Solutions to the problems

Problem 1. With the initial position considered to be the origin, the two coordinates depend on time as $x(t) = u_0 t$, and $y(t) = v_0 t - g t^2/2$. After time T, the coordinates $x(T)$ and $y(T)$ fall on a surface of slope m (tilted from left to right), thus $-y(T)/x(T) = m$, from which T follows. Since the vertical velocity is $v(t) = v_0 - gt$, at $t = T$ the impact velocity is $v' = v(T)$. Note, that the horizontal velocity component remains u_0 at any time.

Problem 2. From (6) $L = Hu_0^2/g$, and by taking $g = 10\ m/s^2$ and $u_0 = 1$ m/s, we obtain $L = 40$ cm. On the other hand, $u_0 = \sqrt{gL/H}$, and with $H = 0.0625 = 1/16$ $u_0 = 8$ m/s follows.

Problem 3. The height of the ball at time t after the nth bounce, measured from the origin (or from the horizontal part) of the step is:

$$y(t)L = y_n L + v_n u_0 t - \frac{g}{2} t^2$$

since we have decided to measure length and velocity in units of L and u_0 respectively. With this choice the time unit should be L/u_0. When replacing t by tL/u_0, where the new t represents dimensionless time, we obtain the dimensionless height as

$$y(t) = y_n + v_n t - \frac{H}{2} t^2.$$

Here parameter H defined in (6) has shown up reinforcing its interpretation as a dimensionless form of g. The horizontal distance from the origin is then $x(t) = x_n + u_n t$. To determine the time $\tilde{t}_n$ passed since the nth collision, we use the fact that at the next collision the ball will hit the step at $y = -m\tilde{N}_n$, i.e., $y_n + v_n \tilde{t}_n - (H/2)(\tilde{t}_n)^2 = -m\tilde{N}_n$, whence

$$\tilde{t}_n = \frac{\sqrt{v_n^2 + 2H(m\tilde{N}_n + y_n)} + v_n}{H}.$$

The collision occurs with horizontal and vertical velocity components

$$\tilde{u}_n' = u_n, \quad \tilde{v}' = v_n - H\tilde{t}_n = -\sqrt{v_n^2 + 2H(m\tilde{N}_n + y_n)},$$

in accordance with (10).

Horizontally, the ball is at a distance of $x_n + u_n \tilde{t}_n$ from the origin, on its right. The jump number $\tilde{N}_n$ is none other than the number of times the unit interval can be included in this interval, i.e. the integer part of the distance. Substituting $\tilde{t}_n$,

$$\tilde{N}_n = \left[x_n + \frac{u_n}{H} \left(v_n + \sqrt{v_n^2 + 2H(m\tilde{N}_n + y_n)} \right) \right],$$

where the square brackets represent the integer part.

As the origin of the coordinate system is placed to the left end of the step on which the nth collision occurs, the coordinate $\tilde{x}_{n+1}$ of the next bounce is the difference of the horizontal displacement and $\tilde{N}_n$, that is:

$$\tilde{x}_{n+1} = x_n + \frac{u_n}{H} \left(v_n + \sqrt{v_n^2 + 2H(m\tilde{N}_n + y_n)} \right) - \tilde{N}_n$$

as stated in (8).

Problem 4. The ball's y and x coordinates as the function of time are

$$y(t) = \tilde{v}'_n t - \frac{H}{2}t^2, \quad x(t) = \tilde{x}_{n+1} + \tilde{u}'_n t.$$

The y coordinate of the curvature at the x coordinate of the ball:

$$y_0(x(t)) = y_0(t) = -r + \sqrt{r^2 - (1 - r + x(t))^2}$$

Substituting $x(t) = \tilde{x}_{n+1} + \tilde{u}'_n t$, and knowing the ball's y coordinate ($y(t) = \tilde{v}'_n t - Ht^2/2$), the vertical distance between the curvature and the ball after time t is:

$$\Delta y(t) = y(t) - y_0(t) = \tilde{v}_n - \frac{H}{2}t^2 + r - \sqrt{r^2 - (1 - r + \tilde{x}_{n+1} + \tilde{u}'_n t)^2}.$$

The flight time after the imagined collision with the rectangular stair is the value t^* for which $\Delta y(t^*) = 0$ with the accuracy prescribed. The full flight time between the nth and $n+1^{st}$ collsions is then $t_n = \tilde{t}_n + t^*$.

