants nonzero in (3) is redundant when the magnetic field is small. Accordingly, the first study of the Almeida-Thouless (AT) [7] instability below eight dimensions considered the simplest model with h^2 the only nonzero coupling in (3) [8].

*temtam@helios.elte.hu

Finding the H dependence of the couplings in Eq. (3) and, in this way, selecting the dominant couplings for small H can be accomplished by two different methods: The first one applies the Gaussian-integral representation (Hubbard transformation) of the original EA model plus additional truncation for small momentum; this was the approach in [6] to derive the generic RS field theory. Here the second method is chosen; namely, neglecting the fluctuations of the order parameter fields provides the field-theoretic representation of the *infinite*-dimensional EA model, which is most easily realized on the complete graph providing the Sherrington-Kirkpatrick (SK) model [9] (see also Ref. [10]). In the following paragraphs, therefore, the Landau free energy of the field theory is compared with the Lagrangian of the SK model $\mathcal{L}_{\rm SK}$, which has an explicit and well-known magnetic field dependence.

The SK model has the Hamiltonian (1) on the complete graph (i.e., $\sum_{(ij)}$ means summation over all the pairs), and the variance of the J_{ij}'s is J^2/N. With the notation $E[\cdots]$ for the average over the J_{ij}'s, the quenched averaged replicated partition function of the SK model can be put into the form

$$E[Z_{SK}^n] \sim \int \left[\prod_{(\alpha\beta)} dq_{\alpha\beta}\right] e^{-\mathcal{L}_{\rm SK}},$$

with

$$\begin{aligned}\frac{1}{N}\mathcal{L}_{\rm SK} = &-\frac{1}{2}\bar{H}^2\sum_{\alpha\beta} q_{\alpha\beta} + \frac{1}{2}(-\bar{\tau}+\bar{H}^2)\sum_{\alpha\beta} q_{\alpha\beta}^2\\ &-\frac{1}{2}\bar{H}^2\sum_{\alpha\beta\gamma} q_{\alpha\gamma}q_{\beta\gamma} - \frac{1}{6}(1-3\bar{H}^2)\sum_{\alpha\beta\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha}\\ &-\frac{1}{3}\bar{H}^2\sum_{\alpha\beta} q_{\alpha\beta}^3 + \bar{H}^2\sum_{\alpha\beta\gamma} q_{\alpha\beta}^2 q_{\beta\gamma}\\ &-\frac{1}{2}\bar{H}^2\sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}q_{\alpha\gamma}q_{\beta\delta} + O(\bar{H}^4,q^4),\end{aligned} \tag{4}$$

where $\bar{\tau} \equiv \frac{1}{2}[1-(J/kT)^{-2}]$ and $\bar{H} \equiv H/kT$. Stationarity of $\mathcal{L}_{\rm SK}$ with respect to $q_{\alpha\beta}$ yields the order parameter in the thermodynamic limit.

In the case of the field theory, it is the Legendre-transformed free energy $F(q_{\alpha\beta})$ that is stationary in the equilibrium state. It is defined by the common rules of the Legendre transformation, namely,

$$F(q_{\alpha\beta}) = -\ln Z(H_{\alpha\beta}) + N\sum_{(\alpha\beta)} H_{\alpha\beta}\, q_{\alpha\beta},$$

$$\frac{\partial \ln Z(H_{\alpha\beta})}{\partial H_{\alpha\beta}} = N\, q_{\alpha\beta},$$

where the partition function $Z(H_{\alpha\beta}) = \int \mathcal{D}\phi\, e^{-\mathcal{L}}$ acquires its dependence on the $H_{\alpha\beta}$'s by adding a source term $-N^{\frac{1}{2}}\sum_{(\alpha\beta)} H_{\alpha\beta}\,\phi_{\mathbf{p}=0}^{\alpha\beta}$ to the RS Lagrangian $\mathcal{L}_{\text{zero-field}} + \delta\mathcal{L}$. [$\sum_{(\alpha\beta)}$ in these formulas means summation over $n(n-1)/2$ pairs of replicas.] Neglecting fluctuations of the fields (tree approximation), i.e., replacing $\phi_{\mathbf{p}}^{\alpha\beta}$ by its average $\langle\phi_{\mathbf{p}}^{\alpha\beta}\rangle = \delta_{\mathbf{p},\mathbf{0}}^{\rm Kr} N^{\frac{1}{2}}\, q_{\alpha\beta}$, provides the mean-field, or Landau, free energy of the model:

$$\begin{aligned}\frac{1}{N}F(q_{\alpha\beta}) = &-\frac{1}{2}h^2\sum_{\alpha\beta} q_{\alpha\beta} + \frac{1}{2}\Bigg[(m+\delta m_1)\sum_{\alpha\beta} q_{\alpha\beta}^2\\ &+ m_2\sum_{\alpha\beta\gamma} q_{\alpha\gamma}q_{\beta\gamma} + m_3\sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}q_{\gamma\delta}\Bigg]\\ &-\frac{1}{6}\Bigg[(w+\delta w_1)\sum_{\alpha\beta\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha} + w_2\sum_{\alpha\beta} q_{\alpha\beta}^3\\ &+ w_3\sum_{\alpha\beta\gamma} q_{\alpha\beta}^2 q_{\beta\gamma} + w_4\sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}^2 q_{\gamma\delta}\\ &+ w_5\sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}q_{\alpha\gamma}q_{\beta\delta} + w_6\sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}q_{\alpha\gamma}q_{\alpha\delta}\\ &+ w_7\sum_{\alpha\beta\gamma\delta\mu} q_{\alpha\gamma}q_{\beta\gamma}q_{\delta\mu} + w_8\sum_{\alpha\beta\gamma\delta\mu\nu} q_{\alpha\beta}q_{\gamma\delta}q_{\mu\nu}\Bigg]\\ &+ O(q^4).\end{aligned} \tag{5}$$

Comparing Eqs. (4) and (5), one can conclude the following for the bare couplings of the field theory:

(i) Writing $m \equiv m_c - \tau$ with $\tau = 0$ at the critical point of the field theory, $m_c \sim (T_c^2 - T_c^{\rm mf2})$ results, where T_c and $T_c^{\rm mf}$ are the critical temperatures of the field theory and its mean-field approximation, respectively. This shows that m_c is the one-loop order.

(ii) The couplings h^2, δm_1, m_2, δw_1, w_2, w_3, and w_5 are of order $\bar{H}^2$, whereas all the other couplings are, at most, of order $\bar{H}^4$.

