THE THERMODYNAMICS OF FRACTALS

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Introduction.

Statistical mechanics, as developed by J.W.Gibbs in the last century, is a theory of great beauty and coherence. By basing it on the concept of *ensembles* Gibbs freed the theory from the ambiguities related to the introduction of probabilities in deterministic systems, and, at the same time, he created a framework which lends itself easily to generalizations - most importantly the subsequent development of quantum statistical mechanics. The advantage of statistical mechanics over many other fields of science is that the number of degrees of freedom is usually enormously large. This means that the "law of large numbers" singles out certain members of the ensemble as "typical" for given external conditions, and their properties are then described by *thermodynamical functions* which contain most of the relevant information about macroscopic systems.

Recently there has been a strong interest in the socalled "thermodynamical description of fractals". A *fractal* is an object having structure on all length scales described by power laws with noninteger exponents.[1] In the analysis of fractals one introduces

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partitions which approximate the structure with a certain accuracy. Regarding the elements of the partitions as an "ensemble" a "statistical mechanics" can now be defined and properties like dimensions and entropies, which describe the fractal, emerge in the limit of infinitely fine partitions in the same way as the thermodynamic limit emerges from usual statistical mechanics.

This formalism goes back to the pioneering work of Sinai[2], Ruelle[3] and Bowen[4] in the seventies, but - until recently - it hasn't attracted much attention among physicists because of the lack of concrete applications and the rather technical presentations available. Within the last few years the situation has changed. The introduction of the concept of "multifractals"[5-9] has given new impulses to the thermodynamical description of fractals both experimentally and theoretically in the study of dynamical systems[9-28] and in other areas where fractals occur, notably the study of aggregates and percolation[29].

The following article is intended as an introduction to certain aspects of this ramified subject. We shall restrict our attention to fractal sets (Cantor sets) generated by chaotic motion of a dynamical system. Due to the existence of natural, hierarchical constructions for such sets the thermodynamical formalism is comparatively well understood and in order to keep the context as simple as possible only fractals embedded in one dimension will be studied, i.e. fractals generated by a one-dimensional map. These sets are either *strange attractors* [30] which characterize the asymptotic motion of generic points, or they can be characterized as *strange repellers* [31-32] which characterize the transient behaviour. In fact, the simplest systems on which the statistical theories can be applied are chaotic repellers of one dimensional maps, and in this article we shall concentrate on such repellers and on the attractors obtained in certain limits. The layout is as follows: Chapter 1 contains general definitions of the type of Cantor sets, which we shall look at, and the different forms of thermodynamics to be considered. Chapter 2 describes the thermodynamics on hyperbolic sets. Here a considerable body of theory is available, and in particular we shall show that the different thermodynamical formalisms are all connected in that case. In chapter 3 we look at a specific example of a non-hyperbolic system at a limiting, completely chaotic point, which is exactly soluble. It introduces the concept of phase transitions and gives hints of more general connections between the thermodynamical functions. Finally in chapter 4 we conclude by discussing results for more general non-hyperbolic systems, concentrating on the general case of "completely chaotic maps".

1. THE THERMODYNAMICAL FORMALISM.

The kind of fractal structures that we shall primarily discuss are generated by a one-dimensional map $x' = f(x)$. Imagine that f maps the unit interval, I, unimodally[33] on an interval which is *larger* as shown in fig.1.1. Thus $f(0) = f(1) = 0$ and f has a single maximum where $f(x) > 1$. The points that are mapped outside the unit interval "escape" and never return again. If there are no attractive periodic orbits almost all initial conditions will eventually lead to escape. There is, however, a Cantor set, the *repeller* $\Lambda \subset I$ which will remain forever and we can construct this Cantor set by looking at the set of points, Λ_n, which remain for n iterates, $n = 1,2,\ldots$ and let $n \to \infty$. The easiest way to obtain the elements of Λ_n is by iterating the map *backwards:* if we denote the two branches of the inverse map f^{-1} by h_0 and h_1 respectively, Λ_n consists of the 2^n intervals obtained by mapping I backwards through any sequence of h_i's, of length n, where each i is either 0 or 1. These small intervals will be called *cylinders,*[4,17] they are disjoint and within each cylinder $f^n(x)$, the n-fold iterate of f, is monotonic.

