Chaotic scattering: An introduction

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In recent years chaotic behavior in scattering problems has been found to be important in a host of physical situations. Concurrently, a fundamental understanding of the dynamics in these situations has been developed, and such issues as symbolic dynamics, fractal dimension, entropy, and bifurcations have been studied. The quantum manifestations of classical chaotic scattering is also an extremely active field, with new analytical techniques being developed and with experiments being carried out. This issue of *Chaos* provides an up-to-date survey of the range of work in this important field of study.

I. WHAT IS CHAOTIC SCATTERING?

In the most general sense, we can define scattering as the problem of obtaining the relationship between an "input" variable (or variables) characterizing an initial condition for some dynamical system (usually Hamiltonian) and an "output" variable (or variables) characterizing a suitably defined "final" state of the system. The simplest example of this type deals with the motion of a point particle in a potential $V(\mathbf{x})$, where $V(\mathbf{x})$ is zero or else very small, outside of some scattering region of finite spatial extent. Thus, outside the scattering region, the orbit moves along a straight line (or an approximately straight line). An orbit approaching the scattering region interacts with the scatterer, and then leaves the scattering region. This is shown schematically in Fig. 1 for the case of a two-dimensional (x,y) scattering problem. Also in Fig. 1, we define an input variable b (the impact parameter) and an output parameter ϕ (the scattering angle). The question addressed is the following: what is the functional relationship between ϕ and b for a given fixed direction of the incoming velocity? The interesting point is that this relationship is qualitatively different for the two cases where the dynamics in the scattering region is chaotic and nonchaotic.

This issue of *Chaos* is devoted to recent developments in the field of chaotic scattering. The purpose is to provide an up-to-date snapshot of the spectrum of activities in the field, ranging from general theory to applications. This article will introduce the general topic of chaotic scattering and will provide elementary background information for those not familiar with chaotic scattering.

To begin, as an example, Fig. 2 shows numerically obtained plots[1] of ϕ versus b for the potential $V(x,y)=x^2y^2\exp[-(x^2+y^2)]$. This potential consists of four hills whose peaks are located at $(x,y)=(\pm 1,\pm 1)$. When the particle energy E is greater than the maximum potential energy at the hill peaks (denoted E_m), the scattering is nonchaotic. As shown in Fig. 2(a), the relation between ϕ and b is a simple smooth curve in this case. Figure 2(b) shows a numerical plot of ϕ versus b for a chaotic case. As is evident from the speckled appearance in this finite resolution plot, there are some regions in which the output variable ϕ varies too rapidly with the input variable b to be resolved. Picking such a region and blowing up its horizontal scale to improve resolution, we see, in Fig. 2(c), that speckled regions still remain. Further blow-ups shown in Figs. 2(d) and 2(e) still fail to resolve the function. This suggests that there are impact parameter values about which ϕ varies by an amount of order one on arbitrarily fine scale in b.

In fact, the set of such b values is numerically found to be fractal with a dimension of approximately $D=0.67$ for this example.[2] In general, for chaotic scattering in two dimensions, there will be a fractal set of b values on which the scattering function ϕ versus b is singular in the manner described above. This fractal set will typically have a dimension greater than zero and less than one in the absence of KAM curves in the scattering region (this is the case for Fig. 2). When there are KAM curves, the singular set typically has dimension one (although we emphasize that the set still has Lebesgue measure zero).[3]

The source of the singularity can, in principle, be seen by following a trajectory from an input at which the scattering function is singular. Such an orbit enters the scattering region, but never exits, continuing to bounce around

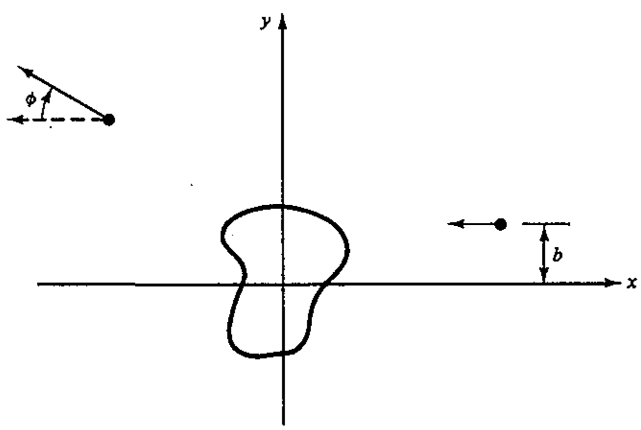

FIG. 1. Schematic illustration of a scattering problem in two dimensions.