A simple three-parameter model was used in Ref. [11] to study, among other things, the AT instability for $6 < d < 8$, $d = 6$, and $d \lesssim 6$; the nonzero bare parameters were $m_1 = m = m_c - \tau$, $w_1 = w$, and h^2. It was found in [11] that the critical field $h_{\rm AT}^2$ behaves continuously while crossing the upper critical dimension 6 for fixed values of the reduced-temperature-like parameter τ and cubic coupling w, and the AT line takes the simple form

$$h_{\rm AT}^2 \approx \frac{4}{(1-w^2\ln\tau)^4}\, w\,\tau^2, \quad d = 6, \tag{6}$$

which is valid if $\tau \ll 1$ and $w^2 \ll 1$, in exactly the upper critical dimension. As the main motivation of the present paper, we want to check whether a suitable extension of this simple model in such a way that, besides h^2, the bare couplings m_2, w_2, w_3, and w_5 are small but nonzero will or will not modify the results of Ref. [11] about the AT instability around the zero-field critical point and for $d \lesssim 6$. In the dimensional regime $6 < d < 8$ where a standard perturbational method is applicable, an extended parameter space with all the couplings of order $\bar{H}^2$ seems to be a convenient extension. Below six dimensions, however, where the simple perturbative method breaks down (due to the more and more infrared divergent graphs as the number of loops increases), it becomes inevitable to apply the RG for the calculation of the asymptotical behavior of physical quantities. In this case, however, it is difficult to define the model by the set of bare couplings (by those, for instance, which are at least of order $\bar{H}^2$), as new couplings will be generated by the RG flow.

In the present paper, we propose to define the model by the set of nonlinear scaling fields: this ensures the closedness of the model under RG flow. The simple three-parameter model of Ref. [11] can be formulated in this way, and its extension will be done by introducing a new (masslike) nonlinear scaling field which, on the level of the bare couplings, leads to a more complicated model. In this more complicated field theory, one can calculate in $6-\epsilon$ dimensions the RS order parameter, the replicon, and longitudinal masses, all in the framework of the first-order RG combined with perturbational analysis. We focus on the asymptotical behavior close to the zero-field critical fixed point. The perturbative zero of the replicon mass defines the onset of the instability of the RS phase (AT surface). The problem of the runaway RG flows along this partially massless, i.e., massless in the replicon sector, manifold (caused by the repulsion of the critical fixed point) is also discussed.

The outline of the paper is as follows: The method of using nonlinear scaling fields for the calculation of physical quantities below six dimensions is discussed in Sec. II. The results in this section are equally valid below and above the critical temperature. The free propagators (replicon and longitudinal) are constructed in Sec. III. The central part of the paper is Sec. IV, where the critical asymptotics of physical quantities, such as the order parameter, the replicon, and longitudinal masses, are elaborated. The more interesting case of $T < T_c$ is presented in Sec. IV A, whereas results for $T > T_c$ are displayed for the sake of completeness and comparison in Sec. IV B. The limitations of the various approximations used to achieve our results are discussed in some detail in the next section. Zeros of the replicon mass are found in a region of the parameter space around the critical fixed point which belongs to the range of applicability of our approximations. There is also a discussion of this Almeida-Thouless critical manifold in Sec. V. Some conclusive remarks and a paragraph about the applied perturbative method are left to Sec. VI. The basic perturbative formulas are displayed in the Appendix.

Many results in this paper, especially the connection between bare parameters and nonlinear scaling fields in Sec. II, are built upon the first-order RG equations of Ref. [12].

II. BELOW SIX DIMENSIONS

The RG equations for the generic cubic field theory defined in Eqs. (2) and (3) can be obtained by integrating out degrees of freedom in a momentum shell at the cutoff Λ. The structure of these flow equations in the one-loop approximation for $n=0$ can be written as

$$\begin{aligned}
\dot{h^2} &= \left[4-\frac{\epsilon}{2}-\mathcal{H}^{(2)}(m_1,m_2,m_3;w_1,\dots,w_8)\right]h^2+\mathcal{H}^{(1)}(m_1,m_2,m_3;w_1,\dots,w_8),\\
\dot{m}_i &= 2\,m_i+\mathcal{M}_i^{(2)}(m_1,m_2,m_3;w_1,\dots,w_8),\quad i=1,2,3,\\
\dot{w}_i &= \frac{\epsilon}{2}\,w_i+\mathcal{W}_i^{(3)}(m_1,m_2,m_3;w_1,\dots,w_8),\quad i=1,\dots,8.
\end{aligned}\tag{7}$$

The functions $\mathcal{F}^{(k)}(m_1,m_2,m_3;w_1,\dots,w_8)$ above (with $\mathcal{F}=\mathcal{H}$, $\mathcal{M}_i$, or $\mathcal{W}_i$) are homogeneous polynomials of degree k in the w's and are analytic in the masses with a nonzero value for $m_1=m_2=m_3=0$. All but the first equation in (7) were published in Ref. [12], although the set of bare couplings was chosen differently there [13]. The flow equation for the magnetic field in the generic case, however, has not been published before:

$$\dot{h^2}=\left(4-\frac{\epsilon}{2}-\frac{1}{2}\,\eta_L\right)h^2+(3g_3+3g_6+2\bar{g}_7)\,\frac{1}{1+2m_1}+(3g_6+2\bar{g}_7)\,\frac{2m_2}{(1+2m_1)(1+2m_1-2m_2)}-2g_6\,\frac{m_2-2m_3}{(1+2m_1-2m_2)^2},\tag{8}$$

with

$$\eta_L=2g_3^2\,\frac{1+6m_1}{(1+2m_1)^4}-\frac{8}{3}\left(g_6^2+g_6\bar{g}_7\right)\frac{1+6m_1-6m_2}{(1+2m_1-2m_2)^4}+\frac{4}{3}\,g_6^2\,(m_2-2m_3)\,\frac{1+18m_1-18m_2}{(1+2m_1-2m_2)^5},\tag{9}$$

where we adopted the notation from Ref. [12] (see also [14]):

$$\begin{aligned}
g_3 &\equiv -w_1+w_2-\tfrac{1}{3}\,w_3,\quad g_6\equiv 2w_1-w_2+w_3-w_5-w_6,\\
\bar{g}_7 &\equiv -\tfrac{3}{2}\,w_1+\tfrac{1}{2}\,w_2-\tfrac{5}{6}\,w_3+\tfrac{2}{3}\,w_4+\tfrac{4}{3}\,w_5+w_6-\tfrac{2}{3}\,w_7.
\end{aligned}$$

One can derive the following information from the RG equations (7):

(i) The zeros on the right-hand side provide the fixed points. In this paper, we are interested in the vicinity of the zero-field critical fixed point: $w^{*2}=\frac{1}{2}\,\epsilon$, $m^*=-\frac{1}{2}\,w^{*2}=-\frac{1}{4}\,\epsilon$, with all the other couplings being zero. We prefer using $2w^{*2}$, instead of ϵ, in the remainder part of the paper.