The construction of this Cantor set is completely analogous to Cantors original example where the middle 1/3 is removed at each stage. Going from Λ_n to Λ_{n+1} a part of each cylinder is indeed removed, but the fraction is different for the different cylinders, depending on the derivative $|f'|$. Such Cantor sets are often referred to as "Cookie cutters". A particularly interesting case is obtained in the limit where the function maps the interval, I, exactly onto itself, since I may then be a chaotic attractor. The map is then said to be "Smale complete"[33] or to be in a fully developed chaotic state and in this case the cylinders fill out I completely. The formalism which we shall develop in this chapter can be applied to such limiting cases without problems, although, as we shall discuss in chapter 3 and 4, the theorems of chapter 2, based on hyperbolicity, break down.

1.1. The Spectrum of Characteristic Exponents.

The points on the Cantor set Λ will remain forever in I and do therefore have well-defined characteristic exponents. If the motion is ergodic almost all points have the same characteristic exponent, but in fact Λ can be decomposed into a set of interwoven Cantor sets each consisting of points with some specific characteristic exponent. Given the cylinders of, say, Λ_n it is very easy to find the different characteristic exponents: each little cylinder expands to the unit interval, I, in precisely n iterates so the length, Δ of the cylinder is related to its characteristic exponent, λ, by

$$\Delta = e^{-n\lambda}. \tag{1.1.1}$$

The fundamental quantity describing this set is the *entropy* $S(\lambda)$, which expresses how many cylinders have given λ (or equivalently, given length). More precisely the meaningfull quantities are the *growth rates* of those numbers so that $e^{nS(\lambda)}\,d\lambda$ is the number of cylinders with characteristic exponent in an interval of size $d\lambda$ around λ. In stead of focussing on a specific value of λ it is usually simpler to use a "Canonical formalism" and introduce a *partition function* $Z_n(\beta)$ as a sum over all cylinders, $I_j^{(n)}$, on a given level[12,17]:

$$Z_n(\beta) = \sum_j \Delta(I_j^{(n)})^\beta. \tag{1.1.2}$$

The parameter β is the analog of inverse temperature in thermodynamics, but one should note that we are interested in the whole interval $-\infty < \beta < \infty$ so "temperatures" can be both positive and negative. The growth rate of the partition function defines the *pressure* $P(\beta)$, i.e.

$$P(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta). \tag{1.1.3}$$

In fact, thermodynamically speaking, $P(\beta)$ is more naturally analog to the *free energy density*, $F(T)$ (times $-\beta$), but since the originators use the word pressure for (1.1.3) we shall stick to it. One value, namely $P(1)$ has a simple physical meaning: it is simply $-\kappa$, where κ is the *escape rate* of points in I[31,32]. The escape rate can be determined by distributing N_0 points uniformly on I and counting the number N_n of survivors after n step. The ratio N_n/N_0 decays exponentially as $N_n/N_0 \sim e^{-n\kappa}$. According to the construction of the cylinders, this ratio is the total length of Λ_n so

$$\sum_i \Delta(I_j^{(n)}) \sim e^{-n\kappa} \tag{1.1.4}$$

from which the relation to $P(1)$ immediately follows. For an attractor, of course, $\kappa = 0$. If we can compute the pressure for all β, the entropy follows by a Legendre transform: The sum over cylinders in (1.1.2) can be replaced by an integral over characteristic exponents by using (1.1.1) and the fact that the number of cylinders of given λ is $\exp(nS(\lambda))$. Thus

$$Z_n(\beta) \approx \int e^{nS(\lambda)-n\beta\lambda}\,d\lambda \tag{1.1.5}$$

and as $n \to \infty$ the saddle-point dominates the integral, which defines the *thermodynamic* value $\lambda = \lambda(\beta)$ through

$$S'(\lambda(\beta)) = \beta \tag{1.1.6}$$

and since the growth rate of (1.1.5) is $P(\beta)$ we get the ralation

$$S(\lambda(\beta)) = P(\beta) + \lambda\beta. \tag{1.1.7}$$

In the language of thermodynamics $\lambda(\beta)$ is analog to the internal energy and (1.1.7) becomes the familiar relation $S = (E-F)/T$. The function $S(\lambda)$ usually looks as shown in fig.1.2. It is non-zero on an interval $[\lambda_1, \lambda_2]$ and has a single maximum. The value of S at the maximum is $\log 2$ since there are 2^n cylinders in Λ_n (note, from (1.1.6) that $S(\lambda)$ is maximal for $\beta = 0$). The *Hausdorff dimension* of the "sub Cantor set" of points with characteristic exponent λ will be called $D(\lambda)$. Obviously this set is covered by $N = e^{nS(\lambda)}$ intervals of length $\Delta = e^{-n\lambda}$ so we find[17].