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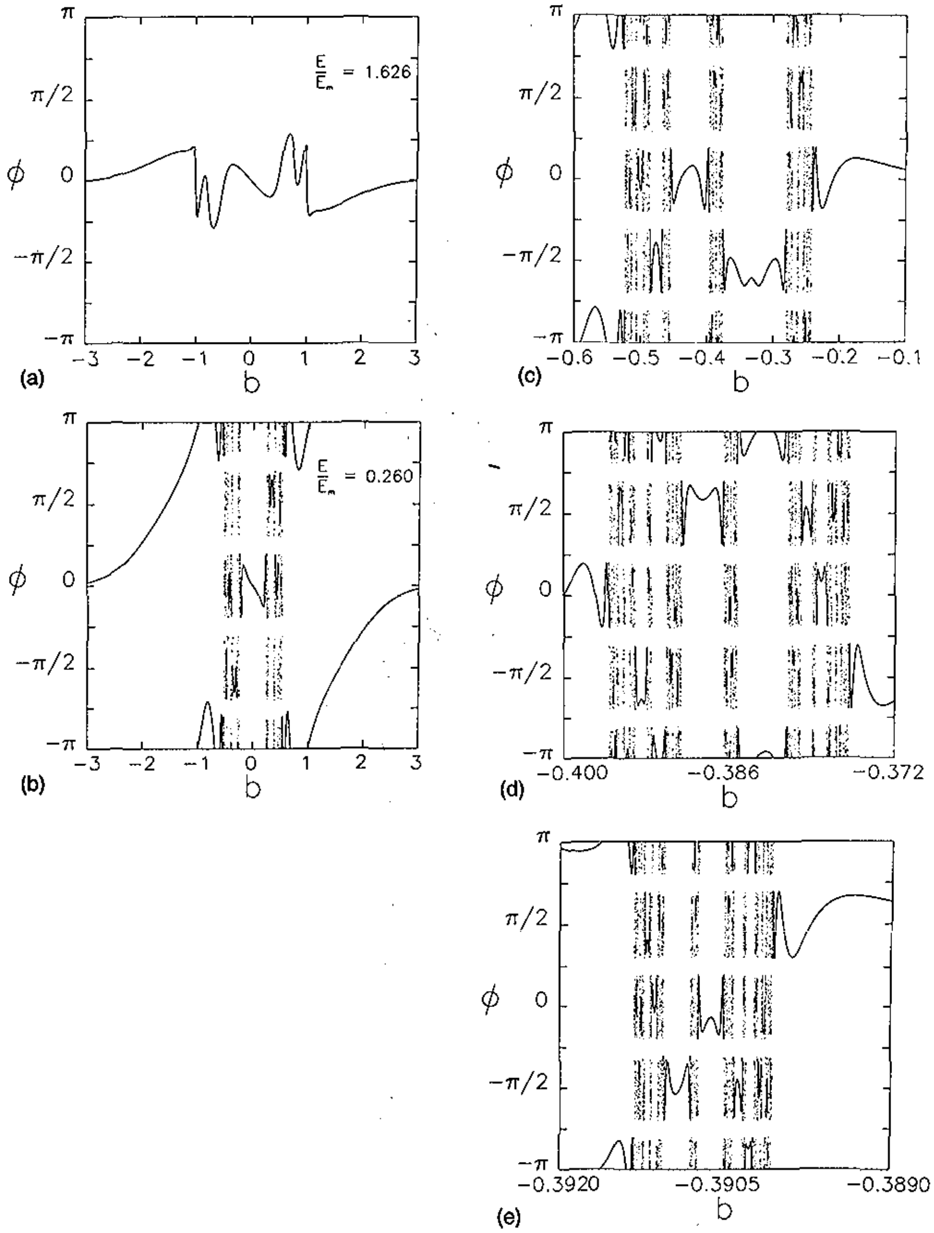

FIG. 2. ϕ versus b for the potential $V(x,y)=x^2y^2\exp[-(x^2+y^2)]$ (a) for a case where $E>E_m$, and (b) for a case with $E<E_m$. (c)–(e) show successive blowups of a small region in b for the case $E<E_m$ shown in (b). (From Ref. 1.)

in the scattering region forever. For example, for the four hill potential $V(x,y)=x^2y^2\exp[-(x^2+y^2)]$ for which Fig. 2 was drawn, an impact parameter value very near to a singular b value might lead to an orbit that experiences some large number of bounces from the hills, say 1000, before exiting the scattering region. A very tiny change in the impact parameter might then result in say 1001 bounces. The tiny change only makes a small relative change in the number of bounces, from 1000 to 1001, but that one extra bounce totally changes the scattering angle. This sensitive dependence on initial conditions and the related fractal structure can very clearly be seen in the everyday-life phenomenon of how three Christmas balls are mirrored in each other.[4] A simplified two-dimensional version of this, the so-called three-disk problem originally considered by Eckhardt[5] and studied in great detail by Gaspard and Rice,[6] provides a nice paradigm model of chaotic scattering.

Alternatively, we can also consider the time delay function that tells us how the time the orbits spend in the scattering region depends on the impact parameter b. In the chaotic case, this function takes on an infinite value whenever there is a singularity in the deflection angle. In the nonchaotic case, Fig. 2(a), $E>E_m$ and no orbit can spend more time in the scattering region than some finite upper bound.

Another way of thinking about the chaotic scattering phenomenon is to consider the invariant set of orbits that bounce around in the potential without ever leaving the scattering region for all time $t\to\pm\infty$. The dynamics on

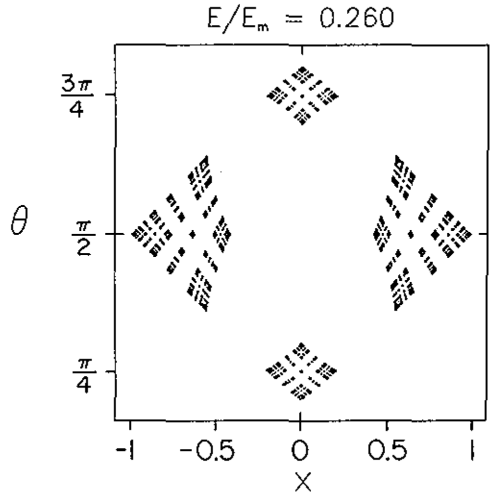

FIG. 3. The chaotic invariant set in the $y=0$ surface of section for the case in Figs. 2(b)–2(e). (From Ref. 1.)