(ii) All the eigenmodes of the linearized RG equations, with the only exception being that belonging to h^2, were published in [12]. In this paper we restrict ourselves to a model with the following four modes:

$$\begin{aligned}
&g_{h^2}\quad\text{with}\ \lambda_{h^2}=4-\tfrac{2}{3}\,w^{*2},\qquad g_{m_1}\quad\text{with}\ \lambda_{m_1}=2-\tfrac{10}{3}\,w^{*2},\quad g_{m_2}\quad\text{with}\ \lambda_{m_2}=2-\tfrac{4}{3}\,w^{*2},\\
&g_w\quad\text{with}\ \lambda_w=-2w^{*2}.
\end{aligned}\tag{10}$$

(iii) The g's above, with subscripts h^2, m_1, m_2, and w referring to the modes to which they belong, are nonlinear scaling fields [15] which satisfy exactly, by definition, the equations $\dot{g} = \lambda g$ and are zero at the fixed point. By means of the RG equations (7) above, one can express the original bare couplings in terms of the g's. Keeping the fields which break the zero-field symmetry (i.e., g_{h^2} and g_{m_2}) linear in these expressions (which is sufficient for a small external field), only the following couplings in $\delta\mathcal{L}$ are generated:

$$\begin{aligned} w^* h^2 &= \big(1 - \tfrac{1}{3}\, g_w - \tfrac{1}{3}\, w^{*2} g_{m_1}\big)\, g_{h^2} + \big(-w^{*2} - \tfrac{7}{3}\, w^{*2}\, g_w + 2\, g_{m_1}\big)\, g_{m_2}, \quad m_2 = \big(1 + \tfrac{4}{3}\, g_w + 5 w^{*2} g_{m_1}\big)\, g_{m_2},\\ w_2/w^* &= \big(-12 w^{*2} - 52\, w^{*2}\, g_w + 48\, w^{*2} g_{m_1}\big)\, g_{m_2}, \quad w_3/w^* = \big(\tfrac{49}{2}\, w^{*2} + \tfrac{637}{6}\, w^{*2}\, g_w - 94\, w^{*2} g_{m_1}\big)\, g_{m_2},\\ w_4/w^* &= \big(-\tfrac{9}{2}\, w^{*2} - \tfrac{39}{2}\, w^{*2}\, g_w + 18\, w^{*2} g_{m_1}\big)\, g_{m_2}, \quad w_5/w^* = \big(-\tfrac{1}{2}\, w^{*2} - \tfrac{13}{6}\, w^{*2}\, g_w - 2 w^{*2} g_{m_1}\big)\, g_{m_2}, \end{aligned} \tag{11}$$

whereas the symmetric couplings m_1 and w_1 are

$$\begin{aligned} m_1 - m^* &= \big[g_{m_1} - w^{*2} g_w + \tfrac{10}{3} g_{m_1} g_w - 2 w^{*2} g_w^2 + \tfrac{16}{3}\, w^{*2} g_{m_1}^2\big] + \big(-1 - \tfrac{4}{3} g_w - 5 w^{*2} g_{m_1}\big) g_{m_2},\\ w_1/w^* - 1 &= \big[5 w^{*2} g_{m_1} + g_w + \tfrac{190}{6} w^{*2}\, g_{m_1}\, g_w + \tfrac{3}{2}\, g_w^2 - 14\, w^{*2}\, g_{m_1}^2\big] + \big(\tfrac{1}{2}\, w^{*2} + \tfrac{13}{6}\, w^{*2}\, g_w + 2\, w^{*2} g_{m_1}\big)\, g_{m_2}. \end{aligned} \tag{12}$$

(The zero-field-symmetric part above has been written up to quadratic order in g_{m_1} and g_w.)

The three-parameter model of Ref. [11] corresponds to the three scaling fields: g_{m_1} and g_w span the symmetric (zero-field) system, whereas g_{h^2} breaks this symmetry. Having a look at Eqs. (11) and (12), one can realize that h^2 is the only coupling of the symmetry-breaking part $\delta\mathcal{L}$ which is generated. Therefore, this model can be equivalently defined by the bare couplings $m_1 = m$, $w_1 = w$, and h^2.

In the present paper, we supplement the model by g_{m_2}, whose introduction considerably complicates the model when it is written as in Eqs. (2) and (3). (One cannot avoid using this representation when, for instance, a scaling function is to be calculated.) The following couplings enter for a small g_{m_2}, according to Eqs. (11) and (12): δm_1, δw_1, m_2, w_2, w_3, w_4, and w_5.

Any observable $\mathcal{O}$ can now be considered as depending on the four scaling fields, and according to the generic theory in Sec. 5.1 of [11], one can write the following *asymptotically exact* expression around the fixed point:

$$\mathcal{O}\big(g_{m_1}, g_w; g_{h^2}, g_{m_2}\big) = \big|g_{m_1}\big|^{\frac{k}{\lambda_{m_1}}}\ \hat{\mathcal{O}}(x,y,z)\bigg[1 + \frac{k_{m_1}}{\lambda_{m_1}} g_{m_1} + \frac{k_w}{\lambda_w}\big|g_{m_1}\big|^{\frac{\lambda_w}{\lambda_{m_1}}} x + \frac{k_{h^2}}{\lambda_{h^2}}\big|g_{m_1}\big|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}} y + \frac{k_{m_2}}{\lambda_{m_2}}\big|g_{m_1}\big|^{\frac{\lambda_{m_2}}{\lambda_{m_1}}} z + \cdots\bigg], \tag{13}$$

where the RG *invariants* are defined as

$$x \equiv g_w\,\big|g_{m_1}\big|^{-\frac{\lambda_w}{\lambda_{m_1}}}, \quad y \equiv g_{h^2}\,\big|g_{m_1}\big|^{-\frac{\lambda_{h^2}}{\lambda_{m_1}}}, \quad z \equiv g_{m_2}\,\big|g_{m_1}\big|^{-\frac{\lambda_{m_2}}{\lambda_{m_1}}}. \tag{14}$$

The $\cdots$ symbol indicates neglected terms, namely, higher powers of the temperaturelike field g_{m_1} and/or quadratic or higher-order monomials of the RG invariants. The k's above are defined for a given $\mathcal{O}$ by its RG flow as

$$\dot{\mathcal{O}} = \big(k + k_{m_1} g_{m_1} + k_w g_w + k_{h^2} g_{h^2} + k_{m_2} g_{m_2} + \cdots\big)\,\mathcal{O}. \tag{15}$$

The scaling function $\hat{\mathcal{O}}$ is not determined by the renormalization group, but auxiliary information is needed (the perturbative method, for instance) to compute it. Hereinafter, we study three observables: the RS order parameter q, the replicon, and longitudinal masses, i.e., Γ_R and Γ_L.

