an incident high energy nucleon and a nucleon in the nucleus is short compared to the time between collisions of the nucleons in the nucleus, he inferred that the high energy reaction could be modelled as a cascade process. Collisions occur between the incident particle and those particles which are directly struck in the nucleus. This model was first investigated in two dimensions in 1948 by Goldberger [Gol48] who performed his calculations by hand for the case of high energy neutrons interacting with heavy nuclei. The first fully three dimensional calculations were done by Metropolis et al. in 1958 [Met58] for incident protons and neutrons using the MANIAC computer; they also added a second stage to the cascade calculation during which the excited residual nucleus evaporates particles, as had also been suggested by Serber.

Many others have contributed to the development of the intranuclear cascade model. The two most commonly used versions of the cascade code in the theory of high energy heavy ion reactions are due to Yariv and Fraenkel [Yar79, 81] and Cugnon et al. [Cug80–84]. What is the intranuclear cascade model as it is used in these codes? It is a microscopic simulation of a nuclear reaction at high bombarding energies. Nuclear collisions are treated as a superposition of independent two body nucleon–nucleon collisions. Nucleons move on straight line trajectories (since there is no field) until they collide with a probability given by the free nucleon–nucleon scattering cross section. The creation of deltas, pions, and other particles and the interaction of all these particles occurs according to experimental cross sections. The intranuclear cascade models incorporate relativistic kinematics. Target and projectile nucleons are initialized in configuration and momentum space with random Fermi momenta and then Lorentz boosted to an appropriate frame, where the collision simulation proceeds. Momentum and energy are conserved in the particle–particle interactions and the evolution of the system is followed up to a time where the majority of interactions have ceased. Collisions are only Pauli blocked according to a simple criterion, say if the total center of mass energy is less than the Fermi energy in ground state nuclear matter or if the outgoing particle would scatter into momentum space regions originally occupied by projectile or target.

Both the Yariv–Fraenkel and the Cugnon cascade satisfy the above criteria. They differ in two respects: (1) the particles in the Yariv–Fraenkel simulation sit in a potential well of constant depth V_0; (2) in the original Yariv–Fraenkel approach the incoming particles (projectile nucleons) are cascading independently through a medium (the target). In the updated version, this scheme has been improved by including the so-called cascade–cascade interactions – for a given cascade particle (a particle which has undergone at least one collision) the other cascade particles are acting as a medium superimposed on the original target medium.

In the Cugnon cascade, one has the problem that the nucleons are not bound; hence one may get spurious effects due to nuclear instability, see fig. II.4 [Mol86]. It is possible to bind the nucleons by letting each nucleon move only with the beam velocity until it interacts with another nucleon, at which point it 'remembers' it's Fermi momentum. Note however that the bound Cugnon cascade is theoretically not very satisfying either, since in real nuclei, nucleons can travel in all directions.

II.2. Nuclear fluid dynamics

II.2.1. Conservation laws and the transition to local equilibrium

Just as in the classical case, one can derive from the VUU equation a general transport equation for a function f [Mar85, Mol85]. Assume for simplicity that the external force $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ is momentum independent. Let us also re-write the VUU equation in the simple form:

$$(\mathrm{d}f/\mathrm{d}t)_{\mathrm{coll}} = \partial f/\partial t + \boldsymbol{v}\cdot\partial f/\partial \boldsymbol{r} + \boldsymbol{F}\cdot\partial f/\partial \boldsymbol{p}\,. \tag{II.46}$$

Integration of this equation over p produces:

$$\partial f/\partial t + \nabla\cdot(\rho \boldsymbol{u}) = \int \mathrm{d}^3p(\mathrm{d}f/\mathrm{d}t)_{\mathrm{coll}}\,. \tag{II.47}$$

The term containing the external force vanishes by partial integration. Now the collision term describes the net rate of gain of particles at position $\boldsymbol{r}$ and with momentum $\boldsymbol{p}$. Since the collisions take place at a point and only redistribute particles in momentum space while conserving their number, the integral over all momenta must vanish. Thus we get the first equation of nuclear fluid dynamics, the continuity equation:

$$\partial f/\partial t + \nabla\cdot(\rho \boldsymbol{u}) = 0\,. \tag{II.48}$$

To obtain an equation for the momentum density, we integrate the VUU equation with a weight of $\boldsymbol{v}$:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla\cdot\int \mathrm{d}^3p\,\boldsymbol{v}\boldsymbol{v}f + \sum_j F_j\int \boldsymbol{v}\frac{\partial f}{\partial \rho_j}\mathrm{d}^3p = 0\,. \tag{II.49}$$

Again the collision term yields a vanishing contribution, because momentum is conserved in the collisions locally. In the second term on the left, one may separate an average and a fluctuating part:

$$\int \mathrm{d}^3p\,\boldsymbol{v}\boldsymbol{v}f = \int \mathrm{d}^3p\,\boldsymbol{u}\boldsymbol{u}f + \int \mathrm{d}^3p\,(\boldsymbol{v}-\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u})f + \int \mathrm{d}^3p\,\boldsymbol{u}(\boldsymbol{v}-\boldsymbol{u})f + \int \mathrm{d}^3p\,(\boldsymbol{v}-\boldsymbol{u})\boldsymbol{u}f\,. \tag{II.50}$$