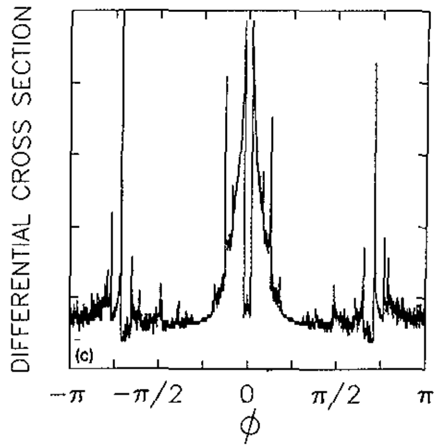

FIG. 4. The differential cross-section for the four hill potential $V(x,y)= x^2y^2 \exp[-(x^2+y^2)]$ for a case with $E<E_m$. (From Ref. 1.)

this invariant set is chaotic, and, for the hyperbolic case, orbits on the set can be encoded by a symbolic dynamics representation. The knowledge of such an encoding is of basic importance[7] since it enables one to distinguish different classes of particle trajectories and to introduce simple characteristic numbers, like, e.g., entropies (see the articles by Troll[8] and Daniels, Vallières and Yuan[9] in this issue). The hyperbolic chaotic scattering set may be envisioned as the intersection of a stable manifold and an unstable manifold, where the stable (unstable) manifold consists of an uncountably many fractal set of roughly parallel surfaces on which orbits approach the invariant chaotic set as $t\rightarrow +\infty$ ($t\rightarrow -\infty$). The fractal set of singular b values for the scattering function corresponds to those b values that lie on the stable manifold.[10] Figure 3 shows a numerical plot of the chaotic invariant set in the surface of section $y=0$ for the case in Fig. 2(a). The intersection structure mentioned above is clearly evident. The fractal dimension of the set along the unstable direction is thus D. Because of the time reversal invariance, however, the other partial dimension, that along the stable direction, has to be D, too. The fractal dimension d_0 of the chaotic set shown in Fig. 3 is thus $d_0=2D(=1.34)$. Note, however, that for chaotic scattering with more than two degrees of freedom it might happen that a randomly chosen initial line does not intersect the stable manifold of the invariant set even if the latter is chaotic. Scattering functions have fractal properties in such cases only if the fractal dimension of the chaotic set is large enough.[11]

Another fundamental aspect of chaotic scattering is the time delay statistics $P(T)$. Imagine that we pick many b values at random in some interval. We then examine the resulting orbit from each value and determine the time T that this orbit spends in the scattering region. The fraction of orbits with "time delays" between T and $T+dT$ is $P(T)\,dT$. For hyperbolic dynamics in the scattering region, one finds that for large T the time delay statistics decays exponentially,

$$P(T)\sim\exp(-T/\tau) \tag{1}$$

where τ is a characteristic time of the scatterer. For chaotic dynamics with bounding KAM surfaces in the scattering region, there is a qualitatively different behavior in which $P(T)$ apparently decays algebraically,[3,12]

$$P(T)\sim T^{-\alpha}. \tag{2}$$

It is worth noting that opening a window on a billiard to allow the escape of particles also represents a kind of scattering problem.[13] In the case of chaotic billiards the decay is typically exponential but the presence of any marginally stable orbits might convert the decay to algebraic even if no KAM surfaces are present.

The differential cross-section is a fundamental characteristic of any scattering process. Figure 4 shows this quantity computed for the four hill problem at a particle energy below E_m.[1] The appearance of a multiplicity of singularities is obvious. They sit on a set of the same fractal dimension D as the singularities of the deflection or time delay functions.[14] In nonchaotic cases, the differential cross-section is either smooth or its singularities are restricted to a set of dimension zero.

The above gives some of the most basic phenomenology of chaotic scattering. Given the generality of the discussion, it should not be surprising that chaotic scattering has found application in a host of fields. We discuss some of these applications in Sec. II, limiting the coverage to classical chaotic scattering (as opposed to quantum chaotic scattering discussed in Sec. IV). Section III reviews some aspects of ergodic motion on invariant chaotic scattering sets, including multifractality, Lyapunov exponents, periodic orbits, etc. Section IV deals with quantum chaotic scattering. The basic problem in quantum chaotic scattering is to analyze and discover the typical behaviors characteristic of quantum scattering problems in the semiclassical regime for situations in which the corresponding Hamiltonian yields chaotic scattering in the classical case.

II. PHYSICAL EXAMPLES OF CHAOTIC SCATTERING

To provide an indication of the wide applicability of chaotic scattering, this section discusses some physical examples where chaotic scattering is relevant. We emphasize that the topics chosen for discussion are only meant as a representative selection and that there may be other significant physical applications that are not mentioned.

A. Celestial mechanics

The birth of the science of chaos was Poincaré's study of the interaction of three gravitationally interacting bodies. Even the one-dimensional three body problem studied recently by Hietarinta and Mikkola[15] has large regions of the phase space connected with chaotic scattering. One classical result is due to Sitnikov[16] who investigated a version of the restricted three body problem. When two heavy masses move on Kepler ellipses around their center of mass and a third body of negligible mass is restricted to move along a line through the center of mass but perpendicular to the plane of the ellipses, Sitnikov was able to rigorously show that the light bodies motion is chaotic even if it escapes the vicinity of the two other masses. A more recent approach using the language of scattering chaos was carried out by Petit and Hénon.[17] They considered the case of two small bodies moving around a very heavy mass. In the planar problem, initially one has the two small bodies moving in a common plane on circles of different radii around the central mass. In the case of close encounters complicated motion can take place, but, with the exception of a set of initial conditions of zero measure, the small particles separate again. This phenomenon is relevant for the moons of large planets or particles in planetary rings, e.g., that of the Saturn.