III. FREE PROPAGATORS OF THE MODEL

When the order parameter q is nonzero, a reorganization of the perturbational series by the shift $\phi_{\mathbf{p}}^{\alpha\beta} \to \phi_{\mathbf{p}}^{\alpha\beta} - \sqrt{N}\, q\, \delta_{\mathbf{p}=0}^{\mathrm{Kr}}$ of the fluctuating fields is useful, as one then gets rid of "tadpole" insertions. As a result, the bare magnetic field and the masses suffer similar shifts:

$$\begin{aligned} h^2 &\to \bar{h^2} = h^2 + (-2m_1 + 2m_2)\, q + (-2w_1 + w_2 - w_3 + w_5)\, q^2,\\ m_1 &\to \bar{m}_1 = m_1 + \big(w_1 - w_2 + \tfrac{1}{3}\, w_3\big)\, q,\\ m_2 &\to \bar{m}_2 = m_2 + \big(-w_1 - \tfrac{2}{3}\, w_3 + w_5\big)\, q,\\ m_3 &\to \bar{m}_3 = m_3 + \big(-\tfrac{2}{3}\, w_4 - \tfrac{1}{3}\, w_5\big)\, q. \end{aligned}$$

In the $n \to 0$ limit, two free propagators emerge in the generic RS theory, namely,

$$\bar{G}_R = \frac{1}{p^2 + 2\bar{m}_1},$$

the replicon propagator, and

$$\bar{G}_L = \frac{1}{p^2 + 2\bar{m}_1 - 2\bar{m}_2},$$

the longitudinal propagator. Any perturbative contribution for some observable will, therefore, depend on q, which must be computed from the equation of state, i.e., from the condition $\langle \phi^{\alpha\beta}_{\mathbf{p}} \rangle = 0$. For the free propagators, we need the tree (zero-loop) approximation of this equation:

$$2w^* q = h^2 q^{-1} - 2m_1 + 2m_2 + [-2(w_1 - w^*) + w_2 - w_3 + w_5]\, q.$$

Using Eqs. (11) and (12) together with the definitions of the RG invariants in (14), the zero-loop order parameter follows, up to first order in x, y, and z, as

$$w^* q = |g_{m_1}| \times \begin{cases} 1 + \frac{7}{3}x + z + \frac{1}{2}y & \text{if } g_{m_1} < 0,\ T < T_c, \\ z + \frac{1}{2}y & \text{if } g_{m_1} > 0,\ T > T_c. \end{cases}$$

This is the point where the calculations above and below T_c separate. Writing the free propagators as

$$\bar{G}_R = \frac{1}{p^2 + |g_{m_1}| \times R}, \quad \bar{G}_L = \frac{1}{p^2 + |g_{m_1}| \times L}, \tag{16}$$

it is obtained in the two respective regimes: for $T < T_c$,

$$R = y, \quad L = 2 + \tfrac{20}{3}x + 2y, \quad g_{m_1} < 0; \tag{17}$$

for $T > T_c$,

$$R = 2 + \tfrac{20}{3}x + y, \quad L = 2 + \tfrac{20}{3}x + 2y, \quad g_{m_1} > 0. \tag{18}$$

Terms higher than first order in x, y, and z are again neglected in the above formulas, in accordance with the smallness of these RG invariants.

IV. ASYMPTOTICAL BEHAVIOR AROUND T_c

A. Below T_c ($g_{m_1} < 0$)

1. The order parameter q

The RG flow for q is simply $\dot{q} = (2 - w^{*2} + \eta_L/2)\, q$ with η_L in (9). Inserting the nonlinear scaling fields with the help of (11) and (12), the k coefficients for q can be read off by the general definition in (15): $k = 2 - \frac{4}{3}\, w^{*2}$, $k_{m_1} = \frac{2}{3}\, w^{*2}$, $k_w = -\frac{2}{3}\, w^{*2}$, $k_{h^2} = 0$, and $k_{m_2} = -\frac{2}{3}\, w^{*2}$. Using the eigenvalues of the various modes from (10), the generic scaling form in (13) becomes

$$q = |g_{m_1}|^{1+w^{*2}}\ \hat{q}(x,y,z)\big[1 + \tfrac{1}{3}\, w^{*2}\, g_{m_1} + \tfrac{1}{3}\, |g_{m_1}|^{-w^{*2}}\, x - \tfrac{1}{3}\, w^{*2}\, |g_{m_1}|^{1+w^{*2}}\, z + \cdots\big]. \tag{19}$$

Comparing this RG formula with its perturbative counterpart [16]

$$\begin{aligned} w^* q = |g_{m_1}| \{ &[1 + (2 + \ln 2)\, w^{*2}] + \tfrac{1}{2}\, [1 + (4 - 2\ln 2)\, w^{*2}]\, y + \tfrac{7}{3}\, [1 + O(w^{*2})]\, x + [1 + O(w^{*2})]\, z \} \\ &+ w^{*2}\, \big(|g_{m_1}|\, \ln |g_{m_1}|\big)\, \big(1 + \tfrac{1}{2}\, y + 2x + z\big) \end{aligned} \tag{20}$$

[which follows from Eq. (A1) by using (11), (12), (10), (14), (A2), and (17)] has a double use: First, the scaling function can be derived as

$$w^* \hat{q} = [1 + (2 + \ln 2)\, w^{*2}] + \tfrac{1}{2}\, [1 + (4 - 2\ln 2)\, w^{*2}]\, y + [2 + O(w^{*2})]\, x + [1 + O(w^{*2})]\, z. \tag{21}$$

Second, the logarithm in (20) should correctly exponentiate in accordance with the asymptotic scaling above: this property is easily checked by comparison.

