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PHENOMENOLOGICAL QUANTUM ELECTRODYNAMICS OF ANISOTROPIC MEDIA*

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The phenomenological quantum theory of the electromagnetic field in anisotropic dielectrica is presented. The Hamiltonian of the field is diagonalized. and its quanta are interpreted as photons polarized in directions corresponding to the classical directions of polarization. Using Abraham's expression we find that the momentum operator cannot be diagonalized simultaneously, thus photons are considered as "quasi-particles" the momenta of which are given as mean values. Nevertheless, these photons are shown to exhibit particle properties in contrast to Minkowski's description, in which case, however, the energy and the momentum can be diagonalized simultaneously.

Introduction

According to quantum electrodynamics the energy and the momentum of the electromagnetic field in vacuum are quantized. The energy of the field is given by

$$U = \sum_i \sum_{r=1}^{2} h\nu_i (n_i^{(r)} + 1/2)\,, \tag{1}$$

while its momentum can be expressed as

$$G = \sum_i \sum_{r=1}^{2} \hbar \mathbf{k}_i n_i^{(r)}\,, \tag{2}$$

where $n_i = 0, 1, 2, \ldots$, $\mathbf{k}_i$ and ν_i stand for the wave-number vector and the frequency of the i-th normal mode (photon), respectively, h denotes Planck's constant and $\hbar = h/(2\pi)$. The superscript r refers to the polarization of the normal modes. Photons specified by energy $h\nu$ and momentum $\hbar\mathbf{k}$ exhibit properties similar to those of particles. Their momentum and energy transform as four-vectors when changing from one inertial system to another one. The rest mass of a photon is zero but it has an inertial mass equal to $h\nu/c^2$, where c denotes the velocity of light in vacuum. The momentum of a photon is related to its mass in the same fashion as in case of particles, i.e. momentum = inertial mass $\times$ velocity. On the other hand the concept of particles arising from the

* Dedicated to Prof. R. GÁSPÁR on his 60th birthday

classical mechanics cannot be applied without some caution since it is meaningless to speak about the position or the path of a photon.

It is an interesting question whether the particle properties discussed above remain valid also for the electromagnetic field in a transparent medium where the field interacts with the charges in the atoms or molecules of the medium and brings them in motion by transferring energy and momentum. The oscillating charges then emit energy and momentum. One might ask then whether the energy and the momentum of the field remain quantized simultaneously under these physical circumstances. If so, do light quanta possess particle properties similar to those in the vacuum or is it possible that the concept of photon is restricted only to vacuum. The answers to these questions can be given by the quantum theoretical treatment of the electromagnetic field in dielectrica. The theoretical investigations of transparent isotropic media were carried out about a quarter of a century ago [1—3].

An important question arising already in the classical description of the electromagnetic field in dielectrica is the choice of the energy-momentum tensor. The two most important candidates, corresponding to different divisions of medium and field, are the expressions proposed by MINKOWSKI and ABRAHAM [4, 5]. Their validity has been discussed for a long time [6—12]. Recent experiments [13, 14] have confirmed the view that at low frequencies ABRAHAM's tensor yields the more plausible results. On the other hand, it is expected that at high frequencies it is a matter of taste which description is used [12]. The quantization procedure based on MINKOWSKI's tensor in isotropic dielectrica led to strange properties for photons [2]. Therefore, we prefer the use of ABRAHAM's expression but the alternative result will be given, too.

It was pointed out earlier that, due to the interaction between the field and the medium, a portion of the energy and the momentum of the radiation appears, in general, in the form of mechanical energy and momentum of molecules. Consequently, when describing the dynamical properties of the radiation field, one must use the so-called radiation tensor arising as a generalisation of ABRAHAM's tensor [7, 8, 15]. For a medium at rest both tensors give the same expressions for the field energy and for the momentum.

Quantum theoretical considerations have been limited only to isotropic dielectrica. Here, we shall extend the phenomenological quantum electrodynamics to anisotropic media. Our investigations show that in anisotropic dielectrica the photon picture holds only in a restricted sense since the momentum of a photon can be given only as a mean value. Therefore, it is perhaps appropriate to call the photons in anisotropic media "quasi-particles".

Before discussing the case of anisotropic dielectrica, in the next Section we summarize the most important results obtained for isotropic media. This will make the picture more complete and easily understandable.