The paper of Boyd and McMillan in this issue[18] considers the chaotic scattering of an incoming star on a binary (two stars circling each other). From the physical point of view the problem is of interest with respect to the heating mechanism for globular clusters: the scattering interaction can result in a more tightly bound binary, thus transferring gravitational energy to kinetic energy.

B. Charged particle trajectories in electric and magnetic fields

Charged particle motion in electric and magnetic fields is an important aspect of a variety of fields including particle accelerators, electromagnetic wave generators (e.g., free electron lasers), and plasma physics in general. In this issue Chernikov and Schmidt[19] discuss the acceleration of a charged particle in a uniform magnetic field in the presence of large amplitude traveling electromagnetic waves. They show that very strong and efficient particle acceleration can result, and that the process is basically one of transient Hamiltonian chaos (i.e., chaotic scattering). As another example, particularly interesting because of recent observational confirmation, we mention work on the interaction of particles with the central plasma sheet in the magnetotail of the Earth's magnetosphere. The magnetic field in this region can be modeled as

$$\mathbf{B}(x,y) = B_0 \tanh(z/\delta)\mathbf{x}_0 + B_n \mathbf{z}_0, \tag{3}$$

where $\mathbf{x}_0$ and $\mathbf{z}_0$ are unit vectors, and δ, B_0 and B_n are constants. For $|z| \gg \delta$ the field is uniform, and particles spiral about the field. Since $B_n \neq 0$ the guiding center of a particle spiraling along the field can move toward the layer $|z| \sim \delta$. Particle motion inside this layer can be chaotic. The dependence of the particle trajectory exiting the layer on the particle motion in the homogeneous field region before entry is a chaotic scattering problem. It was predicted theoretically on the basis of solutions of this problem that the particle velocity distribution function for ions would display resonant peaks corresponding to resonances in the chaotic scattering problem. Recent space probe measurements beautifully confirm these predictions (see Ref. 20 for a review of this subject).

C. Magnetic field line trajectories

In plasmas, magnetic field line trajectories, given by the equation,

$$d\mathbf{x}/ds = \mathbf{B}(\mathbf{x}), \tag{4}$$

are often important for determining physical properties of the system. In Eq. (4) s represents a parametric variable measuring distance along the magnetic field line. Since $\nabla \cdot \mathbf{B} = 0$, Eq. (4) is analogous to the motion of a passive particle convected by an incompressible fluid (described by $d\mathbf{x}/dt = \mathbf{v}$ and $\nabla \cdot \mathbf{v} = 0$; see Sec. II D). Recent work by Lau and Finn [20] on reconnection of three-dimensional magnetic fields in solar plasmas, has shown that chaotic scattering for the conservative dynamical system (4) provides the essential ingredient necessary for understanding their problem. In magnetic reconnection the topology of the magnetic field configuration changes and magnetic energy is converted to heat. However, in infinitely conducting plasmas, field lines are frozen into the plasma, and topology change is then ruled out. Thus for field lines to reconnect, in a highly conductive situation (as in the plasma above the surface of the Sun), an energy dissipation mechanism must be found. Lau and Finn consider a situation where typical magnetic field lines leave and then return to the Sun's surface. Some of these take longer (in field line length) to do this, and they find that there is a fractal set which takes forever to return. This is clearly the same phenomena as noted in Sec. I. The necessary dissipation is associated with these infinitely long lines because an inductive electric field then has an infinitely long time to accelerate electrons along these magnetic field lines, and electrons can then gain large energy.

D. Hydrodynamical processes

It has long been known that the dynamics of ideal linear vortices is chaotic if the number of participating vortices is greater than three (see Refs. 22 and 23). When vortices are started far away from each other, they can come close due to their mutual interaction. For an infinite homogeneous fluid, a bounded configuration for all of the participating vortices does not exist, and the distance between some of them will sooner or later grow without any limit. Such processes are examples of scattering, and, in case of four or more vortices, they are typically chaotic. The motion of two vortex pairs has been studied in detail by Aref, Eckhardt and co-workers.[24] They pointed out that in the course of interaction, the pairs exchange partners for a finite amount of time. The new couples have different vorticities and move, as long as they are far away from each other, on approximately circular orbits. Close encounters of these couples strongly perturb the orbits, but this "metastable" configuration of two couples can last very long, until a new collision finally leads to an exchange again. This results in the recovery of the two original vortex pairs which then leave each other along straight line orbits. This is in complete qualitative agreement with the experiments by van Heist and Flor[25] but, unfortunately, no quantitative analysis of experimental data has yet been performed. The motion in the "metastable" configuration is chaotic and the process exhibits all properties of chaotic scattering mentioned in Sec. I. A related problem with potential practical applications is the motion of a single vortex in the vicinity of a boundary.[26] The contribution of Kadtke and Novikov in this issue investigates this phenomenon in detail.[27]

It is also well known that the motion of a passive tracer particle in a simple nonstationary, say time periodic, velocity field can be chaotic (see, e.g., Ref. 28). Incompressibility implies the Hamiltonian character of the Lagrangian dynamics. If the nonstationarity of the flow is restricted to a central region outside of which the velocity field is practically stationary, the Lagrangian advection problem in an incompressible fluid provides another example of scattering motion. As pointed out by Aref,[29] the tracer dynamics in the space of ideal vortices can be chaotic even if the motion of vortices is integrable, e.g., in the presence of three vortices. The transport by the vortical flow generated by a vortex pair in a wavy-walled channel has been investigated by Rom-Kedar, Leonard, and Wiggins.[30] These early studies have not yet used the tools of chaotic scattering theory, therefore, a reconsideration of them in this spirit would be interesting.