2. The replicon mass

The replicon mass satisfies the equation $\dot{\Gamma}_R = (2 - \eta_R)\, \Gamma_R$, with η_R computed in Ref. [12]. (See also [14].) Instead of providing the complete formula for η_R here again, we show it expressed and linearly truncated in terms of the nonlinear scaling fields:

$$\eta_R = -\tfrac{2}{3}\, w^{*2}\big(1 - 2g_{m_1} + g_{m_2} + 2g_w\big).$$

The k coefficients (of Γ_R) follow then from (15): $k = 2 + \frac{2}{3}\, w^{*2}$, $k_{m_1} = -\frac{4}{3}\, w^{*2}$, $k_w = \frac{4}{3}\, w^{*2}$, $k_{h^2} = 0$, and $k_{m_2} = \frac{2}{3}\, w^{*2}$. The generic result (13) can then be translated to the case of the replicon mass [see also (10)] as

$$\Gamma_R = |g_{m_1}|^{1+2w^{*2}}\ \hat{\Gamma}_R(x,y,z)\big[1 - \tfrac{2}{3}\, w^{*2}\, g_{m_1} - \tfrac{2}{3}\, |g_{m_1}|^{-w^{*2}}\, x + \tfrac{1}{3}\, w^{*2}\, |g_{m_1}|^{1+w^{*2}}\, z + \cdots\big]. \tag{22}$$

The corresponding perturbative formula follows from (A3) and the use of Eqs. (11), (12), (10), (14), (A4), and (17):

$$\Gamma_R = |g_{m_1}| \{-4w^{*2} + [1 + (-8 + 3\ln 2 - 4\ln y) w^{*2}] y + O(w^{*2})\, x + O(w^{*2})\, z\} + w^{*2}\, \big(|g_{m_1}|\, \ln |g_{m_1}|\big)\, 2y. \tag{23}$$

Matching these two expressions of the replicon mass provides the scaling function:

$$\hat{\Gamma}_R(x,y,z) = -4w^{*2} + [1 + (-8 + 3\ln 2 - 4\ln y)\, w^{*2}]\, y + O(w^{*2})\, x + O(w^{*2})\, z, \tag{24}$$

and it is easy to check that the criterion of proper exponentiation is satisfied.

3. The longitudinal mass

The k coefficients, defined in (15), for Γ_L follow from its RG equation $\dot{\Gamma}_L = (2 - \eta_L)\,\Gamma_L$ and Eqs. (9), (11), and (12): $k = 2 + \frac{2}{3}\, w^{*2}$, $k_{m_1} = -\frac{4}{3}\, w^{*2}$, $k_w = \frac{4}{3}\, w^{*2}$, $k_{h^2} = 0$, and $k_{m_2} = \frac{4}{3}\, w^{*2}$. Just like for the replicon case, one can easily conclude the scaling form of the longitudinal mass:

$$\Gamma_L = |g_{m_1}|^{1+2w^{*2}}\, \hat{\Gamma}_L(x,y,z)\big[1 - \tfrac{2}{3} w^{*2} g_{m_1} - \tfrac{2}{3} |g_{m_1}|^{-w^{*2}} x + \tfrac{2}{3} w^{*2}\, |g_{m_1}|^{1+w^{*2}} z + \cdots\big], \tag{25}$$

which can be compared with (A5) in the Appendix:

$$\begin{aligned}\Gamma_L = |g_{m_1}|\big\{&[2 + (-8 + 4\ln 2)\, w^{*2}] + [2 + (1 + 4\ln 2 - 6\ln y)\, w^{*2}]y + \tfrac{20}{3}[1 + O(w^{*2})]x + O(w^{*2})z\big\}\\ &+ 4w^{*2}\left(|g_{m_1}|\, \ln|g_{m_1}|\right)\left(1 + y + \tfrac{11}{3}\, x\right).\end{aligned} \tag{26}$$

[Using Eqs. (11), (12), (10), (14), (A6), and (17) is necessary to put (A5) into this form.] The scaling function can now be read off as

$$\hat{\Gamma}_L(x,y,z) = [2 + (-8 + 4\ln 2)\, w^{*2}] + [2 + (1 + 4\ln 2 - 6\ln y)\, w^{*2}]\, y + [8 + O(w^{*2})]\, x + O(w^{*2})\, z, \tag{27}$$

and exponentiation can be checked.

B. Results for $T > T_c$ ($g_{m_1} > 0$)

For the sake of completeness and a possible comparison with the $T < T_c$ case, results for the three observables above the critical temperature (in a small but finite magnetic field) are presented in this section. Their scaling forms in Eqs. (19), (22), and (25) are equally valid in this high-temperature asymptotical regime; the scaling functions, however, are different. Due to the change in the free propagators according to (16) and (18), the one-loop perturbative results are now [instead of (20), (23), and (26)]

$$\begin{aligned}w^* q &= g_{m_1}\big\{\tfrac{1}{2}\,[1 - (1 + 2\ln 2)\, w^{*2}]\, y + [1 + O(w^{*2})]\, z\big\} + w^{*2}\left(g_{m_1}\, \ln g_{m_1}\right)\left(\tfrac{1}{2}\, y + z\right),\\ \Gamma_R &= g_{m_1}\big\{2[1 + (1 + 2\ln 2)\, w^{*2}] + \tfrac{1}{2}[2 + (1 - 2\ln 2)\, w^{*2}]\, y + \big[\tfrac{20}{3} + O(w^{*2})\big]x + O(w^{*2})z\big\}\\ &\quad + w^{*2}\left(g_{m_1}\, \ln g_{m_1}\right)\big[4 + 2y + \tfrac{44}{3}\, x\big],\\ \Gamma_L &= g_{m_1}\big\{2[1 + (1 + 2\ln 2)\, w^{*2}] + \tfrac{1}{2}[4 + (5 + 2\ln 2)\, w^{*2}]y + \big[\tfrac{20}{3} + O(w^{*2})\big]\, x + O(w^{*2})z\big\}\\ &\quad + w^{*2}\left(g_{m_1}\, \ln g_{m_1}\right)\big[4 + 4y + \tfrac{44}{3}x\big].\end{aligned}$$

After a comparison with the scaling forms in Eqs. (19), (22), and (25), the scaling functions above the critical temperature can be concluded:

$$w^* \hat{q} = \tfrac{1}{2}[1 - (1 + 2\ln 2)\, w^{*2}]\, y + [1 + O(w^{*2})]z, \tag{28}$$

$$\hat{\Gamma}_R(x,y,z) = 2[1 + (1 + 2\ln 2)\, w^{*2}] + \big[1 + \tfrac{1}{2}\,(1 - 2\ln 2)\, w^{*2}\big]\, y + [8 + O(w^{*2}]\, x + O(w^{*2})\, z, \tag{29}$$

$$\hat{\Gamma}_L(x,y,z) = 2[1 + (1 + 2\ln 2)\, w^{*2}] + \big[2 + \tfrac{1}{2}\,(5 + 2\ln 2)\, w^{*2}\big]\, y + [8 + O(w^{*2}]\, x + O(w^{*2})\, z. \tag{30}$$

One can make the following observations about the behavior of the three quantities around the critical point:

(i) The high-temperature ($g_{m_1} > 0$) and zero-external-magnetic-field ($y = z = 0$) phase possesses a higher symmetry with a zero-order parameter [see (28)] and a single mass [due to the degeneration between the replicon and longitudinal masses; see Eqs. (29) and (30)].