$$D(\lambda) = \lim_{n\to\infty} -\frac{\log N}{\log \Delta} = \frac{S(\lambda)}{\lambda} \tag{1.1.8}$$

The maximal dimension of such subsets is the Hausdorff dimension D_0 of the whole set. Thus $D_0 = D(\lambda_0)$, where λ_0 maximizes $D(\lambda)$. According to (1.1.8) this means that $S(\lambda_0) = \lambda_0 S'(\lambda_0)$. In terms of the pressure function this translates into the well-known relation[2] $P(D_0) = 0$.

Another way of defining a spectrum of characteristic exponents for a repellor or an attractor is via the (unstable) *periodic points* [34-36] i.e. points satisfying

$$f^n(x^*) = x^* \tag{1.1.9}$$

Note that any fixed point or cycle of a length that devides into n satisfies (1.1.9). The stability of such a cycle point is determined by the derivative $|f^{n}{}'(x^*)|$ and a natural thermodynamics can be formed by

$$Z_{fix,n}(\beta) = \sum_{x^* \in fix f^n} e^{-\beta \log|f^{n}{}'(x^*)|} \tag{1.1.10}$$

whose growth rate defines a "Free energy"

$$\beta F_{fix}(\beta) = -\lim_{n\to\infty} \frac{1}{n} \log Z_{fix,n}(\beta) \tag{1.1.11}$$

in analogy with the pressure $P(\beta)$, except for the sign and a factor of β conventional in thermodynamics. In general F_{fix} and P are not necessarily related, but for *hyperbolic systems* they are, as we shall see in chapter 2. This formalism is often easy to implement and extends easily to higher dimensional systems. It has gained additional actuality after the recent appearance of algorithms that locate periodic trajectories in experimental time series[37].

1.2. Cantor Sets with Measure: Generalized Entropies and the $f(\alpha)$ Spectrum.

The notion of a *measure* doesn't seem directly to apply to the Cantor set generated above. One can think of all the different cylinders of Λ as "equally probable". This is not the case if one wants to describe e.g. the motion on a set generated by (forward) iteration of a map $x' = f(x)$. If f is a map *on* the interval (i.e. such that all points remain in it) each segment of the interval can be assigned a measure telling how often it is visited when the map is iterated. This is called[33] the *natural* measure, μ. As we shall see a natural measure also exists for chaotic repellers.

The natural measure defining the probabilities of each element of the partition (in our case the cylinders) is seldomly known in an analytic way i.e. there is no simple connection between the lengths, Δ_i, of the cylinders and the probabilities $p_j = \mu(I_j^{(n)})$. For hyperbolic systems such a relation does exist as will be explained in the next chapter and in that case, therefore, the different "thermodynamic" quantities introduced in this chapter are all related.

Beside the natural measure one can define many other invariant measures. The generalized entropies, with respect to a measure ν, are defined as follows. For the set of n-cylinders with probabilities p_j given by some invariant measure ν the partition function

$$Z_{\nu,n}(q) = \sum_j p_j{}^q \tag{1.2.1}$$

defines the Renyi entropies[38]. The generalized entropies[30,39] are the growth rates of these Renyi entropies

$$h_q(\nu) = \lim_{n\to\infty} \frac{1}{1-q} \frac{1}{n} \log Z_{\nu,n}(q) \tag{1.2.2}$$

Here we have used the fact that the partition function defined by the cylinders becomes infinitely fine everywhere, it is a "generator"[30]. In terms of the "Free energy"

$$qF_\nu(q) = -\lim_{n\to\infty} \frac{1}{n} \log Z_{\nu,n}(q) \tag{1.2.3}$$

we get

$$h_q(\nu) = \frac{qF_\nu(q)}{q-1} \tag{1.2.4}$$

The denominator $q-1$ secures the correct limit as $q \to 1$:

$$h_1(\nu) = \lim_{q\to 1} h_q(\nu) = \lim_{n\to\infty} -\frac{1}{n} \sum_j p_j \log p_j \tag{1.2.5}$$

The case where the measure ν is the natural measure μ is of particular importance and to avoid confusion we use K for the corresponding generalized entropies i.e. $K_q = h_q(\mu)$. The entropy K_1 is called the *metric entropy* or "Kolmogorov - Sinai invariant" and describes the rate at which the chaotic dynamics generates "information".