Photons in transparent isotropic media

The electromagnetic wave passing through a transparent isotropic medium produces a varying electric and magnetic polarization in the dielectricum and the resulting radiation modifies the wave itself. Even if an entirely transparent medium is considered, the incident energy of radiation is present at a given moment only partly in the form of electromagnetic energy, since partly it appears as the kinetic and potential energies of the polarized molecules. In a periodical wave, e.g. the field transfers energy and momentum to the dielectricum and recovers them in the next half period. In general, one can picture the interaction as an exchange of energy and momentum. If the dielectricum is at rest, the electromagnetic force acting on the medium cannot cause macroscopic displacements, instead it produces stresses in the material which compensate the forces causing molecular deformations. Consequently, the energy and momentum of the radiation passing through the medium is partly of electrical and partly of mechanical origin. Accordingly, the energy-momentum tensor, $S_{\alpha,\beta}$, characterizing the radiation consists of two parts: ABRAHAM's tensor, $T_{\alpha,\beta}$, of the electromagnetic field and in addition the tensor $t_{\alpha,\beta}$ describing the mechanical energy and momentum as well as the stresses caused by the field, i.e.:

$$S_{\alpha,\beta} = T_{\alpha,\beta} + t_{\alpha,\beta}, \qquad \alpha, \beta = 0, 1, 2, 3 . \tag{3}$$

It was shown earlier that $S_{\alpha,\beta}$ is divergence free and symmetric. One can easily check also that $S_{\alpha,\beta}$ obeys the MØLLER criterion [16], i.e. when changing from one inertial system to another, the velocity of the propagation of the radiant energy transforms in the same way as that of a particle. Thus the energy and the momentum of the radiation in a medium are to be calculated by means of the radiation tensor $S_{\alpha,\beta}$. Since in a coordinate system fixed to the dielectricum, in the so-called rest frame, only the space-like components of $t_{\alpha,\beta}$ are non-vanishing, the expressions of the energy and the momentum of the radiation in this system coincide with those of ABRAHAM's tensor. In terms of the field vectors they are expressed as

$$U = \frac{1}{2}\int (\mathbf{ED} + \mathbf{HB})\, dV, \tag{4}$$

$$G = \frac{1}{c}\int (\mathbf{E}\times\mathbf{H})\, dV . \tag{5}$$

Following the procedure of quantum electrodynamics, one considers these quantities as operators and calculates their eigenvalues. The latter are given by [3]

$$U = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} (n_{\mathbf{k}}^{(a)} + n_{\mathbf{k}}^{(b)} + 1) , \tag{6}$$

$$G = \sum_{\mathbf{k}} \mathbf{g}_{\mathbf{k}} (n_{\mathbf{k}}^{(a)} + n_{\mathbf{k}}^{(b)}) , \tag{7}$$

with

$$\varepsilon_{\mathbf{k}} = \frac{\hbar k c}{\sqrt{\varepsilon\mu}} , \qquad \mathbf{g}_{\mathbf{k}} = \frac{\hbar \mathbf{k}}{\varepsilon\mu} , \tag{8}$$

where ε and μ denote the dielectric coefficient and the magnetic permeability of the medium, while $n_{\mathbf{k}}^{(a)}$ and $n_{\mathbf{k}}^{(b)}$ can be any non-negative integer number. The indices a and b refer to the two independent polarizations. Eqs. (6) and (7) show that the energy and the momentum of the radiation are quantized quantities in isotropic dielectrica as well, they are integer multiples of the quanta $\varepsilon_{\mathbf{k}}$ and $\mathbf{g}_{\mathbf{k}}$, respectively. Thus, the quantized structure of the radiation turns out to be valid not only in vacuum but also in transparent isotropic dielectrica.

Between the phase velocity

$$\mathbf{v} = \frac{c}{\sqrt{\varepsilon\mu}} \frac{\mathbf{k}}{k} \tag{9}$$

and the frequency ν the following relation holds

$$kv = 2\pi\nu . \tag{10}$$

The energy $\varepsilon_{\mathbf{k}}$ and the momentum $\mathbf{g}_{\mathbf{k}}$ of a photon can be expressed through $\mathbf{v}$ and ν as

$$\varepsilon_{\mathbf{k}} = h\nu , \tag{11}$$

$$\mathbf{g}_{\mathbf{k}} = \frac{h\nu}{c^2} \mathbf{v} . \tag{12}$$

For a dielectricum at rest, the phase velocity $\mathbf{v}$ coincides with the classical expression of the propagation velocity of the energy of a plane wave with wave-number $\mathbf{k}$:

$$\mathbf{v}^* = \mathbf{S}/u . \tag{13}$$

($\mathbf{S}$ denotes the Poynting vector, and u stands for the energy density.) The momentum of a photon can be expressed in isotropic dielectrica, too, as inertial mass $\times$ velocity. From (12) the inertial mass of a photon is found to be

$$m = h\nu/c^2 \tag{14}$$

as it is expected from the theory of relativity.