(ii) In zero-external-magnetic-field ($y = z = 0$) below the critical temperature ($g_{m_1} < 0$), the order parameter is nonzero (Eq. (21); this is the RS spin-glass phase invented by Edwards and Anderson [2]). However, according to Eq. (24), the replicon mass becomes negative due to the one-loop term, showing that this phase is unstable, just as in mean-field theory [9], and replica symmetry must be broken.

(iii) There is a slight splitting between the replicon and longitudinal masses in a small magnetic field above T_c [Eqs. (29) and (30)], whereas the longitudinal mass is definitely massive below T_c [Eq. (27)] and therefore separates from the replicon one.

(iv) It is obvious from Eq. (24) that stability of the RS phase is restored for $y > y_0 \sim O(w^{*2})$ and $g_{m_1} < 0$.

V. DISCUSSION: RANGE OF APPLICABILITY AND ASYMPTOTICALLY DETECTED ALMEIDA-THOULESS INSTABILITY

In deriving our basic results for the scaling forms and scaling functions of the three observables (q, Γ_R, and Γ_L), several approximations were applied in the previous section. To see clearly the limits of these approximations, it might be useful to give an overall list of them here:

(i) The RG equations and the auxiliary perturbative calculations have the one-loop character, and therefore, $w^{*2} = \epsilon/2 \ll 1$.

(ii) The multiplicative factor (which is analytic in the fields g_{m_1}, g_w, g_{h^2}, and g_{m_2}) in the, in principle, exact scaling formula (13) was truncated to linear order in the nonlinear scaling fields. We must have, therefore, $|g_{m_1}|$, $|g_w|$, $|g_{m_2}|$, and g_{h^2} much smaller than unity. In fact, the normalization of the nonlinear scaling fields (which is not fixed originally) was chosen in such a way that their asymptotic regime around the fixed point is independent of ϵ.

(iii) Quadratic and higher-order terms in the RG invariants were neglected in the scaling functions, i.e., $|x| \ll 1$, $|z| \ll 1$, and $y \ll 1$. The first of them is automatically fulfilled if $|g_w| \ll 1$ since g_w is an irrelevant field. The other two fields are relevant, and therefore, we have the stronger conditions $|g_{m_2}| \ll |g_{m_1}|^{\frac{\lambda_{m_2}}{\lambda_{m_1}}}$ and $g_{h^2} \ll |g_{m_1}|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}}$.

(iv) Up to this point, we have conditions for the parameters of the effective field theory representing the physical spin glass. Translating the above results as a requirement between temperature and magnetic field, we observe that $|g_{m_1}|$ is proportional to the reduced temperature, $g_{h^2} \approx w^* h^2 \sim H^2$, and $|g_{m_2}| \approx |m_2| \sim H^2$ [see Eqs. (4) and (5)]. As λ_{h^2} is the leading relevant eigenvalue, we arrive at

$$H^2 \ll |g_{m_1}|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}} \sim \left|\frac{T - T_c}{T_c}\right|^{\frac{\lambda_{h^2}}{\lambda_{m_1}}}.$$

An important consequence of the above analysis is that the ratio z/y is independent of H^2 and $z \ll y$: this justifies the simple three-parameter model in [11] with the fields $|g_{m_1}|$, g_w, and g_{h^2} (or, equivalently, m_1, w_1, and h^2). Anyway, z entered only the scaling function for q in (21).

The scaling functions in Eqs. (21), (24), and (27) for $T < T_c$ [and also Eqs. (28), (29), and (30) for $T > T_c$] constitute our basic result: they are the leading part of a perturbative series, and one could calculate, in principle, any higher-order terms in ϵ and/or in the invariants (say y). These series belong completely to the critical fixed point; in other words, they are characteristics of the zero-magnetic-field fixed point. Their validity is therefore independent of the fate of the relevant couplings (like h^2, m_2, and w_i, $i = 2, \ldots, 5$) under the iteration of the renormalization group, i.e., whether they approach another fixed point (perturbative or nonperturbative) or flow away to infinity.

As a matter of fact, the question is what information you can extract from these perturbative series. Let's make this point clearer with the case of the longitudinal mass in (27). (For the sake of simplicity, invariants other than y are neglected in the following discussion.) $\hat{\Gamma}_L$ is positive for $y = 0$; that is, the zero-field spin-glass phase is longitudinally massive. Although it is physically plausible that Γ_L remains massive in an external field too, this cannot be verified using (27) (or from a longer series), as a nonperturbative zero of $\hat{\Gamma}_L$ is not available from such a series.

The situation is fundamentally different for the replicon mass $\hat{\Gamma}_R(y)$ below T_c, as it has a *perturbative* zero: $y_0 = 4w^{*2} + \cdots$, whereas the longitudinal mode is massive, $\Gamma_L = 2 + 4\ln 2\, w^{*2} + \cdots$, along this Almeida-Thouless instability surface. $\hat{\Gamma}_R(y)$ will probably be singular at this zero:

$$\hat{\Gamma}_R(y) \sim (y - y_0)^{\dot{\gamma}},$$

with some exponent. This asymptotic form, however, cannot be verified from the series (24) due to the lack of proper exponentiation. The exponent $\dot{\gamma}$ cannot be extracted from (24), as it does not belong to the critical fixed point but possibly to some, at this moment unknown, zero-temperature fixed point. (The scenario drafted above follows closely the crossover behavior at a bicritical point presented in Ref. [17].)

VI. FINAL REMARKS

It has been shown how one can detect the critical surface with zero replicon mass (the Almeida-Thouless critical manifold) asymptotically in the close vicinity of the zero-magnetic-field fixed point *perturbatively* just below the upper critical dimension. Nevertheless, this AT critical surface is spanned by relevant scaling fields like g_{h^2} and g_{m_2}, which break the symmetry of the critical zero-magnetic-field fixed point, and runaway RG flows toward infinite couplings follow [10]. The lack of an attractive perturbative fixed point governing the AT instability surface [18] and the runaway flows can be understood by the schematic phase diagrams from Refs. [11,19]: RG flows along the AT line terminate into a zero-temperature fixed point, and the effective cubic field theory (fitted to the asymptotics around the zero-magnetic-field critical transition) is, in fact, *not* appropriate for representing the zero-temperature spin glass. A field theory for the low-temperature spin glass is obviously sorely needed for the understanding of the critical asymptotics along the AT line.