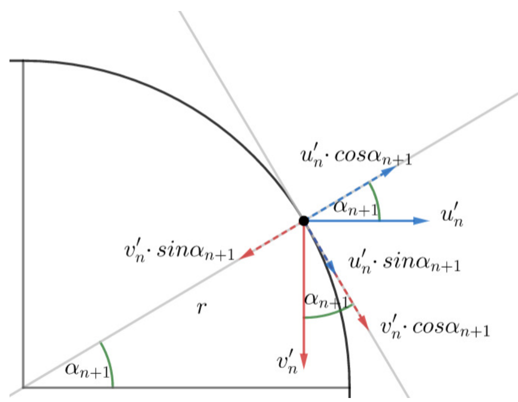

Figure 1: Transforming the impact velocity from perpendicular to polar components. Remember that v'_n is negative, and this should be taken into account when writing down the polar components.

Problem 5. The impact velocity components are $u'_n > 0$, $v'_n < 0$, but we would like to express this vector in polar coordinates. The radial component v_r is considered positive if it points outward, while the tangential one, v_t, we choose to be positive if it points in the clockwise direction (as this component remains positive during the collision).

One can read off from Fig. 1 that the polar components are

$$v'_{r,n} = u'_n \cos\alpha_{n+1} + v'_n \sin\alpha_{n+1}, \quad v'_{t,n} = u'_n \sin\alpha_{n+1} - v'_n \cos\alpha_{n+1}.$$

Next we apply the collision rule (provided $v'_{r,n}$ is inward, i.e., $v'_{r,n} < 0$)

$$v_{r,n+1} = -kv'_{r,n} = -k(u'_n \cos\alpha_{n+1} + v'_n \sin\alpha_{n+1}),$$

$$v_{t,n+1} = jv'_{t,n} = j(u'_n \sin\alpha_{n+1} - v'_n \cos\alpha_{n+1}),$$

to obtain the rebound velocities in polar components. The transformation leading back to perpendicular components is:

$$u_{n+1} = v_{r,n+1}\cos\alpha_{n+1} + v_{t,n+1}\sin\alpha_{n+1}, \quad v_{n+1} = v_{r,n+1}\sin\alpha_{n+1} - v_{t,n+1}\cos\alpha_{n+1}.$$

After substituting here $v_{r,n+1}$, $v_{t,n+1}$ we recover (14), (15).

Problem 6. The smallest permitted angle α_c is the one that belongs to a *tangent* impact velocity. The slope of the tangent in a point along the circular arc seen under angle α_{n+1} is $-\cot(\alpha_{n+1})$. In the case of a tangential impact this equals v'_n/u'_n, thus $\alpha_c = arccot(\mid v'_n \mid /u'_n)$. Note that this condition also follows from the vanishing of the radial impact velocity, $v'_{r,n}$ (see solution to problem 5.). Hence, angle α_{n+1} falls in the interval $[\alpha_c, \frac{\pi}{2}]$.

Problem 7. For impacts with intermediate α_{n+1} values the vertical rebound velocity is typically smaller than on a horizontal surface.

Problem 8. Dividing (14) by u'_n and taking $j = 1$ we have

$$\frac{u_{n+1}}{u'_n} = \sin^2\alpha_{n+1} - k\cos^2\alpha_{n+1} + \frac{\mid v'_n \mid}{u'_n}(k+1)\sin\alpha_{n+1}\cos\alpha_{n+1} = 1-(1+k)\cos^2\alpha_{n+1} + (1+k)\frac{\mid v'_n \mid}{u'_n}\sin\alpha_{n+1}\cos\alpha_{n+1}.$$