All these results are valid in a coordinate system fixed to the dielectricum. In a system moving as compared to the medium, the corresponding expressions can be obtained by means of the Lorentz transformation. For example, in a system moving along the x-axis with velocity V, one finds

$$\varepsilon'_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \frac{1 - \beta/n}{\sqrt{1 - \beta^2}} = h\nu \frac{1 - \beta/n}{\sqrt{1 - \beta^2}}, \tag{15}$$

where $\beta = V/c$, and $n = c/\sqrt{\varepsilon\mu}$ denotes the refractive index of the medium.

Using the transformation formula of the frequency

$$\nu' = \nu \frac{1 - \beta n}{\sqrt{1 - \beta^2}} \tag{16}$$

one can write

$$\varepsilon'_{\mathbf{k}} = h\nu' \frac{1 - \beta/n}{1 - \beta n}. \tag{17}$$

This shows that the energy of the photon in isotropic media cannot be expressed, in general, as $h\nu$, the formula $\varepsilon = h\nu$ is valid only in the rest frame.

Since the velocity v of a photon in dielectrica is smaller than the velocity of light c in vacuum the photon is characterized by a non-zero rest mass

$$m_0 = \frac{h\nu}{c^2 n} \sqrt{n^2 - 1}, \tag{18}$$

which is a positive real quantity. The rest energy ε_0 of a photon can be obtained from (15) for $\beta = 1/n$

$$\varepsilon_0 = \frac{h\nu}{n} \sqrt{n^2 - 1}. \tag{19}$$

Comparing ε_0 with (18) one finds $\varepsilon_0 = m_0 c^2$ in accordance with the theory of relativity. The inertial mass (14) and the rest mass (18) of a photon are connected by the equation

$$m = \frac{m_0}{\sqrt{1 - v^{*2}/c^2}}, \tag{20}$$

which is well-known for particles.

The results of the phenomenological quantum electrodynamics for isotropic media are thus entirely consistent with the corpuscular picture of the radiation. The concept of photon proves to be correct in this case, too.

We have summarized here the particle properties of the photons in dielectrica in detail since these quantum theoretical results played an important role in the debate about the energy-momentum tensor. As a matter of fact, a phenomenological quantum theory of the electromagnetic field in transparent isotropic media was worked out already in 1948 by JAUCH and WATSON [2] but they based their theory on the so-called canonical energy-momentum tensor. Since this differs from MINKOWSKI's one [4] only in spatial divergencies, their results coincide essentially with those of a theory based on MINKOWSKI's tensor. They obtained also quantized values for the field energy and momentum, but their photons exhibited properties which did not fit into a realistic physical picture. For example, the energy of a photon turned out to be negative in coordinate systems where the velocity of the dielectricum was larger than c/n. Furthermore, the rest mass of a photon was imaginary, and in the rest frame of a photon its rest energy was zero but its momentum was not. The reason behind these properties was the following. If one applies MINKOWSKI's tensor the momentum of the closed system formed by the dielectricum and the electromagnetic field is divided between the medium and the field in an unnatural way and therefore the momentum of the photon obtained in this way contains a contribution depending on the momentum of the medium, too. As we have seen, the description based on ABRAHAM's interpretation is free from such non-physical consequences.

Phenomenological quantum theory of the electromagnetic field in anisotropic media

It is assumed that the medium is electrically anisotropic but its magnetic permeability $\mu = 1$. The equation relating the electric field vectors is

$$\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E} \tag{21}$$

where $\boldsymbol{\epsilon}$ denotes the symmetric tensor of the dielectric coefficient. The coordinate system is chosen to be the principal axis system fixed to the medium. Then

$$D_i = \varepsilon_i E_i, \qquad i = 1, 2, 3, \tag{22}$$

where ε_i stand for the eigenvalues of $\boldsymbol{\epsilon}$.

The free radiation field is described by the Maxwell equations, the corresponding Lagrangian density of which is given as

$$\mathfrak{L} = \frac{1}{2}(\mathbf{ED} - \mathbf{H}^2), \tag{23a}$$

where the scalar and vector potentials, Φ and $\mathbf{A}$, defined by

$$\mathbf{E} = -\operatorname{grad}\Phi - \frac{1}{c}\dot{\mathbf{A}} \tag{23b}$$

and

$$\mathbf{H} = \text{rot}\, \mathbf{A}, \tag{23c}$$

are considered as the canonical variables. The conjugated momentum, $\mathbf{P}$, of $\mathbf{A}$ is defined by

$$P_i = \frac{\partial \mathfrak{L}}{\partial \dot{A}_i} = -\frac{1}{c} D_i, \qquad i = 1, 2, 3. \tag{24}$$

The energy and the momentum of the field according to ABRAHAM's tensor are

$$U = \frac{1}{2}\int (\mathbf{ED} + \mathbf{H}^2)\, dV \tag{25}$$

and

$$G = \frac{1}{c}\int (\mathbf{E} \times \mathbf{H})\, dV, \tag{26}$$

respectively.