Another broad class of problems is related to viscous flows. The passive particle motion in any open incompressible flow can also be an example of scattering.[31] Shariff, Pulliam, and Ottino, and Jung and Ziemniak demonstrated[32] that particles advected by a flow generating a von Kármán-street behind a cylinder can be trapped behind the cylinder, and exhibit there a sensitive dependence on initial conditions. This is the region where a strange set exists responsible for the chaos of the scattering process. The contribution of Jung, Tél and Ziemniak[33] in this issue is a more quantitative investigation of this problem emphasizing the important role of the hydrodynamical boundary conditions in making the particle dynamics nonhyperbolic.

E. Models of chemical reactions

In many chemical reactions the number of molecules in the initial and finite states is the same, i.e., no chemical complex is created. Nevertheless, intermediate complexes of finite average lifetime might be present.

In the simplest case, one has initially two atoms, say A and B, bound in a molecule and a third atom C interacting with the other atoms. Reaction takes place when these two components collide with each other. The qualitatively different outcomes of the interaction lead either to the creation of new molecules AC or BC and free atoms B or A or to the survival of molecule AB and atom C. This system is an example of a three body problem, and is not integrable even in restricted geometries like one-dimensional motion.

The iteration of classical trajectories in such restricted geometries lead, much before the concept of chaos became widespread, to surprising results. Rankin and Miller found[34] that small changes in the initial conditions lead to drastic differences in the final states, so that a nonreactive trajectory might exist in the vicinity of a reactive one or vice versa. Later work[35] showed that this property persists to the smallest numerically accessible scales. Noid, Gray and Rice pointed out[36] the underlying fractal structure in the scattering process. The importance of unstable periodic orbits to the understanding of these chemical reactions has been emphasized by Pechukas, Pollak and Child.[37] By now it has become clear that there is typically an infinity of unstable periodic orbits present[38] forming the backbone of a nonattracting chaotic set. Most recent investigations use the complete theory of dynamical system and chaotic scattering.[39] Since chaotic motion is present as long as all atoms are close to each other, the average lifetime of chaotic trajectories τ can be interpreted in classical models of chemical reactions as the average lifetime of the intermediate complex. In this focus issue, the contribution by Koch and Bruhn[40] deals with a model of chemical reactions.

An interesting new development is the control of chaos in chemical reactions[41] which means that a certain type of nonchaotic motion can be stabilized by appropriately choosing an external perturbation. This opens the possibility of stabilizing metastable complexes.

F. Scattering in atomic and nuclear physics

The classical dynamics of two-electron atoms has much similarity to that of $A-B-C$ chemical reactions with the essential difference that the interaction is of long range type due the Coulomb force. Much attention has been focused on the problem of helium. It has long been believed that the classical helium atom would be unstable apart from a set of initial conditions of measure zero, and would spontaneously autoionize. Recently, however, it has been shown by Richter and Wintgen,[42] and later by Yamamoto and Kaneko[43] that the phase space is stable around certain periodic orbits corresponding to, e.g., asymmetrical stretching motions of the two electrons around the nucleus.[44] Nevertheless, only a small part of the phase space corresponds to stable regular motion.[42] Outside of

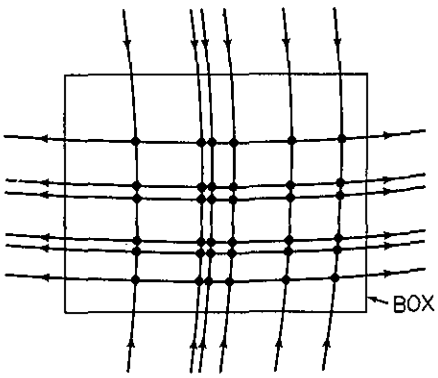

FIG. 5. Schematic illustration in a Poincaré surface of section of a chaotic invariant set as the intersections of stable and unstable manifolds which each consist of a Cantor set of roughly parallel lines.

this region, there exists an infinity of unstable periodic orbits embedded in a nonattracting chaotic set, and, therefore, nearly the entire phase space can be explored by means of scattering processes. As the first investigations indicate,[45] the tori around the stable periodic orbits can successfully be used for carrying out semiclassical quantization. In this volume, Yuan and Gu[46] investigate the scattering of an electron on a helium ion.

Another interesting phenomenon connected with both chemical reactions and atomic processes is the interaction of the constituents with an external electromagnetic field. Even if the motion is simple in the field-free case, external driving generically destroys integrability and leads to fragmentation or ionization. The study of such systems has also attracted recent interest and serves as a useful model for dissociation of molecules[47] or ionization of atoms.[48] The contribution of Beeker and Eckelt[49] in this issue is devoted to a periodically driven scattering system and points out that parabolic orbits also generate power law decays.