This expression is larger equal unity if

$$-\cos^2\alpha_{n+1} + \frac{\mid v'_n \mid}{u'_n}\sin\alpha_{n+1}\cos\alpha_{n+1} \geq 0,$$

i.e.,

$$\frac{\mid v'_n \mid}{u'_n} \geq \cot\alpha_{n+1}.$$

In the notation of problem 6, the left hand side is just $\cot\alpha_c$, and we have

$$\cot\alpha_c \geq \cot\alpha_{n+1},$$

i.e. $\alpha_{n+1} \geq \alpha_c$. This is, however, the condition of validity for (04), and is thus always fulfilled. As a consequence, $u_{n+1} \geq u'_n = u_n$ is also always valid. Equality is found only for a tangent bounce occuring with $\alpha_{n+1} = \alpha_c$ (and, of course, for $\pi/2$).

Problem 9. Rewrite (14) as

$$\frac{u_{n+1}}{u'_n} = j - (k+j)\cos^2\alpha_{n+1} + (k+j)\frac{\mid v'_n \mid}{u'_n}\frac{1}{2}\sin 2\alpha_{n+1}.$$

Its derivative with respect to α_{n+1} is

$$(k+j)\left(\sin 2\alpha_{n+1} + \frac{\mid v'_n \mid}{u'_n}\cos 2\alpha_{n+1}\right).$$

This vanishes (with negative second derivative) at an α^* for which

$$\tan 2\alpha^* = -\frac{\mid v'_n \mid}{u'_n}.$$

Accepting $\alpha^* = \pi/4 + \alpha_c/2$ implies $2\alpha^* = \pi/2 + \alpha_c$, but since $\tan(\pi/2 + \alpha_c) = -\cot\alpha_c$ and $\cot\alpha_c = \mid v_n' \mid / u_n'$, this is an identity (see solution to problem 6.).

The value of u_{n+1}/u_n' at $\alpha_{n+1} = \alpha_c$ is

$$\frac{u_{n+1}}{u_n'} = j - (k+j)\left(\cos^2\alpha_c - \frac{\mid v_n' \mid}{u_n'}\sin\alpha_c\cos\alpha_c\right) = j - (k+j)\cos\alpha_c\sin\alpha_c\left(\cot\alpha_c - \frac{\mid v_n' \mid}{u_n'}\right).$$

The expression in the parantheses is, however, zero, as follows from the definition of α_c. Thus $u_{n+1} = ju_n'$. For $\alpha_{n+1} = \pi/2$ this can be seen by direct substitution.

Problem 10. Using (16) we find for α_c

$$\cot\alpha_c = \frac{2m}{1-k}.$$

With $k = 0.75$ $\cot\alpha_c = 4$ follows, i.e., $\alpha_c = 0.245$, about $14°$. This critical angle is the starting point of the graphs in Fig. 5 of the paper.

Problem 11. Relation (5) can be used as an estimate since long stretches of bounces occur as if the steps were rectangular. Remember that after the nth bounce on the curved surface, parameter H in the expression of $\bar{N}$ should be taken with the last value of horizontal velocity, u_n. Since H is inversely proportional to u^2 (see (6)), after the nth bounce

$$\bar{N} = \frac{2mu_n^2}{H}\frac{1+k}{1-k}$$

holds, where H is the dynamical parameter evaluated with the initial u_0. Applying this with N_{th} and u_{th} instead of $\bar{N}$ and u_n, we have

$$u_{th}^2 = N_{th}\frac{H}{2m}\frac{1-k}{1+k}.$$

With our baseline parameters $m = 1/2$, $H = 4$ with $k = 0.75$ and $N_{th} = 100$ we find $u_{th}^2 = 400/7$ from which $u_{th} = 7.6$.

Problem 12. Although flying away could be avoided this way, the dissipation is so strong that sticking down would dominate. We have to fight against both types of escape, hence the choice for a "balanced" pair of COR values producing about the same amount of cases with flying away and with sticking down.