In a quantum mechanical description the possible values which U and G might assume are specified by the eigenvalues of the corresponding operators. Therefore in the following we shall determine the operators* of the field energy and momentum and then calculate their eigenvalues.

The commutation relations of our basic quantities, i.e. the potentials and the conjugated momenta are as follows:

$$[P_\alpha(\mathbf{r}, t), A_\beta(\mathbf{r}', t)] = \frac{\hbar}{i}\, \delta_{\alpha,\beta}\, \delta(\mathbf{r} - \mathbf{r}'),$$

$$[P_\alpha(\mathbf{r}, t), P_\beta(\mathbf{r}', t)] = [A_\alpha(\mathbf{r}, t), A_\beta(\mathbf{r}', t)] = 0, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{27}$$

The operators A_α, P_α are considered here as four-vectors defined by $A_\alpha = (A_0, \mathbf{A})$, $P_\alpha = (P_0, \mathbf{P})$ where $A_0 \equiv \Phi$. The complication due to the fact that in a classical calculation the conjugated momentum associated to A_0 turns out to be zero can be avoided by using the quantization procedure described in [3]. The operators $A_\alpha(\mathbf{r}, t)$ and $P_\beta(\mathbf{r}, t)$ are functions of space and time. One might introduce space-independent operators by expanding them, as usual, in a series of orthogonal functions. In order to do it periodic boundary conditions will be assumed in a cube of length L. The size of the cube can be chosen arbitrarily since the final result will not depend on it. We expand the operators A_α and P_β in Fourier series as

$$A_\alpha(\mathbf{r}, t) = L^{-3/2} \sum_{\mathbf{k}} q_{\alpha,\mathbf{k}}\, e^{i\mathbf{kr}},$$

$$P_\beta(\mathbf{r}, t) = L^{-3/2} \sum_{\mathbf{k}} p_{\beta,\mathbf{k}}\, e^{-i\mathbf{kr}}, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{28}$$

* We shall use the same notation for operators as for the corresponding classical quantities.

Here the components of the wave-number vector $\mathbf{k}$ are multiples of $2\pi/L$. The operators $q_{\alpha\mathbf{k}} \equiv (q_{0,\mathbf{k}}, \mathbf{q}_\mathbf{k})$, $p_{\alpha\mathbf{k}} = (p_{0\mathbf{k}}, \mathbf{p}_\mathbf{k})$ obey the following commutation relations

$$[p_{\alpha\mathbf{k}}, q_{\beta\mathbf{k}'}] = \frac{\hbar}{i}\,\delta_{\alpha,\beta}\,\delta_{\mathbf{k},\mathbf{k}'},$$
$$[p_{\alpha\mathbf{k}}, p_{\beta\mathbf{k}'}] = [q_{\alpha\mathbf{k}}, q_{\beta\mathbf{k}'}] = 0\,. \tag{29}$$

The operator of the field energy can be expressed in terms of $\mathbf{q}_\mathbf{k}$ and $\mathbf{p}_\mathbf{k}$:

$$U = \frac{c^2}{2}\sum_\mathbf{k}\left[\sum_{j=1}^{3}\frac{p_{j\mathbf{k}}\,p_{j,-\mathbf{k}}}{\varepsilon_j} + \frac{1}{c^2}(\mathbf{k}\times\mathbf{q}_\mathbf{k}, \mathbf{k}\times\mathbf{q}_{-\mathbf{k}})\right].$$

It follows from the Maxwell equation div $\mathbf{D} = 0$ and from (27) that

$$(\mathbf{k}, \mathbf{p}_\mathbf{k}) = 0. \tag{30}$$

The vector operator $\mathbf{p}_\mathbf{k}$ is therefore orthogonal to the wave-number vector $\mathbf{k}$, and $\mathbf{p}_\mathbf{k}$ can be decomposed into two independent components. Let $\mathbf{e}_\mathbf{k}^{(a)}$ and $\mathbf{e}_\mathbf{k}^{(b)}$ denote two unit vectors orthogonal to $\mathbf{k}$ and to each other

$$(\mathbf{e}_\mathbf{k}^{(a)}, \mathbf{k}) = 0, \quad (\mathbf{e}_\mathbf{k}^{(b)}, \mathbf{k}) = 0, \quad (\mathbf{e}_\mathbf{k}^{(a)}, \mathbf{e}_\mathbf{k}^{(b)}) = 0\,,$$
$$|\mathbf{e}_\mathbf{k}^{(a)}| = 1, \qquad |\mathbf{e}_\mathbf{k}^{(b)}| = 1\,. \tag{31}$$