The last two terms vanish because the average of $\boldsymbol{v}-\boldsymbol{u}$ is zero. The fluctuating part defines the kinetic stress tensor:

$$\mathbf{P}(\boldsymbol{r}, t) = \int \mathrm{d}^3p\,(\boldsymbol{v}-\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u})f(\boldsymbol{r}, \boldsymbol{p}, t)\,. \tag{II.51}$$

Note that the pressure is identically zero only if all the particles have exactly the mean velocity $\boldsymbol{u}$. The external force contribution can be rewritten using

$$p_i\frac{\partial f}{\partial p_j} = \frac{\partial}{\partial p_j}(p_i f) - \delta_{ij}f \tag{II.52}$$

and then the first term on the right hand side vanishes in partial integration. The momentum conservation equation is then:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla\cdot(\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla\cdot\mathbf{P} + \frac{\rho}{m}\boldsymbol{F}. \tag{II.53}$$

Define the energy density as:

$$\rho E = \frac{m}{2} \int \mathrm{d}^3p\, v^2 f(\boldsymbol{r}, \boldsymbol{p}, t)\,. \tag{II.54}$$

Then through the introduction of the average velocity ρE may be split up into a flow kinetic energy and a thermal contribution:

$$\rho E = \frac{m}{2} \rho u^2 + \frac{m}{2} \int (\boldsymbol{v} - \boldsymbol{u})^2 f(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}^3p\,. \tag{II.55}$$

Integrating the VUU equation with a weight of $mv^2/2$ gives

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot \int \mathrm{d}^3p \frac{m}{2} v^2 \boldsymbol{v} f + \boldsymbol{F} \cdot \int \mathrm{d}^3p \frac{m}{2} v^2 \frac{\partial f}{\partial \boldsymbol{p}} = 0 \tag{II.56}$$

where again the collision term gives no contribution because it conserves kinetic energy of the particles. The second term is split up into a number of parts:

$$\int \mathrm{d}^3p \frac{m}{2} \boldsymbol{v} v^2 f = \int \mathrm{d}^3p \frac{m}{2} (\boldsymbol{v} - \boldsymbol{u})^2 (\boldsymbol{v} - \boldsymbol{u}) f + \int \mathrm{d}^3p f \frac{m}{2} u^2 \boldsymbol{u} + m \int \mathrm{d}^3p\, \boldsymbol{u} \cdot (\boldsymbol{v} - \boldsymbol{u})(\boldsymbol{v} - \boldsymbol{u}) f$$
$$+ \int \mathrm{d}^3p \frac{m}{2} (\boldsymbol{v} - \boldsymbol{u})^2 \boldsymbol{u} f = \boldsymbol{q} \cdot \rho E \boldsymbol{u} + \boldsymbol{u} \cdot \mathbf{P} \tag{II.57}$$

wherein the first term $\boldsymbol{q}$ describes the transport of thermal energy by thermal motion (thermoconductivity). The second and third term are combined to give the transport of total energy by collective flow and the last term describes the work done against the stresses. Finally the external force contribution can be re-written by partial integration [Mol85]:

$$\boldsymbol{F} \cdot \int \mathrm{d}^3p\, \tfrac{1}{2} v^2 \frac{\partial f}{\partial \boldsymbol{p}} = -\boldsymbol{F} \cdot \int \mathrm{d}^3p\, \boldsymbol{v} f = -\rho \boldsymbol{u} \cdot \boldsymbol{F} \tag{II.58}$$

and the conservation of energy equation reads:

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot (\rho E \boldsymbol{u}) = -\nabla \cdot (\boldsymbol{u} \cdot \mathbf{P}) + \rho \boldsymbol{u} \cdot \boldsymbol{F}\,. \tag{II.59}$$

These then are the equations of nuclear fluid dynamics, which assume a local equilibrium, in contrast to the more general VUU theory which includes non-equilibrium phenomena. These are macroscopic equations which can be solved in three dimensions for two interacting nuclei.

II.2.2. Fluid dynamics and the nucleon mean free path

For the applicability of the fluid dynamical concepts it has to be ensured that fast equilibration and thermalization of the incident momentum and energy occurs in high energy heavy ion collisions. This is the case if the mean free path λ is small compared to the typical dimension, L, of the system

$$\lambda/L \ll 1 . \tag{II.60}$$

The mean free path λ is given by

$$\lambda = 1/(\sigma \cdot \rho)$$

where σ is the total nucleon–nucleon scattering cross section and ρ is the actual nuclear density. For normal nuclear density ρ_0 and a free n–n scattering cross section $\sigma_{NN} \sim 30$ mb at high energies ($E_{Lab} > 200$ MeV/n), the mean free path is $\lambda \sim 2$ fm, which is not too small against the nuclear dimensions $L \sim 10$ fm [Sch68, Sch74a, Sch74b].