Scattering experiments are essential in studying the nuclear structure, too. Therefore, it is natural that chaotic scattering also has relevance to this branch of physics[50] which is reviewed in the contribution to this issue by Baldo, Lanza and Rapisarda.[51]

III. ERGODIC ASPECTS OF CHAOTIC SCATTERING SETS

The case where there are two degrees of freedom and the chaotic invariant set is hyperbolic allows for a rather complete theoretical picture. We review results for this case in this section.

As discussed in Sec. I, we can regard the chaotic scattering set S as the intersection of its stable and unstable manifolds. This is illustrated schematically in a Poincaré surface of section in Fig. 5. Now let B be a rectangle containing S. We assume that under the action of the Poincaré map f, all points in B except for those on the invariant set and its stable manifold leave B and never return. Thus part of B must be mapped out of B, and the Lebesgue measure (area) remaining in B must decay. Say we randomly sprinkle a large number N_0 of initial points in B. After t iterates, N_t points remain. The quantity τ in Eq. (1) is

$$\frac{1}{\tau}=\lim_{t\to\infty}\lim_{N_0\to\infty}\frac{1}{t}\ln\left(\frac{N_0}{N_t}\right).$$

If t is large, the N_t point remaining at time t must have started near the stable manifold. After t iterates these points now line up along the unstable manifold. In fact, for any large time t, one finds that the remaining points essentially lie in thin strips covering the unstable manifold segments in B. Iterating these strips backward in time to $t=0$, the strips along the unstable manifold map to strips along the stable manifold. Since f is area preserving and the dimensions of the box B are order one, we see that the thickness $\ell_i^{(t)}$ of the ith strip along the stable manifold at time t is of the same order as the thickness of the unstable manifold strip that it originates from.

We can operationally define natural measures of a set A for the stable and unstable manifolds as

$$\mu_s(A)=\lim_{t\to\infty}\lim_{N_0\to\infty}N_s(A)/N_t,$$

$$\mu_u(A)=\lim_{t\to\infty}\lim_{N_0\to\infty}N_u(A)/N_t,$$

respectively. Here $N_s(A)$ denotes the number of the remaining N_t trajectories in B whose initial conditions lie in A; while $N_u(A)$ denotes the number of the remaining N_t trajectories in B lying in A at time t. Similarly, we define the natural measure of the chaotic set S itself as

$$\mu(A)=\lim_{t\to\infty}\lim_{N_0\to\infty}N_\xi(A)/N_t,$$

where $N_\xi(A)$ is the number of orbits which do not leave B by time t and which lie in A at time ξt, where ξ is a number satisfying $0<\xi<1$ [the definition gives the same result for $\mu(A)$ for *any* value of ξ greater than zero and less than one].

Lyapunov exponents $\lambda_1>\lambda_2$ associated with the natural measure on the chaotic invariant set can be defined by considering the Jacobian matrix of partial derivatives $Df^t(x)$ of the t times iterated map f^t for the N_t initial points x whose orbits have not left B at time t. In particular, the matrix

$$\lim_{t\to\infty}\lim_{N_0\to\infty}\langle[Df^t(x)]^*Df^t(x)\rangle_{N_t}^{1/2t},$$

where $*$ denotes the adjoint, and $\langle\ldots\rangle_{N_t}$ denotes an average over the N_t remaining initial conditions, has eigenvalues $\exp(\lambda_{1,2})$. Because f is area preserving $\lambda_1=-\lambda_2$.

The information dimension of the stable and unstable manifold measures are equal and are given in terms of the Lyapunov exponent and the characteristic decay time τ by[52,53]

$$d_{s,u}=2-\frac{1}{\tau\lambda_1}. \tag{5}$$

The information dimension d_1 of the chaotic invariant set is

$$d_1 = d_u + d_s - 2 = 2\left(1 - \frac{1}{\tau\lambda_1}\right), \tag{6}$$

where the first equality results from the fact that S is the intersection of its stable and unstable manifolds. The metric entropy of the natural measure μ is[52,53]

$$h_\mu = \lambda_1 - 1/\tau. \tag{7}$$

The paper in this issue by Gaspard[54] uses Eq. (5) in his study of the connection between the statistical treatment of diffusive transport (an irreversible process) and the dynamics of chaotic scattering (which is reversible).

Multifractal properties of the measure μ are readily available from numerical calculations of the set of strip widths $\{\ell_i^{(t)}\}$ as shown in Ref. 55, where the following partition function is introduced

$$\sum_i \ell_i^{(t)\beta} \sim \exp[-\beta F(\beta)t]. \tag{8}$$

Here $F(\beta)$ plays a role analogous to that of a free energy density and β is analogous to the inverse temperature. The function $F(\beta)$ yields a convenient characterization of ergodic properties of the system. In particular the decay time is given by

$$F(1) = \tau^{-1}; \tag{9}$$

the dimension spectrum d_q of the measure μ is given by the solution of

$$\hat{\beta}(q)F(\hat{\beta}(q)) = q/\tau, \tag{10}$$

where $\hat{\beta}(q) = q - \frac{1}{2}(q-1)d_q$ (the factor 1/2 follows from the fact that the dimension spectrum is the same $d_q/2$ along both the stable and the unstable manifold due to the Hamiltonian character); and the metric entropy is

$$h_\mu = F'(1). \tag{11}$$

In view of Eqs. (5) and (7) the Lyapunov exponent λ_1 is obtained as the derivative of $\beta F(\beta)$ taken at $\beta = 1$. Similar results can be obtained using a partition function based on averages of finite time Lyapunov numbers.[56]

Another very fruitful approach to the ergodic properties of the chaotic scattering set S is based on the dense set of periodic orbits embedded in S. Again a relevant partition function can be defined, this time as a sum over all periodic orbits of given period T. In principle, results follow from the large T limit. In practice, it is difficult to obtain many large period orbits, and techniques of obtaining greatly improved convergence have been formulated (see Ref. 57 for a survey). The periodic orbit formulation is particularly important in problems of quantum chaotic scattering (see Sec. IV). (In that case the results are asymptotic rather than convergent.)