The vectors $\mathbf{e}_\mathbf{k}^{(a)}$, $\mathbf{e}_\mathbf{k}^{(b)}$ and $\mathbf{k}$ are chosen to form a right handed system. Consequently $\mathbf{e}_\mathbf{k}^{(a)}$ changes its sign when reflecting $\mathbf{k}$: $\mathbf{e}_\mathbf{k}^{(a)} = -\mathbf{e}_{-\mathbf{k}}^{(a)}$ but $\mathbf{e}_\mathbf{k}^{(b)}$ remains unchanged: $\mathbf{e}_\mathbf{k}^{(b)} = \mathbf{e}_{-\mathbf{k}}^{(b)}$. The vectors $\mathbf{q}_\mathbf{k}$ and $\mathbf{p}_\mathbf{k}$ are decomposed into the three orthogonal basis vectors as

$$\begin{aligned}\mathbf{q}_\mathbf{k} &= Q_\mathbf{k}^{(a)}\,\mathbf{e}_\mathbf{k}^{(a)} + Q_\mathbf{k}^{(b)}\,\mathbf{e}_\mathbf{k}^{(b)} + Q_\mathbf{k}^{(0)}\,\mathbf{k}/k\,,\\ \mathbf{p}_\mathbf{k} &= P_\mathbf{k}^{(a)}\,\mathbf{e}_\mathbf{k}^{(a)} + P_\mathbf{k}^{(b)}\,\mathbf{e}_\mathbf{k}^{(b)}\,.\end{aligned} \tag{32}$$

From (29) follows that the operators $P_\mathbf{k}$, $Q_\mathbf{k}$ obey the canonical commutation rules:

$$[P_\mathbf{k}^{(a)}, Q_{\mathbf{k}'}^{(a)}] = [P_\mathbf{k}^{(b)}, Q_{\mathbf{k}'}^{(b)}] = \frac{\hbar}{i}\,\delta_{\mathbf{k},\mathbf{k}'}\,, \tag{33}$$

while all other quantities commute. In terms of $P_\mathbf{k}$ and $Q_\mathbf{k}$, the energy operator reads

$$\begin{aligned}U = \frac{c^2}{2}\sum_\mathbf{k}\Bigg[&-P_\mathbf{k}^{(a)}\,P_{-\mathbf{k}}^{(a)}\sum_{j=1}^{3}\frac{e_{j\mathbf{k}}^{(a)2}}{\varepsilon_j} + P_\mathbf{k}^{(b)}\,P_{-\mathbf{k}}^{(b)}\sum_{j=1}^{3}\frac{e_{j\mathbf{k}}^{(b)2}}{\varepsilon_j} -\\ &-(P_{-\mathbf{k}}^{(a)}\,P_\mathbf{k}^{(b)} - P_\mathbf{k}^{(a)}\,P_{-\mathbf{k}}^{(b)})\sum_{j=1}^{3}\frac{e_{j\mathbf{k}}^{(a)}\,e_{j\mathbf{k}}^{(b)}}{\varepsilon_j} - \frac{k^2}{c^2}(Q_\mathbf{k}^{(a)}\,Q_{-\mathbf{k}}^{(a)} - Q_\mathbf{k}^{(b)}\,Q_{-\mathbf{k}}^{(b)})\Bigg].\end{aligned} \tag{34}$$

This expression can be diagonalized only if the coefficient of the term $P^{(a)}_{-\mathbf{k}} P^{(b)}_{\mathbf{k}} - P^{(a)}_{\mathbf{k}} P^{(b)}_{-\mathbf{k}}$ vanishes i.e. if

$$\sum_{j=1}^{3} \frac{e^{(a)}_{j\mathbf{k}} e^{(b)}_{j\mathbf{k}}}{\varepsilon_j} = 0 . \tag{35}$$

Eqs. (35) and (31) determine two unit vectors. It can easily be seen that Eqs. (35) and (31) are equivalent to the condition of the classical crystal optics specifying the directions of polarization (of **D**) in the medium, quoted in several textbooks [17, 18]. Consequently, if $\mathbf{e}^{(a)}_{\mathbf{k}}$ and $\mathbf{e}^{(b)}_{\mathbf{k}}$ are chosen to coincide with the directions of polarization of an electromagnetic plane wave (with wave-number **k**) in the crystal, the energy operator is obtained as