High relative momenta between nuclei, signifying no overlap in phase space, as well as the large longitudinal momentum decay length calculated from the free n–n scattering cross section were interpreted as a complete transparency for the two nuclei at high energies and as the death for compression (shock) waves at energies above 1 GeV/n [Sob75]. However, in the "formation flight" of ensembles of nucleons, collective scattering phenomena, namely the enlargement of the cross section due to precritical scattering close to phase transitions [Gyu77, Ruc76] and compression effects cannot be neglected, so that the scattering cross section and the density can be modified drastically leading to a decrease of the mean free path

$$\lambda \approx 1.4 \frac{\sigma_{NN}}{\sigma_{coll}} \cdot \frac{\rho_0}{\rho} \text{fm} . \tag{II.61}$$

The precritical scattering of nucleons in the vicinity of a phase transition is in analogy to the critical opalescence, which is characterized by the great enhancement of the scattering cross section of light near a liquid–gas phase transition, or of the critical scattering of neutrons in ferromagnets near the Curie point [Sta71] or – as a last example – the critical scattering appearing in two colliding plasma beams: When the drift velocity of the two plasmas exceeds a critical value, unstable plasmon modes appear, resulting in a growth of strong electric fields, which greatly reduce the penetration depth of the two plasmon beams in comparison to values estimated from free two body collisions. Thus the vicinity of a phase transition – e.g. the onset of pion condensation or gluon condensation is expected to be marked by the occurrence of critical nucleon–nucleon scattering, i.e. a large enhancement (a factor of 2–4 for pion condensation) of the density dependent n–n cross section [Gyu77, Ruc76].

This would mean that even at bombarding energies above one GeV/n nuclei do not become transparent to each other: On the contrary, very violent collisions can be expected. One should keep in mind, however, that nucleus–nucleus collisions are a quantum mechanical process. Hence – in the sense of quantum mechanical fluctuations – under the same initial conditions processes with violent randomization (i.e. the occurrence of pronounced shock waves) may occur as well as processes with less pronounced interaction. It is a formidable experimental task to separate the former from the latter.

Indeed, recent experiments (which we shall discuss later) show that up to lab-energies of 4 GeV/n a considerable part (~30%) of the total cross section is due to violent events with very high multiplicities and large momentum transfer.

II.2.3. Nuclear fluid dynamics and quantum mechanics

Immediately after the discovery of quantum mechanics, the formal analogy of quantum mechanics to

fluid dynamics had been discovered [Mad26]. One uses the Schrödinger equation

$$-\frac{\hbar^2\Delta}{2m}\psi + V(r)\psi = \mathrm{i}\,\hbar\frac{\partial\psi}{\partial t} \tag{II.62}$$

and the separation of a phase S in the wave function

$$\psi(r,t) = \phi(r,t)\cdot\exp\{\mathrm{i}mS(r,t)/\hbar\} \tag{II.63}$$

one gets

$$\left(-\frac{h^2}{2m}\nabla^2\phi + \frac{m}{2}\phi(\nabla S)^2 - \mathrm{i}\,h\nabla\phi\cdot\nabla S - \frac{\mathrm{i}\,\hbar}{2}\phi\nabla^2 S + \nabla\phi\right)\cdot\exp[\mathrm{i}mS/h]$$

$$= \left(\mathrm{i}\,h\frac{\partial\phi}{\partial t} - m\frac{\partial S}{\partial t}\phi\right)\cdot\exp[\mathrm{i}mS/h]\,. \tag{II.64}$$

Multiplication by $\psi^* = \phi\cdot\exp(-\mathrm{i}\,mS/\hbar)$ and separation of real and imaginary part results in

$$\frac{m}{2}\phi(\nabla S)^2 - \frac{h^2}{2m}\nabla^2\phi + V\phi = -m\phi\frac{\partial S}{\partial t} \tag{II.65}$$

and

$$\phi\partial\phi/\partial t + \phi\nabla\phi\cdot\nabla S + \tfrac{1}{2}\phi\nabla^2 S = 0 \tag{II.66}$$

respectively. The last equation may be transformed to

$$\frac{\partial}{\partial t}\phi^2 + \nabla\cdot(\phi^2\nabla S) = 0\,. \tag{II.67}$$

This is nothing else than the well known continuity equation describing the conservation of probability density in quantum mechanics. According to (II.63) the probability density is

$$\rho(r,t) = \phi^2(r,t) \tag{II.68}$$

and the probability current is

$$j(r,t) = \phi^2(r,t)\cdot\nabla S(r,t) \tag{II.69}$$

following from the definition

$$j(r,t) = \frac{\hbar}{2\mathrm{i}\,m}[\psi^*\nabla\psi - \psi\nabla\psi^*]\,. \tag{II.70}$$

From (II.68) and (II.69) a velocity field

$$v(r, t) = \nabla S(r, t)$$

can be deduced with $\nabla \times v = 0$. Equation (II.65) may be treated similarly. After some manipulations and using eq. (II.67) one finds

$$\frac{\partial}{\partial t}(m\phi^2 \nabla S) + \nabla \cdot (m\phi^2 \nabla S \nabla S) = -\phi^2 \nabla V + \phi^2 \nabla \left(\frac{h^2}{2m} \frac{\phi \nabla^2 \phi}{\phi^2} \right) \tag{II.71}$$

and with the definitions of ρ and v one finally obtains

$$\frac{\partial}{\partial t}(m\rho z) + \nabla \cdot (m\rho v v) = -\rho \nabla V - \rho \nabla \left(-\frac{h^2}{2m} \frac{\sqrt{\rho} \nabla^2 \sqrt{\rho}}{\rho} \right), \tag{II.72}$$

that is identical with the conservation equation of the momentum with an external force due to a potential V. Only the last term on the right-hand side is new. It depends solely on density and may be interpreted as an inner pressure caused by quantum mechanical effects. It disappears in the classical limit of $\hbar \to 0$.