IV. QUANTUM CHAOTIC SCATTERING

Since many scattering experiments are done on microscopic systems where quantum effects are essential, a question which naturally arises is whether one can find any sign of an underlying classical chaos in a quantum scattering process. Since the pioneering work of Blümel and Smilansky[58] it has been known that chaos clearly manifests itself for scattering in the quantum world. In particular, chaotic and nonchaotic quantum scattering processes exhibit different properties in the semiclassical regime.

The central object in quantum scattering theory is the S-matrix. At fixed particle energy, chaotic scattering is characterized by fluctuations in the dependence of the S-matrix on the initial and final states. These fluctuations are so strong that the random matrix theory proves an appropriate framework for their description. Consequently, the nearest neighbor distribution of the S-matrix's eigenphases is universally a Wigner distribution[58,59] if the underlying classical scattering is chaotic.

Another manifestation of scattering chaos is the presence of typical fluctuations in the energy dependence of a given transition probability or cross-section. Taking the semiclassical correlation function of a fixed transition probability at two different energy values, it should be a Lorentzian in the energy difference,[58,59] provided the classical dynamic is hyperbolic. Furthermore, the width of the Lorentzian turns out to be $\hbar$ divided by τ, the classical average chaotic lifetime defined by (1). Thus, an important classical characteristic of the motion appears in the quantum context. It is interesting to note that the same type of fluctuations were found in the S-matrix of compound nuclei reactions, and was called Ericson fluctuations, in the 1960s.[60] Similar fluctuations are expected in the transmission probabilities in a beam of CsI molecules passing through a region of inhomogeneous electrostatic field. This later result is shown in the contribution of Blümel to this issue[61] where he also shows that such an experiment is realistic.

The Lorentzian form of the correlation function strongly relies on the exponential decay form of the delay time statistics. In nonhyperbolic chaotic scattering processes the decay is algebraic. The same semiclassical argument then leads to the result that a power law behavior of the transmission probability correlation function is expected at low energy and the exponent in this power law is determined by the exponent α characterizing the stickiness of KAM surfaces as introduced in (2).[62]

Another feature characterizing quantum chaotic scattering is the distribution of the poles of the S-matrix in the complex energy plane. In hyperbolic cases, poles are excluded from a strip: they cannot come closer from below the real axis than a finite distance. The width of the strip is, in the semiclassical theory, $\hbar$ times a number which is on the same order of magnitude as the classical escape rate, $1/\tau$.[6]

The differential cross-section is a basic characteristic of quantum scattering, too. Its semiclassical form[63] exhibits singularities exactly in those directions where the classical counterpart does. More interestingly, away from classical singularities, due to interference oscillations, other singularities show up.[14] They are on a fractal set having the same dimensionality D as the classical singularities, e.g., in the deflection function. From the energy dependence of the differential cross-section in a given direction, the classical lifetime τ can be extracted again.

Although classically chaotic scattering processes have random matrix properties in the eigenphases of S, the reverse is not true. There exist simple classically nonchaotic examples whose S-matrix exhibits random properties.[64] This paradox can be resolved by using the concept of Poincaré scattering maps invented by Jung.[65] This type of map is defined by introducing a reinjection and considering the outgoing conditions of a scattering trajectory to be the incoming conditions for the next iterate of the map. In other words, successive applications of the Poincaré scattering map can be understood as subjecting the particle to another scattering on the same object as before, in which the previous final state serves as a new initial condition. Thus, the motion on the Poincaré scattering map can only be more complicated than in the original problem. Semiclassical arguments show[66] that it is the Poincaré scattering map that is the classical analog of the quantum S-matrix. If this map is chaotic, then S has random properties and vice versa. There are cases for which there is no chaos in the original classical problem, i.e., no chaotic set exists, but the Poincaré scattering map possesses such a set. Then random matrix properties characterize the quantum process although its classical counterpart is nonchaotic.[67]

An interesting example is provided by the scattering on a single hard wall billiard that is obviously classically nonchaotic. A simple geometrical observation shows, however, that there is a one-to-one correspondence between trajectories of the iterated Poincaré scattering map and trajectories of the dynamics inside the billiard, because of the simple rules of reflection. Thus, the scattering and the inner Poincaré maps are equivalent, and, in any case, when the billiard is chaotic, the scattering map will be chaotic, too. Even more is true: because of this inner–outer duality, the energy spectrum of the closed billiard problem and the eigenphase spectrum of the S-matrix are equivalent. This observation opens a new way for quantizing chaotic billiards via scattering methods[67,68] as discussed in detail by Dietz and Smilansky in this issue.[69]

A related problem is the quantum scattering on billiards with elastic walls produced by a very high and thin potential barrier along the billiard's perimeter.[70] Because of quantum mechanical tunneling, the incoming wave penetrates into the billiard and might be trapped there for a while leading to a resonancelike behavior which obviously has no classical counterpart. Fluctuations in the S-matrix and the cross-section are nevertheless present and might differ from those in the case of a finite classical lifetime as discussed in the paper by Csordás and Seba.[71]

A quite different way that chaos is relevant to waves is discussed by Pikovsky in this issue[72] who studies how nondispersive linear waves are scattered on discrete nonlinear elements.