$$U = \frac{c^2}{2} \sum_{\mathbf{k}} \left(-\alpha_{\mathbf{k}} P^{(a)}_{\mathbf{k}} P^{(a)}_{-\mathbf{k}} + \beta_{\mathbf{k}} P^{(b)}_{\mathbf{k}} P^{(b)}_{-\mathbf{k}} - \frac{k^2}{c^2} Q^{(a)}_{\mathbf{k}} Q^{(a)}_{-} + \frac{k^2}{c^2} Q^{(b)}_{\mathbf{k}} Q^{(b)}_{-\mathbf{k}} \right) \tag{36}$$

with

$$\alpha_{\mathbf{k}} \equiv \sum_{j=1}^{3} \frac{e^{(a)2}_{j\mathbf{k}}}{\varepsilon_j}, \qquad \beta_{\mathbf{k}} \equiv \sum_{j=1}^{3} \frac{e^{(b)2}_{j\mathbf{k}}}{\varepsilon_j} . \tag{37}$$

Finally, the creation and the annihilation operators $a^+_{\mathbf{k}}, a_{\mathbf{k}}$ and $b^+_{\mathbf{k}}, b_{\mathbf{k}}$ are introduced by defining

$$\begin{aligned} Q^{(a)}_{\mathbf{k}} &= \sqrt{\frac{\hbar c}{2k}}\, \delta_{\mathbf{k}} (a_{\mathbf{k}} - a^+_{-\mathbf{k}}) , & Q^{(b)}_{\mathbf{k}} &= \sqrt{\frac{\hbar c}{2k}}\, \gamma_{\mathbf{k}} (b_{\mathbf{k}} + b^+_{-\mathbf{k}}) , \\ P^{(a)}_{\mathbf{k}} &\quad \sqrt{\frac{\hbar k}{2c}}\, \frac{i}{\delta_{\mathbf{k}}} (a^+_{\mathbf{k}} + a_{-\mathbf{k}}) , & P^{(b)}_{\mathbf{k}} &= \sqrt{\frac{\hbar k}{2c}}\, \frac{i}{\gamma_{\mathbf{k}}} (b^+_{\mathbf{k}} - b_{-\mathbf{k}}) . \end{aligned} \tag{38}$$

It is assumed that $\delta_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ do not change their signs when reflecting **k**. Then the commutation relations of the new operators are g ven as

$$\begin{aligned} [a_{\mathbf{k}}, a^+_{\prime}] &= [b_{\mathbf{k}}, b^+_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} , \\ [a_{\mathbf{k}}, b_{\mathbf{k}'}] &= [a_{\mathbf{k}}, b^+_{\mathbf{k}'}] = 0 . \end{aligned} \tag{39}$$

With the choice $\delta_{\mathbf{k}} = \sqrt[4]{\alpha_{\mathbf{k}}}$, $\gamma_{\mathbf{k}} = \sqrt[4]{\beta_{\mathbf{k}}}$ the energy operator takes the following diagonal form

$$U = \sum_{\mathbf{k}} \hbar k c \left[\sqrt{\alpha_{\mathbf{k}}}\, (N^{(a)}_{\mathbf{k}} + 1/2) + \sqrt{\beta_{\mathbf{k}}}\, (N^{(b)}_{\mathbf{k}} + 1/2) \right] , \tag{40}$$

where $N^{(a)}_{\mathbf{k}} \equiv a^+_{\mathbf{k}} a_{\mathbf{k}}$, $N^{(b)}_{\mathbf{k}} \equiv b^+_{\mathbf{k}} b_{\mathbf{k}}$. Since the eigenvalues of the latter operators are $n^{(a)}_{\mathbf{k}}, n^{(b)}_{\mathbf{k}} = 0, 1, 2, \ldots$, the eigenvalues of the energy operator U are obtained as

$$U = \sum_{\mathbf{k}} \hbar k c \left[\sqrt{\alpha_{\mathbf{k}}}\,(n_{\mathbf{k}}^{(a)} + 1/2) + \sqrt{\beta_{\mathbf{k}}}\,(n_{\mathbf{k}}^{(b)} + 1/2)\right]. \tag{41}$$

Thus, the energy of the electromagnetic radiation is quantized also in anisotropic media but now the energy quantum (energy of a photon) becomes different for different directions of polarization:

$$\begin{aligned} \varepsilon_{\mathbf{k}}^{(a)} &= \hbar k c \sqrt{\alpha_{\mathbf{k}}}\,, \\ \varepsilon_{\mathbf{k}}^{(b)} &= \hbar k c \sqrt{\beta_{\mathbf{k}}}\,. \end{aligned} \tag{42}$$

The refractive index of the medium for polarization a (or b) can be shown to be

$$\frac{1}{n_{a(b)}^2} = \sum_{j=1}^{3} \frac{(e_{j\mathbf{k}}^{(a)/(b)/})^2}{\varepsilon_j}\,. \tag{43}$$

Using (43), (37) and (42) the energy of a photon in anisotropic media is obtained in terms of the refractive index as

$$\varepsilon_{\mathbf{k}}^{(a)/(b)/} = \frac{\hbar k c}{n_{a(b)}}\,. \tag{44}$$

Eq. (44) is of the same form as the expression of the energy of a photon in an isotropic medium, $\varepsilon = \hbar k c/n$, buth there the refractive index is independent of the direction of polarization. When taking the isotropic limit $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ the results go over into those of [3].