Equations (II.67) and (II.72) have been obtained for the probability density of *one* single particle. Thus the conclusion of a quantum mechanical *many* body system, behaving like interpenetrating fluids with interaction, is obvious. It is important to note that (e.g. in contrast to kinetic theory) each single quantum mechanical particle is a continuum itself. Therefore the problem of granulation of the microscopical density does not occur. However, the main problem will be a reasonable definition for the many particle densities and velocities. An important question will be, whether all quantities entering the equations of motion may be described as functions of these macroscopic variables (and a temperature) or not.

II.2.4. Nuclear fluid dynamics and TDHF

For many-particle systems it is also possible to derive continuity equations. However, the procedure is very lengthy as has been seen above. For reasons of simplicity, we will therefore restrict ourselves to a deviation by analogy of the hydrodynamic equations from the TDHF equations. That is less general but the principal argument [Won77, Mar77] is the same as the one used above in chapter II.1. The TDHF equations are well known:

$$\begin{aligned} \mathrm{i}\, h \frac{\partial}{\partial t} \psi_\alpha = & -\frac{h^2}{2m} \nabla^2 \psi_\alpha(r, t) + \psi_\alpha(r, t) \int \mathrm{d}^3 r'\, V(r - r') \sum_\beta \psi_\beta^*(r', t)\, \psi_\beta(r', t) \\ & - \sum_\beta \psi_\beta(r, t) \int \mathrm{d}^3 r'\, V(r - r')\, \psi_\beta^*(r', t)\, \psi_\alpha(r', t) \end{aligned} \tag{II.73}$$

where ψ_α is the single particle wave function and $V(r - r')$ is the two particle interaction. In analogy to the results of the previous section we define a many-particle density and a many-particle current density

$$\rho(r,t)=\sum_{\alpha}\psi_{\alpha}^{*}(r,t)\psi_{\alpha}(r,t)=\sum_{\alpha}\phi_{\alpha}^{2}(r,t) \tag{II.74}$$

$$j(r,t)=\sum_{\alpha}j_{\alpha}(r,t)=\sum_{\alpha}\phi_{\alpha}^{2}(r,t)\,\nabla S_{\alpha}(r,t)\,, \tag{II.75}$$

with a separation of the one particle wave function as above,

$$\psi_{\alpha}(r,t)=\phi_{\alpha}(r,t)\cdot\exp[\mathrm{i}\,mS_{\alpha}(r,t)/h]\,. \tag{II.76}$$

In complete analogy to the previous section, we then obtain again a continuity equation with the mean collective velocity being

$$v(r,t)=j(r,t)/\rho(r,t)=\sum_{\alpha}\phi_{\alpha}^{2}(r,t)\,\nabla S_{\alpha}(r,t)\Big/\sum_{\alpha}\phi_{\alpha}^{2}(r,t)=\Big\{\sum_{\alpha}\rho_{\alpha}v_{\alpha}\Big\}\Big/\Big\{\sum_{\alpha}\rho_{\alpha}\Big\}\,. \tag{II.77}$$

ρ_{α} and v_{α} again are the single particle quantities. The collective velocity therefore is the weighted average of the single particle velocities. Due to the average the velocity field v is no longer curl-free.

The derivation of the momentum equation again is completely analogous and can be written

$$\frac{\partial}{\partial t}j(r,t)+\nabla\cdot\sum_{\alpha}(\phi_{\alpha}^{2}\nabla S_{\alpha}\nabla S_{\alpha})=-\frac{1}{m}\sum_{\alpha}\phi_{\alpha}^{2}\nabla\Big\{\frac{\phi_{\alpha}(-(h^{2}/2m)\nabla^{2})\phi_{\alpha}}{\phi_{\alpha}^{2}}\Big\}-\frac{\rho}{m}\nabla\int \mathrm{d}^{3}r'\;V(r-r')\,\rho(r')$$

$$+\frac{1}{m}\int \mathrm{d}^{3}r'\,\nabla V(r-r')\,\Big|\sum_{\alpha}\psi_{\alpha}^{*}(r)\psi_{\alpha}(r')\Big|^{2}\,. \tag{II.78}$$

This form is not yet usable for a fluid dynamical description. For the transition to hydrodynamics all one particle quantities have to be expressed by ρ, j and by a temperature field T.

We will start with the second term of eq. (II.78). Introducing the collective velocity v (II.77) we get

$$\nabla\cdot\sum_{\alpha}(\phi_{\alpha}^{2}\nabla S_{\alpha}\nabla S_{\alpha})=\nabla\cdot\Big(\rho\boldsymbol{vv}+\sum_{\alpha}\phi_{\alpha}^{2}(\nabla S_{\alpha}-\boldsymbol{v})(\nabla S_{\alpha}-\boldsymbol{v})\Big)$$

$$=\nabla\cdot\Big(\rho\boldsymbol{vv}+\frac{1}{m}\,\mathbf{P}\Big)\,, \tag{II.79}$$

with $\mathbf{P}$ being – in analogy to the kinetic theory, cf. II.2.1 – the stress tensor

$$\mathbf{P}=m\sum_{\alpha}\phi_{\alpha}^{2}(\nabla S_{\alpha}-v)(\nabla S_{\alpha}-v)\,, \tag{II.80}$$

which results from the statistical deviations of the single particle velocities from the mean value v.