Problem 13. The problem is similar to that in geometrical optics, but we have to take into account the effect of velocity change due to dissipation, expressed by the presence of both types of CORs. In contrast to optics, the angles of incidence and of reflection will not be the same (with the exception of the rather special case of $k = j$). Let γ and β denote the angles of incidence and reflection, respectively, belonging to the larger angle of impact. Those belonging to the smaller one are denoted by γ' and β' (see Fig. 2).

As seen in section 3.5, the impact velocity components in polar components are $v_r' < 0$ and v_t' (index n is suppressed here for clarity, see also Fig. 3 of the paper. For the angle of incidence then holds $\tan\gamma = v_t'/\mid v_r' \mid$. According to the collision rule (12), the rebound components are $v_r = -kv_r'$ and $v_t = jv_t'$, from which the tangent of the angle of deflection β is simply $\tan\beta = v_t/v_r = j/k \cdot v_t'/\mid v_r' \mid = j/k\tan\gamma$. In an explicit form

$$\beta(\gamma) = \tan^{-1}\left(\frac{j\tan\gamma}{k}\right).$$

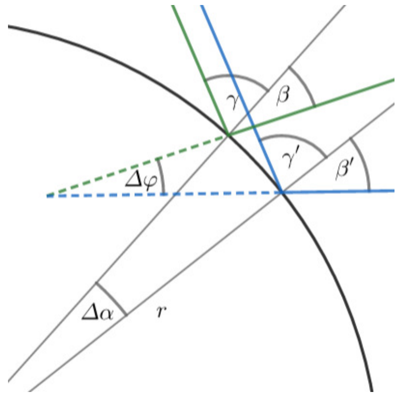

Figure 2: Schematic diagram illustrating the reflection of two parallel paths and their angles of incidence and deflection.

The angle of incidence belonging to the smaller impact angle is $\gamma' = \gamma + \Delta\alpha$ as can be read off from Fig. 2. The angle difference between the two rebound paths is $\Delta\varphi = \gamma' + \beta' - (\gamma + \beta) = \Delta\alpha + \beta' - \beta$. Here β' is of course the same function of γ' as β is of γ. The difference $\Delta\beta = \beta' - \beta$ is taken with an angle difference $\Delta\alpha$. Pulling out $\Delta\alpha$, we can write $\Delta\varphi = \Delta\alpha(1 + \Delta\beta/\Delta\alpha)$. For small differences, the second term is the derivative of β with respect to α: $d\beta/d\alpha$. We thus obtain

$$\Delta\varphi = \Delta\alpha \left(1 + \frac{d}{d\alpha} \tan^{-1}\left(\frac{j \tan\gamma}{k}\right)\right).$$

Evaluating the derivative ($d/dx \tan^{-1}(x) = 1/(1 + x^2)$) we obtain

$$a = 1 + \frac{j/k}{(j/k)^2 \sin^2\gamma + \cos^2\gamma}.$$

With the exception of $j = k$ this quantity depends on the angle of incidence. In our estimation we shall consider it to be a typical value γ^* belonging to the mean impact angle α^*. We can intuitively say that typical particles arrive at the curved surface under an angle α_c relative to the vertical (as they are perpendicular to the radius at α_c), their angle of incidence is $\gamma^* = \pi/2 + \alpha_c - \alpha^*$. Upon substituting $\alpha^* = \pi/4 + \alpha_c/2$ we find that γ^* coincides with α^*. With our base values, $\alpha_c = 0.245$ (see solution to Problem 10) giving $\gamma^* = \alpha^* = \pi/4 + \alpha_c/2 = 0.91$, i.e. $\gamma^* = 52°$. Substituting this, as well as $k = 0.75$ and $j = 0.2$ to the expression above, we can obtain a typical value for the constant: $a = 1.63$.

Problem 14. Our model assumes that the tangential COR j is velocity-dependent according to the same law for any $x \geq 1 - r$. However, close to the start of the curvature the surface is still nearly horizontal, but the tangential breaking is as intensive here as anywhere on the curvature. In a more refined model, one could apply a transitional region around $x = 1 - r$ with a gradually increasing tangential dissipation, i.e. an α-dependent exponent δ in (21).