The scattering on a surface of constant negative curvature has been another paradigm of both classical and quantum chaotic scattering.[73] The contribution of Gutzwiller to this issue invents a new method for the quantum description of the problem based on the construction of a quantum Poincaré map and knowledge of the classical motion.[74]

The semiclassical quantization of scattering problems in terms of classical periodic orbit properties has attracted much recent interest. The starting point is the famous Gutzwiller trace formula[75] relating the trace of the quantum mechanical Green function to the unstable classical periodic orbits on the nonattracting chaotic set. Earlier works have shown that by means of zeta function methods and cycle expansions[57,76] very accurate results can be obtained. The importance of these problems is reflected in the fact that a group of papers is devoted to them in this issue. Alonso and Gaspard improve the validity of the trace formula by determining a correction to it in powers of $\hbar$ leading to an order of magnitude improvement in the accuracy.[77] A new type of periodic orbit quantization procedure is suggested by Cvitanovic *et al.*[78] who show that their quantum Fredholm determinant has better convergence than other zeta functions. The contribution of Eckhardt[79] is devoted to the semiclassical study of the time delay correlation function based on classical periodic orbits. Theoretical conjectures of these papers are supported by numerical evidence in the three disk problem or other hyperbolic models.

One of the most appealing applications of quantum chaotic scattering is provided by the motion of electrons in semiconductor microstructures. Recent developments in lithography and material growth now allow for fabricating semiconductor devices of arbitrary geometrical shapes on micron scales. Since this size is considerably smaller than the elastic mean free path of electrons at low temperatures, the motion inside such microstructures is ballistic. Scattering is then determined by the shape of the device and elastic collisions on its boundary. Of particular interest are two-probe microstructures of circular or stadium shape and cross-shaped junctions with four probes. The classical motion in the first and second cases corresponds to that in leaky billiards and to that between four disks, respectively. Because, however, the Fermi wavelength is not completely negligible on micron scales, a semiclassical description is necessary. Chaotic properties and fluctuations in the conductance of such devices have been observed both in numerical models[80,81] and in experiments[82] and have been interpreted as Ericson fluctuations. For example, as a consequence of the general theory, the conductance correlation function as a function of wave-number difference should be a Lorentzian with a width inversely proportional to the classical chaotic lifetime. This form only holds for hyperbolic cases, and for billiards having periodic orbits of marginal stability, which induce nonhyperbolic effects, the algebraic decay of the time delay statistics generates another type of correlations. Again a group of papers discusses this important topic in our focus issue. Baranger, Jalabert and Stone[83] give an overview of the field and point out a surprising failure of the semiclassical methods in nonchaotic cases. Lin, Delos and Jensen[84] deal with properties of the time delay statistics in hyperbolic and nonhyperbolic cases and with their consequences on semiclassical quantities. They also show that experimental observations reviewed here by Marcus *et al.*[85] can only be understood if scattering noise is also taken into account.

In a more general setting, quantum chaotic scattering can be observed not only in cases when the classical counterpart (or the Poincaré scattering map) is chaotic but also in any wave (e.g., light) scattering problem when the ray dynamics (geometrical optics) is chaotic. A convincing experimental observation is a microwave experiment carried out by Doron, Smilansky and Frenkel.[86] Reflection of microwaves has been studied in an elbow-shaped cavity corresponding to a symmetry reduced four-disk problem. As the wavelength was comparable with the cavity size, the authors of Ref. 86 observed transmission fluctuations of the Ericson type in complete agreement with the semiclassical theory of chaotic scattering. Another application of a similar nature is the contribution to this issue by Mortessagne, Legrand and Sornette[87] who give a nice review of chaotic scattering in room acoustics. They also point out that the acoustic wave dynamics can be understood as a semiclassical theory based on the underlying classical nonattracting chaotic set.

V. CONCLUSION

As mentioned in Sec. II, the topics chosen for discussion in this review are a representative selection rather than a full list of all significant aspects. Nevertheless, we would like to mention that nonlinear wave scattering in media is a subject of great relevance. In particular, the chaotic scattering of kink-antikink systems and solitons studied by Campbell and co-workers[88] and Gorshkov and co-workers,[89] respectively, deserves further attention. In a more general setting, phenomena that may, at first glance, have nothing to do with a scattering process, but lead to a recursion of Hamiltonian character with an asymptotically simple behavior, can be investigated by the methods sketched above. Notable examples are models of an optical memory[90] and of the quantum Hall effect.[91]

We believe that chaotic scattering represents an unusual and appealing slice through nonlinear dynamics which might, however, be a still undiscovered field to many chaos researchers.

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[2]One interesting question evident from Fig. 2(a) at large energy and Fig. 2(b) at lower energy is that of how the change from regular scattering to chaotic scattering is accomplished as the energy is lowered. This question is considered in Ref. 1 and more generally in the paper of Tél *et al.* in this issue [T. Tél, C. Grebogi, and E. Ott, Chaos **3**, 495 (1993)].

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