In order to reduce the potential terms in eq. (II.78) we will use a separation of the n–n force into a short range, density dependent part V_{s} and into a long range part V_{l} [Bet68].

$$V(r-r') = V_s(\rho(r))\delta(r,r') + V_l(r-r')\,. \tag{II.81}$$

The first (direct) potential term in (II.78) then becomes

$$\int \mathrm{d}^3r'\; V(r-r')\,\rho(r') = V_s[\rho(r)]\,\rho(r) + U_l(r)\,. \tag{II.82}$$

U_l is the mean potential, stemming from the long range interaction

$$U_l(r) = \int \mathrm{d}^3r'\; V_l(r-r')\,\rho(r')\,. \tag{II.83}$$

For the exchange part, the Slater approach is used, yielding [Mar77]

$$\frac{1}{m}\int \mathrm{d}^3r'\,\nabla V(r-r')\left|\sum_\alpha \psi_\alpha^*(r)\,\psi_\alpha(r')\right|^2 \simeq -\frac{\rho}{m}\nabla\frac{\partial}{\partial\rho}W_x(\rho) + F_{Cx}\,. \tag{II.84}$$

$W_x(\rho)$ stands for a contribution to the exchange force stemming from the binding energy per nucleon at a given density ρ, whereas F_{Cx} reflects the Coulomb part of the exchange force. F_{Cx} cannot be further simplified. Fortunately, it is small in most cases and can be neglected. Particularly in high energy collisions the Coulomb interaction does not play an important role.

The direct short range term as well as the exchange term are not explicitly calculated because they are contained in the effective part of the density dependent binding energy resulting from nuclear matter calculations [Sto79]. That is one way quantum effects can be included effectively in the equation of state.

The Navier–Stokes equations of motion of a viscous fluid, i.e. a fluid with some non-equilibrium properties (thermal conductivity and viscosity, which can be physically associated with finite mean free path effects) can be written as continuity equations for the baryon number ρ, momentum density $\rho\boldsymbol{v}$, and energy density ρE with the gradients of the pressure P, the potentials V, and the viscous stress tensor, $\mathbf{P}$, as source terms,

$$\begin{aligned}
&\partial_t\rho + \partial_i(\rho v_i) = 0\,,\\
&\partial_t(\rho v_i) + \partial_j(\rho v_j v_i) = -\frac{1}{m}\,\partial_i P + \frac{1}{m}\,\partial_j[\eta(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\partial_k v_k) + \zeta\delta_{ij}\partial_k v_k] - \frac{\rho}{m}\,\partial_i V\,,\\
&\partial_t(\rho E_T) + \partial_j(\rho E_T v_j) = \kappa\partial_j^2 T + \partial_i v_j[-P_T\delta_{ij} + \eta(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\partial_k v_k) + \zeta\delta_{ij}\partial_k v_k]\,.
\end{aligned} \tag{II.85}$$

The indices i, j and k are running over the space coordinates and there is a summation for indices occurring twice in one term. The coupled non-linear equations for the density fields $\rho(r,t)$, momentum density fields $\rho(r,t)\boldsymbol{v}(r,t)$ and energy density fields $\rho(r,t)E(r,t)$ are solved simultaneously. The source terms p, V, $\kappa\,\partial_j^2 T$ and the viscosity part will be discussed below.

II.2.5. Relativistic fluid dynamics

We have only discussed the non-relativistic equations of motion up to now. However, as the

bombarding energy is increased, relativistic effects become increasingly important. The range of validity of the non-relativistic formalism is not sharply defined, but at bombarding energies $E_{\mathrm{Lab}} >$ 500 MeV/nucleon, the relative velocity of projectile and target exceeds the speed of light, c, and at best qualitative results may be obtained.

There are two ways in which a system may become relativistic. It may be

(a) macroscopically relativistic in that the flow velocity becomes large or

(b) microscopically relativistic in that the excitation energy is non-negligible in comparison to the rest energy.

Case (a) is reflected by the equations of motion and case (b) by the equation of state.

As in the non-relativistic case (eqs. II.85), the relativistic equations of motion reflect the conservation of baryon number, momentum and energy, and may be brought into the continuity equation form:

$$\begin{aligned} &\partial_t \rho_{\mathrm{L}} + \partial_k(\rho_{\mathrm{L}} v_k) = 0 \\ &\partial_t \boldsymbol{M}_i + \partial_k(\boldsymbol{M}_i v_k) = -\partial_i P \\ &\partial_t e_{\mathrm{L}} + \partial_k(e_{\mathrm{L}} v_k) = -\partial_k(P v_k)\,, \end{aligned} \tag{II.86}$$

where long range potentials and dissipative terms have been dropped.

The quantities ρ_{L}, $\boldsymbol{M}$ and e_{L} are the densities of baryon number, momentum and energy respectively as specified in a fixed ("lab") reference frame. These are related to the "proper" or "co-moving" densities in the local rest frame by

$$\begin{aligned} &\rho_{\mathrm{L}} = \gamma\rho \\ &M_i = \gamma^2(e + P)v_i \\ &e_{\mathrm{L}} = \gamma^2(e + P) - P \end{aligned} \tag{II.87}$$

where

$\gamma = (1 - v^2)^{-1/2}$ is the usual Lorentz factor,

ρ is the proper baryon number density,

$e = \rho(m_0c^2 + E_{\mathrm{c}}(\rho) + E_{\mathrm{th}}(\rho, T))$ is the proper internal energy density, including the rest energy and

P is the pressure.