Now we turn to the calculation of the momentum of the radiating field. Starting with (26) and repeating the procedure described above, one obtains the following expression as the momentum operator

$$\begin{aligned} G = \sum_{\mathbf{k}} \Bigg[& \hbar \left(\mathbf{k}\,\alpha_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(a)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(a)} k_j}{\varepsilon_j}\right) N_{\mathbf{k}}^{(a)} + \\ & + \hbar \left(\mathbf{k}\,\beta_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(b)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(b)} k_j}{\varepsilon_j}\right) N_{\mathbf{k}}^{(b)} - \\ & - \frac{\hbar}{2} \sqrt[4]{\frac{\beta_{\mathbf{k}}}{\alpha_{\mathbf{k}}}}\, \mathbf{e}_{\mathbf{k}}^{(b)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(a)} k_j}{\varepsilon_j} (a_{\mathbf{k}}^{+} b_{\mathbf{k}} + a_{\mathbf{k}}^{+} b_{-\mathbf{k}}^{+} + a_{-\mathbf{k}} b_{\mathbf{k}} + a_{-\mathbf{k}} b_{-\mathbf{k}}^{+}) - \\ & - \frac{\hbar}{2} \sqrt[4]{\frac{\alpha_{\mathbf{k}}}{\beta_{\mathbf{k}}}}\, \mathbf{e}_{\mathbf{k}}^{(a)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(b)} k_j}{\varepsilon_j} (b_{\mathbf{k}}^{+} a_{\mathbf{k}} - b_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+} - b_{-\mathbf{k}} a_{\mathbf{k}} + b_{-\mathbf{k}} a_{-\mathbf{k}}^{+}) \Bigg]. \end{aligned} \tag{45}$$

The last two sums containing both types of polarization are non-diagonal. Note that due to (31) in the isotropic case the non-diagonal terms vanish and G goes over to the expression obtained in [3].

The energy eigenstates of the electromagnetic field are not eigenstates of the momentum of the field in anisotropic media. If, however, one takes the quantum mechanical mean value of (45) in an eigenstate of $N_{\mathbf{k}}^{(a)}$ and $N_{\mathbf{k}}^{(b)}$ the non-diagonal terms vanish:

$$\langle \mathbf{G} \rangle = \sum_{\mathbf{k}} \left[\hbar \left((\mathbf{k}\,\alpha_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(a)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(a)} k_j}{\varepsilon_j} \right) n_{\mathbf{k}}^{(a)} + \right. \\ \left. + \hbar \left(\mathbf{k}\,\beta_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(b)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(b)} k_j}{\varepsilon_j} \right) n_{\mathbf{k}}^{(b)} \right], \tag{46}$$

i.e. the mean value of the momentum of the electromagnetic field becomes a multiple of momentum quanta. An interpretation of this result can be that in anisotropic dielectrica the momentum of a photon is given by one of the following expressions depending on the polarization

$$\mathbf{g}_{\mathbf{k}}^{(a)} = \hbar \left(\mathbf{k}\,\alpha_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(a)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(a)} k_j}{\varepsilon_j} \right), \tag{47}$$

$$\mathbf{g}_{\mathbf{k}}^{(b)} = \hbar \left(\mathbf{k}\,\beta_{\mathbf{k}} - \mathbf{e}_{\mathbf{k}}^{(b)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(b)} k_j}{\varepsilon_j} \right). \tag{48}$$

It is worth-while noting that the relation between the momentum of a photon $\mathbf{g}_{\mathbf{k}}$, its energy $\varepsilon_{\mathbf{k}}$, and the velocity of the propagation of the energy $\mathbf{v}^*$ defined by (13) is the same as for vacuum or isotropic media:

$$\mathbf{g}_{\mathbf{k}}^{(a)/(b)/} = \frac{\varepsilon_{\mathbf{k}}^{(a)/(b)/}}{c^2} \mathbf{v}_{a(b)}^* . \tag{49}$$

The combination $\varepsilon_{\mathbf{k}}^{(a)/(b)/}/c^2 \equiv m_{a(b)}$ is the inertial mass of a photon, the expression of $\mathbf{v}_{a(b)}^*$ is given by the formula (54). Eq. (49) shows that the direction of the photon momentum corresponds to that of the Poynting vector but in anisotropic dielectrica it does not coincide with the direction of the wave-number $\mathbf{k}$ characterizing the propagation of the light.