What is claimed above, namely, that the existence of a spin-glass transition in an external magnetic field may be possible even if the RG trajectories run away from the critical fixed point without terminating in a perturbative novel fixed point, has been demonstrated with a simpler model in which the interaction depends on the hierarchical distance between the Ising spins, i.e., in the hierarchical Edwards-Anderson (HEA) model. A first-order RG analysis of the generic replica symmetric phase [20] in Ref. [21] found no relevant fixed point governing the transition in a field: the couplings renormalize toward infinite values. Notwithstanding that, a careful Monte Carlo simulation on a modified version of the HEA [22] provided evidence for a transition in nonzero external field through a study of the spin-glass susceptibility and the correlation length associated with it. Most importantly, Ref. [22] found a transition in nonzero field in the non-mean-field region $\sigma \gtrsim 2/3$, where σ, the parameter of the HEA analogous to the spatial dimension d of the short-range model in Euclidean space, was within 2% of the upper critical value $\sigma = 2/3$. This clearly shows that the AT instability persists while traversing the analog of the upper critical dimension from the mean-field to the non-mean-field region, in spite of the absence of a perturbative fixed point governing the AT critical surface [21]. One point is still lacking here, namely, the observation of the transition perturbatively by computing the asymptotical behavior of the spin-glass susceptibility (or, equivalently, the replicon mass) around the critical fixed point. This is left to a subsequent work.

As for the short-range model, it has been advocated for some time [23,24] that the lower critical dimension for the AT line should be $d = 6$, i.e., that the spin-glass transition in an external field disappears just at the upper critical dimension of the zero-field model. The fault in the arguments of Ref. [23] about the behavior of the AT line (computed perturbatively for $d > 6$), namely, that it disappears while approaching $d = 6$ from above, was pointed out in [11]. The issue was reconsidered in Ref. [25], admitting now that the six-dimensional AT line cannot be derived by a limiting process from the perturbative result in $d > 6$. Yeo and Moore [25], however, incorrectly claimed that the calculation of the six-dimensional AT line in [11] was performed by just this wrong limiting process. In fact, the $d = 6$ case was studied separately in Ref. [11], as it must be, by the special one-loop perturbative RG at the upper critical dimension where the scaling exponent of the cubic coupling constant is zero (see also Refs. [26,27]).

From the discussion in the last two paragraphs and also from the results of the present paper, it follows that the lower critical dimension for the spin-glass transition of the Ising spin glass in an external magnetic field is probably less than $d = 6$. One must, however, emphasize that the perturbative RG is not able to make predictions about the existence of the AT line far below $d = 6$. Numerical simulation results in $d = 3$ and $d = 4$ (or in the corresponding long-range one-dimensional model as a "proxy" for the short-range system) in this regard are controversial (see [28,29] and references therein).

Finally, we have some notes about the perturbative method: The calculations of physical quantities were performed in the present paper with the combined use of the renormalization group and a series expansion in terms of the coupling constants. This method is absolutely conventional *around* a perturbative fixed point: a perturbative result like (20), for instance, is interpreted by the RG ansatz in (19), and the scaling function can be identified as in (21). In the meantime, a consistency check is available with the proper exponentiation of the logarithms of the temperature-like scaling field. Two peculiarities, however, occur: The first one is due to the quadratic symmetry breaking caused by the nonzero RS order parameter which leads to the two distinct free propagators, with the replicon mode being almost massless in a small magnetic field below T_c. The other one is related to the replicated nature of the field theory, which may cause problems in the $n \to 0$ spin-glass limit. Although this limit proved to be quite smooth in our model with four scaling fields, the behavior and physical meaning of the remaining modes, like the third mass mode, are not clear.

APPENDIX: SUMMARY OF SOME ONE-LOOP RESULTS FOR THE GENERIC REPLICA SYMMETRIC THEORY

In this Appendix, we provide results which are equally valid in the high- and low-temperature regimes, assuming that the proper values of R and L [see Eqs. (18) and (17)] must be inserted.

1. The equation of state

The order parameter q satisfies the implicit equation

$$2w^*q = h^2 q^{-1} - 2m_1 + 2m_2 + [-2(w_1 - w^*) + w_2 - w_3 + w_5]\, q + q^{-1}\frac{1}{N}\sum_{\vec{p}} Y(p), \tag{A1}$$

with the one-loop graph

$$Y(p) = \big(w_2 + \tfrac{1}{3}w_3 + \tfrac{4}{3}w_4 - \tfrac{1}{3}w_5 - w_6 - \tfrac{4}{3}w_7\big)\,\bar{G}_R + \big(3w_1 - 2w_2 + \tfrac{4}{3}w_3 + \tfrac{4}{3}w_4 - \tfrac{1}{3}w_5 - w_6 - \tfrac{4}{3}w_7\big)\,2\bar{m}_2\,\bar{G}_R\bar{G}_L$$
$$+ (4w_1 - 2w_2 + 2w_3 - 2w_5 - 2w_6)(-\bar{m}_2 + 2\bar{m}_3)\bar{G}_L^2.$$

This one-loop integral can be computed, and one gets

$$w^*\frac{1}{N}\sum_{\vec{p}} Y(p) = w^{*2}\,|g_{m_1}|^2(1-2x)^{-1/2}\Big[\tfrac{1}{2}(R-L)\,|g_{m_1}|^{-1} + \tfrac{1}{2}(L-R)(L-3R)\ln|g_{m_1}| + \tfrac{3}{2}R^2\ln R$$
$$+ \tfrac{1}{2}L(L-4R)\ln L + L(L-R)\Big] + O(w^{*4}). \tag{A2}$$

2. The replicon mass

The one-loop formula for the replicon mass was published in [6] [see Eqs. (49a)–(49h) and (62) therein]. Here we reproduce it in terms of the set of bare parameters used throughout the present paper and for $n = 0$:

$$\Gamma_R = 2m_1 + 2w^*q + 2\big[(w_1 - w^*) - w_2 + \tfrac{1}{3}w_3\big]\,q - \frac{1}{N}\sum_{\vec{p}}\sigma_R, \tag{A3}$$

with the replicon self-energy

$$\sigma_R = \big(-2w_1^2 - \tfrac{4}{3}\,w_1w_3 - \tfrac{16}{3}\,w_1w_4 - \tfrac{8}{3}\,w_1w_5 + 2w_2^2 + \tfrac{8}{3}\,w_2w_3 + \tfrac{16}{3}\,w_2w_4 + \tfrac{4}{3}\,w_2w_5 + \tfrac{2}{9}\,w_3^2 - \tfrac{16}{9}\,w_3w_4 - \tfrac{8}{9}\,w_3w_5 + \tfrac{4}{9}\,w_5^2\big)\bar{G}_R^2 + \big(-2w_1^2 + 12w_1w_2 + \tfrac{4}{3}\,w_1w_3 - \tfrac{16}{3}\,w_1w_4 - \tfrac{16}{3}\,w_1w_5 - 8w_2^2 + \tfrac{4}{3}\,w_2w_3 + \tfrac{16}{3}\,w_2w_4 + \tfrac{8}{3}\,w_2w_5 + \tfrac{6}{9}\,w_3^2 - \tfrac{16}{9}\,w_3w_4 - \tfrac{16}{9}\,w_3w_5 + \tfrac{8}{9}\,w_5^2\big)2\bar{m}_2\bar{G}_R^2\bar{G}_L + \big(-8w_1^2 + 16w_1w_2 - \tfrac{16}{3}\,w_1w_3 - 8w_2^2 + \tfrac{16}{3}\,w_2w_3 - \tfrac{8}{9}\,w_3^2\big) \times(-\bar{m}_2 + 2\bar{m}_3)\,\bar{G}_R\bar{G}_L^2 + \tfrac{1}{9}\,(6w_1 - 3w_2 + 2w_3 - 2w_5)^2 4\bar{m}_2{}^2\,\bar{G}_R^2\bar{G}_L^2.$$