The proper densities ρ, e and the velocity v, which are the physically interesting quantities, must be obtained from the lab densities ρ_{L}, M, e_{L} by inverting the non-linear equations (II.87) with $P = P(\rho, e)$ from the equation of state. This non-trivial technical problem is a complication over the non-relativistic case, where the velocity can be calculated directly from ρ and ρv.

We show in figs. II.13 and 14 the relativistic nuclear fluid dynamical model calculations [Gra84] for

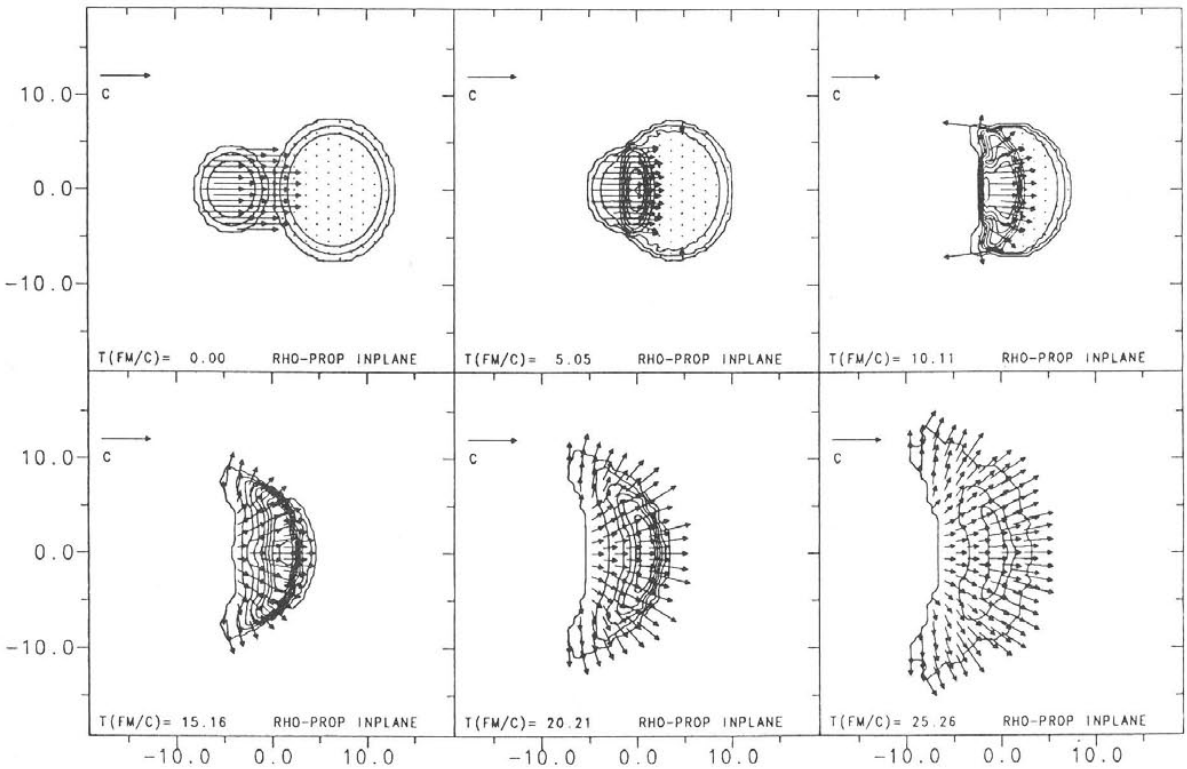

Fig. II.13. Collision of Ar (770 MeV/nucleon) + Pb at $b = 0$ fm in the Nuclear Fluid Dynamic model. Note the remarkable similarity to the VUU theory [Gra84].

the reaction Ar (770 MeV/nucleon) + Pb at $b = 0$ and 4 fm studied above in the VUU non-equilibrium theory. Note the surprising similarities of the fluid dynamical results with the VUU calculation. Similar results are, however, only obtained when heavy projectiles and targets are studied, because only in this case does the VUU approach predict the approach to thermal equilibrium. For light systems like C + C the fluid dynamical model does not apply – there is simply not enough target material in the way of the projectile to result in nuclear stopping.

II.2.6. One dimensional shocks

The above equations describe fluid dynamical processes completely. However, it is often advantageous to gain more insight into the physical processes by solving more simplified, schematic models, which can be solved (at least to some extent) analytically. In this case another set of equations is applied in the more schematic treatment of the fluid dynamical description of high energy heavy ion collisions, namely the shock equations: Shock waves have to be clearly distinguished from sound waves. In contrast to sound waves, shock waves are connected with a strong, density dependent mass flow velocity v_f. The shock front itself propagates with the shock velocity $v_s > v_f$ and also depends strongly on the compression amplitude [Sch74, Bau75]. Shock waves are non-linear phenomena – for large amplitudes $\rho \gg \rho_0$ both v_s and v_f tend to the velocity of light (see fig. II.15), while for small perturbations $\rho \sim \rho_0$ they approach the linear limit of sound waves. Shock waves imply a large entropy production: The matter flow through the shock front is highly irreversible, it is not only connected with strong compression, but also with large thermal excitation [Hof76, Stö77, Stö77a, Stö78].