The non-diagonal terms in the expression of the momentum (45) are the results of a mixing of waves with the same wavelength but different frequencies and polarizations. They depend on time as $\exp[i(\varepsilon_{\mathbf{k}}^{(a)} - \varepsilon_{\mathbf{k}}^{(b)})t/\hbar]$. The average of these terms is zero, therefore no measurable contribution is given by them. It can be easily seen that this phenomenon coming from the anisotropy of the medium is not of quantum origin since it appears also in the classical theory of crystal optics. If one calculates the Poynting vector of two superposed electromagnetic plane waves with wave number **k** and different polarizations, then it also contains "interference terms" with a time dependence as above. Since the time average of the "interference terms" vanishes the light intensity is composed of two diagonal terms corresponding to the two possible directions of polarization.

If we consider an electromagnetic wave of a given (a or b) polarization then, of course, no mixing can occur, consequently the energy and momentum operators are diagonal and their eigenvalues can uniquely be given as sums of photon energies and photon momenta, respectively. In this sense the corpuscular nature of the electromagnetic radiation holds also in anisotropic dielectrica. In the general case, however, we have to consider photons of anisotropic media as "quasi-particles" the momenta of which can be given only as mean values.

Finally, for the sake of comparison we summarize the results which would arise from a calculation based on MINKOWSKI's tensor. According to MINKOWSKI the momentum of the field for $\mu = 1$, is given by

$$\mathbf{G}^{(M)} = \frac{1}{c}\int (\mathbf{D} \times \mathbf{H})\, dV, \tag{50}$$

while the expression of the energy remains (26).

Following the quantization procedure described above, one obtains

$$\mathbf{G}^{(M)} = \sum_{\mathbf{k}} \hbar\, \mathbf{k}(a_{\mathbf{k}}^{+} a_{\mathbf{k}} + b_{\mathbf{k}}^{+}\, b_{\mathbf{k}}) \tag{51}$$

as the momentum operator the eigenvalues of which are

$$\mathbf{G}^{(M)} = \sum_{\mathbf{k}} \hbar\, \mathbf{k}(n_{\mathbf{k}}^{(a)} + n_{\mathbf{k}}^{(b)}), \tag{52}$$

where $n_{\mathbf{k}}^{(a)}$ and $n_{\mathbf{k}}^{(b)}$ denote non-negative integer numbers. Accordingly, MINKOWSKI's photons possess momentum

$$\mathbf{g}_{\mathbf{k}}^{(M)} = \hbar\, \mathbf{k}.$$

MINKOWSKI's photons, however, exhibit the same strange properties as those in isotropic media. For example, the rest mass of these photons is imaginary: $m_{0a(b)}^{(M)} = \hbar k \sqrt{1 - n_{a(b)}^2}/(c n_{a(b)})$. Furthermore, the direction of the photon momentum does not coincide with that of the propagation velocity v^* of the energy defined by (13) and consequently the momentum cannot be expressed as inertial mass $\times$ velocity.

On the contrary, for photons arising from a theory based on ABRAHAM's tensor the formula momentum = inertial mass $\times$ velocity holds (see (49)), the rest mass given by

$$m_{0\,a(b)} = \frac{\hbar k}{c n_{a(b)}} \sqrt{1 - v^{*2}_{a(b)}/c^2} \tag{53}$$

with

$$\mathbf{v}^*_{a(b)} = c \left(\frac{\mathbf{k}}{k} \frac{1}{n_{a(b)}} - \mathbf{e}_{\mathbf{k}}^{(a)/(b)/} n_{a(b)} \sum_{j=1}^{3} \frac{e_{j\mathbf{k}}^{(a)/(b)/} k_j}{\varepsilon_j k} \right) \tag{54}$$

is real and the rest mass and the inertial one $m_{a(b)}$ are related by the formula

$$m_{a(b)} = \frac{m_{0\,a(b)}}{\sqrt{1 - v^{*2}_{a(b)}/c^2}} \,. \tag{55}$$

Eqs. (55) and (49) suggest that in anisotropic media $v^*_{a(b)}$ is to be considered as the velocity of a photon of polarisation a (or b).

Concluding, we can say that in spite of the fact that ABRAHAM's photons are only "quasi-particles" in the sense discussed above, they possess a number of properties characterizing the realistic particles. On the other hand, the advantage of the use of MINKOWSKI's expression lies in the fact that the energy and the momentum operators of the field commute and, consequently, they can be diagonalized simultaneously.

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