Performing the momentum integral provides

$$\frac{1}{N}\sum_{\vec{p}} \sigma_R = w^{*2}\,|g_{m_1}|(1-2x)^{-1}\bigg[-|g_{m_1}|^{-1} + (L-3R)\,\ln|g_{m_1}| + \frac{R(4L+3R)}{L-R}\ln R + \frac{L(L-8R)}{L-R}\ln L + 2(R+2L)\bigg] + O(w^{*4}). \tag{A4}$$

3. The longitudinal mass

The first-order expression for the longitudinal mass takes the form (for $n = 0$)

$$\Gamma_L = 2m_1 - 2m_2 + 4w^*q + 2\,[2(w_1 - w^*) - w_2 + w_3 - w_5]\,q - \frac{1}{N}\sum_{\vec{p}}\sigma_L, \tag{A5}$$

with the longitudinal self-energy (which is identical to the anomalous one when $n = 0$; see Eqs. (49a)–(49h), (63), and (64) of [6])

$$\sigma_L = \big[6\big(w_1 - w_2 + \tfrac{1}{3}w_3\big)^2 - \tfrac{4}{3}(2w_1 - w_2 + w_3 - w_5 - w_6)(3w_1 - 3w_2 + w_3 + 4w_4 + 2w_5 - 4w_7)\big] \times\bar{G}_R^2 - \tfrac{8}{3}\,(2w_1 - w_2 + w_3 - w_5 - w_6)\,(3w_1 - 3w_2 + w_3 + 4w_4 + 2w_5 - 4w_7)\big(2\bar{m}_2\,\bar{G}_R^2\bar{G}_L + 2\bar{m}_2{}^2\,\bar{G}_R^2\bar{G}_L^2\big) - 8(-2w_1 + w_2 - w_3 + w_5 + w_6)^2(-\bar{m}_2 + 2\bar{m}_3)\,\bar{G}_L^3.$$

After integration it becomes

$$\frac{1}{N}\sum_{\vec{p}}\sigma_L = w^{*2}\,|g_{m_1}|\,(1-2x)^{-1}\big[-|g_{m_1}|^{-1} - 2R\,\ln|g_{m_1}| + 6R\ln R - 8R\ln L + (8L - 9R)\big] + O(w^{*4}). \tag{A6}$$

[1] K. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).

[2] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[3] A. B. Harris, T. C. Lubensky, and J.-H. Chen, Phys. Rev. Lett. **36**, 415 (1976).

[4] E. Pytte and J. Rudnick, Phys. Rev. B **19**, 3603 (1979).

[5] A. J. Bray and M. A. Moore, J. Phys. C **12**, 79 (1979).

[6] T. Temesvári, C. De Dominicis, and I. R. Pimentel, Eur. Phys. J. B **25**, 361 (2002).

[7] J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).

[8] J. E. Green, M. A. Moore, and A. J. Bray, J. Phys. C **16**, L815 (1983).

[9] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).

[10] T. Temesvári, Phys. Rev. B **78**, 220401(R) (2008).

[11] G. Parisi and T. Temesvári, Nucl. Phys. B **858**, 293 (2012).

[12] I. R. Pimentel, T. Temesvári, and C. De Dominicis, Phys. Rev. B **65**, 224420 (2002).

[13] For the sake of ease for the reader, we give here the precise citations where the linear connection between the two sets of couplings can be found: For the masses (i.e., m_R, m_A, and m_L in [12] versus m_1, m_2, and m_3 here) see Eq. (32) of [12] and Eqs. (22)–(24) of [6], whereas for the cubic couplings (i.e., g_i, $i = 1,\ldots,8$, in [12] versus w_i, $i = 1,\ldots,8$, here) see Eqs. (49a)–(49h) of [6].

[14] The last term in $\eta_L = \eta_A$ is incorrectly missing in Eq. (87) of Ref. [12]. A similar term proportional to $m_2 - 2m_3$ was also left out from expression (86) for η_R.

[15] The general theory of the application of nonlinear scaling fields was briefly summarized in Sec. 5.1 of Ref. [11]. The concept of nonlinear scaling fields was introduced by Wegner [30].

[16] The notation $O(w^{*2})$ will be consistently used in this section whenever the corresponding correction is not available in this first-order RG calculation.

[17] D. J. Amit and Y. Y. Goldschmidt, Ann. Phys. (NY) **114**, 356 (1978).

[18] A. J. Bray and S. A. Roberts, J. Phys. C **13**, 5405 (1980).

[19] M. C. Angelini and G. Biroli, Phys. Rev. Lett. **114**, 095701 (2015).

[20] An analogous study for the short-range model in Euclidean d-dimensional space can be found in Ref. [12], where the eigenmodes of the linearized RG around the critical fixed point are also presented.

[21] M. Castellana and C. Barbieri, Phys. Rev. B **91**, 024202 (2015).

[22] M. Castellana and G. Parisi, Sci. Rep. **5**, 8697 (2015).

[23] M. A. Moore and A. J. Bray, Phys. Rev. B **83**, 224408 (2011).

[24] M. Moore, Phys. Rev. E **86**, 031114 (2012).

[25] J. Yeo and M. A. Moore, Phys. Rev. B **91**, 104432 (2015).
[26] T. Temesvári, Phys. Rev. B **94**, 176401 (2016).
[27] J. Yeo and M. A. Moore, Phys. Rev. B **94**, 176402 (2016).
[28] R. A. Baños, A. Cruz, L. Fernandez, J. Gil-Narvion, A. Gordillo-Guerrero, M. Guidetti, D. Iniguez, A. Maiorano, E. Marinari *et al.*, Proc. Natl. Acad. Sci. USA **109**, 6452 (2012).
[29] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Phys. Rev. B **87**, 024414 (2013).
[30] F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).