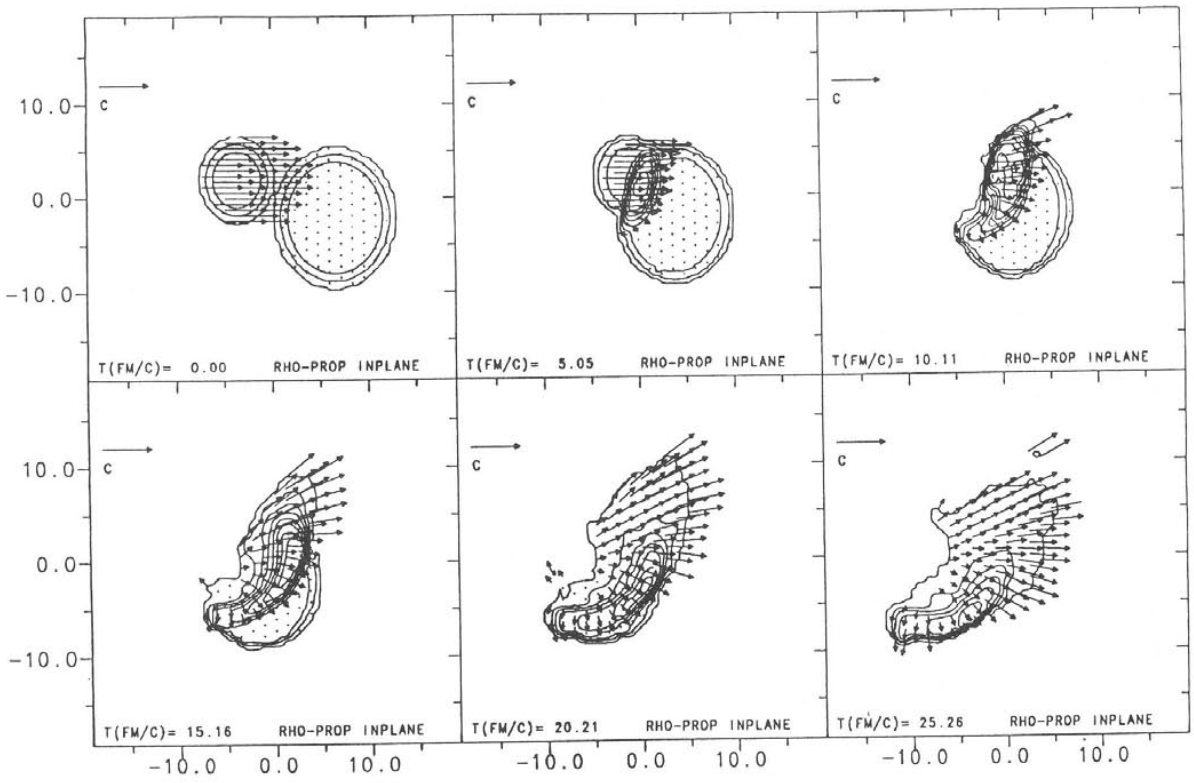

Fig. II.14. The same system at $b = 4$ fm in the Nuclear Fluid Dynamic model [Gra84].

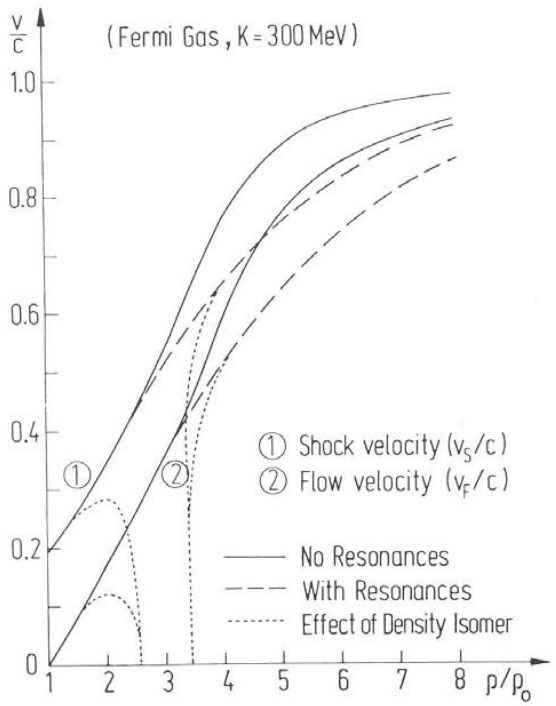

Fig. II.15. The dependence of the velocity of the shock front and the matter flow velocity on the density. Dashed lines: influence of hadronic resonances; short dashes: influence of a density isomer at $\rho = 3\rho_0$ [Hof76, Stö77].

The shock calculations have to be viewed as an idealization assuming a zero width of the shock front together with the discontinuous jump of the state variables (e.g. ρ, T, e, P). However, comparison of the nuclear shock wave calculations with the result of the full Navier Stokes calculations [Stö79b] shows that the resulting compression rates and temperatures are very similar, although in the Navier Stokes calculations the compression front is smeared out over 1–2 fm due to the viscosity. Such a width seems to be rather realistic, as the width of a shock front is approximately given by 2–3 mean free path, which can be less than a fermi in high energy nuclear collisions. For a large nuclear transparency, the shock front width may be of the order of the nuclear radius. However, no indication for transparency has been found in the high energy experiments up to now.

The relativistic Rankine–Hugoniot equations can be derived from the continuity of the

$$\begin{aligned} &\text{particle flux density} && [j^0] = [\rho u_x] = 0 \\ &\text{energy flux density} && [T_{0x}] = [\mathrm{i}\, u_0 u_x] = 0 \\ &\text{and momentum flux density} && [T_{xx}] = [\mathrm{i}\, u_x^2 + P] = 0 \end{aligned} \tag{II.88}$$

where [ ] denotes the jump of the respective variable across the shock front, and x gives the direction normal to the shock front as seen from the shock front's rest frame.

Eliminating the velocities u_x from the continuity equations yields the relativistic shock equation

$$i_0^2/\rho_0^2 - i^2/\rho^2 + (P - P_0)(i_0/\rho_0^2 - i/\rho^2) = 0 \tag{II.89}$$

which gives a unique connection between the free enthalpy i, pressure P, and density ρ within the respective rest frame of the matter (subscript $_0$ stands for the undisturbed matter in front of the shock wave, quantities without subscript refer to matter in the compressed state). When we insert $i = \rho W + P$ and $i_0 = \rho_0 W_0$ the equation

$$W^2 - W_0^2 + P(W/\rho - W_0/\rho_0) = 0 \tag{II.90}$$

is obtained. This is the Rankine–Hugoniot equation, which determines the temperature T as a function of ρ, i.e. $T(\rho)$. If W is equated with the center of mass kinetic energy one can solve for the density $\rho(E_{\mathrm{Lab}})$ and therefore for the bombarding energy dependence of any other thermodynamic quantity. Here $W(\rho, T)$ is the energy per baryon, which characterizes the nuclear equation of state. Neglecting pions and resonances and regarding the pure nucleon fluid only, the relation $P_{\mathrm{T}} = \alpha\rho E_{\mathrm{T}}$ obtained above can be used to obtain a quadratic equation in E_{T}, which can be solved in terms of the nucleon density ρ analytically. E_{T} is the temperature-dependent part of $W(\rho, T)$.

The shock velocities v_{s} and v_{f} can be determined by the continuity of the energy and momentum flux density. From the relative velocities of the matter with respect to the shock front, the relative matter flow velocity v_{f} is obtained by covariant summation (see fig. II.12),

$$\frac{v_{\mathrm{s}}}{c} = \left\{\frac{PW\rho}{(W\rho - W_0\rho_0)(W_0\rho_0 + P)}\right\}^{1/2}, \qquad \frac{v_{\mathrm{f}}}{c} = \left\{\frac{P(\rho W - \rho_0 W_0)}{\rho W(P + \rho_0 W_0)}\right\}^{1/2}. \tag{II.91}$$

A simple illustrative model can be constructed to calculate the shock compression and temperature

in the central collision of two heavy nuclei as a function of the bombarding energy [Bau75, Stö78]. This model assumes the compressed fluid to be at rest in the center of momentum system (equal velocity frame). Three dimensional fluid dynamical calculations show that this requirement is fulfilled fairly well for non-peripheral collisions of heavy nuclei near the collision axis: A quasi-stationary compression stage develops. That means, that practically all of the incident kinetic energy is transformed into internal energy (compression and excitation). As v_f denotes the flow velocity of the shocked matter, which is at rest in the c.m. frame, relative to the target matter, we can calculate the laboratory bombarding energy via

$$E_{\text{Lab}} = [(1-(v_p/c)^2)^{-1/2} - 1]W_0 \tag{II.92}$$

where $v_p = 2v_f/(1+(v_f/c)^2)$ is the projectile velocity.

Though this model will, due to the lack of kinetic energy of the compressed matter and due to the outflow of matter perpendicular to the collision axis, give too large values for compression and temperatures as a function of the bombarding energy (as compared to three dimensional calculations), it is sufficient to give an overview about the expected compression and thermal excitation. The influence of the beam energy and the nuclear equation of state (e.g. different compressibility constants) and the importance of resonance and pion production in the collision dynamics can thus be studied nearly analytically.

III. Confrontation of the Theory with Experimental Data – Extraction of the Nuclear Equation of State

III.1. Expansion, fragment formation and the entropy puzzle

III.1.1. Compression and expansion

In the preceding chapter the fluid dynamical model has been introduced. Historically fluid dynamics has been the first approach to be applied to high energy nuclear collisions. This model is most intuitive in that it refers directly to the thermodynamical concepts developed in chapter I. Therefore the underlying physics of high compression and excitation can be discussed in a macroscopic language. The spatial and temporal distributions of the density, velocity, temperature, etc. as obtained from this approach are analyzed in the following. The dependence of the thermodynamic variables on the impact parameter, bombarding energy, and projectile–target combination is studied. The fluid dynamical model is the only one of the above discussed approaches for which the nuclear equation of state serves as immediate input – the VUU approach also includes the compression energy, but since it is a non-equilibrium theory the statistical variables temperature, thermal pressure, etc. are not easily accessible. We use the fluid dynamical model in the following as a reference case – not only to compare to the data, but also to the microscopic theories.

The fluid dynamical development of a heavy ion collision typically proceeds in the following way (fig. III.1): When two nuclei collide at high energy the overlap zones are stopped and a strong non-linear shock wave is formed. High density, pressure and temperature are created in this region. This stage is described by the "compression" line in fig. III.1. The numbers on that line indicate the energies in GeV/n at which the corresponding maximum values for ρ/ρ_0 and T are reached. At $E_{\text{Lab}} = 200$ MeV/n, for example, a maximum compression of three times equilibrium density and a temperature of about 40 MeV can be obtained. During that stage the entropy of the system rises up to a certain saturation