The generalized entropies, K_q contain information about the dynamics of the system, e.g. how far into the future we can predict it with a given initial knowledge. The so-called $f(\alpha)$ *spectrum* [8] contains, in contrast, metric-geometric information. It generalizes the Hausdorff dimension and is related to the generalized dimensions[6], D_q. In every point, x, (or family of cylinders containing x) we define the *pointwise dimension* or *crowding index* by looking at the scaling of the probability with the length of the intervals. If the pointwise dimension is α it means that

$$p_j = \nu(I_j^{(n)}) \sim (\Delta(I_j^{(n)})^\alpha \tag{1.2.6}$$

where ν is some measure, usually the natural one. The function $f(\alpha)$ then describes the "density" of cylinders with that α. More precisely, $f(\alpha)$ is the Hausdorff dimension of the set of points with pointwise dimension α analogos to the function $D(\lambda)$ defined above. As shown in ref.8 this function can be obtained from a generating function, Γ, whose terms are like the *ratio* of two Boltzmann factors:

$$\Gamma_n(q,\tau) = \sum_i \frac{p_i^q}{\Delta_i^\tau} \tag{1.2.7}$$

For each q, $\tau(q)$ is found by requiring that $\Gamma = \lim_{n\to\infty} \Gamma_n$ is finite. For large n (1.2.7) can be written as an integral and

$$\Gamma(q,\tau) \approx \lim_{\Delta\to 0} \int d\alpha \Delta^{\alpha q-\tau} \Delta^{-f(\alpha)} \tag{1.2.8}$$

The condition for this to remain finite can be found by saddle point arguments analogous to (1.1.5), and one obtains[8]

$$f(\alpha) = \alpha q - \tau(q) \tag{1.2.9}$$

where

$$\alpha(q) = \tau'(q) \tag{1.2.10}$$

The generalized dimensions, D_q are then simply

$$D_q = \frac{-\tau(q)}{1-q}. \tag{1.2.11}$$

In special cases where something is known about the relation between the measure (p_i) and length scales (Δ_i) there is a direct thermodynamical formalism for $f(\alpha)$ analogous to section 1.1. In particular[12,15], if the 2^n intervals Δ_i of the repellor of section 1.1 are given equal probability (2^{-n}) it is easy to see that the generating function becomes

$$\Gamma = 2^{-nq} \sum \Delta_j^{-\tau} \tag{1.2.12}$$

For this to remain finite as $n \to \infty$ demands that $P(-\tau) = q \log 2$, which implicitely defines $\tau(q)$. The relation (1.1.1) together with $p_j = 2^{-n}$ gives us by definition

$$\alpha = \frac{\log 2}{\lambda} \tag{1.2.13}$$

and further, for this λ

$$f(\alpha) = S(\lambda)/\lambda = D(\lambda) \tag{1.2.14}$$

A relation between $f(\alpha)$ and $D(\lambda)$ might have been expected, since they are both Hausdorff dimensions of a subset of the Cantor set.

This formalism has been succesfully applied to fractals appearing at the borderline of chaos for which a Renormalization Group exists[11,13,19]. Those fractals can be looked upon as cookie cutters generated by a map (f in section 1.1) which is related to the "universal function" coming from the Renormalization Group[17]. The map (e.g. unimodal map or circle map), which is becoming chaotic, maps cylinders onto each other precisely at this point and therefore the "balanced" measure $p_j = 2^{-n}$ is the dynamically relevant one.

1.3. Measure and Density: The Generalized Frobenius-Perron Equation.

One might ask whether there is a "natural" way to assign a measure to a cookie cutter. If we regard the function f generating the Cantor set as a formal device there is of course no a priori reason to choose one measure in stead of another, but if we want the measure to represent probabilities with respect to the dynamics generated by f, this is not the case. It is true that (almost) all points will eventually escape so the measure must be concentrated on the invariant Cantor set, Λ. But if we start out close enough to Λ we can stay for as many iterates as desired and this maps out a measure[32].

To find analytic methods of describing this measure let us begin by reminding ourselves how invariant measures are constructed for an *attractor*.

The natural measure μ of a one dimensional strange attractor can be described by its *density*, $\rho(x)$, such that the measure of the set A is

$$\mu(A) = \int_A \rho(x)dx \tag{1.3.1}$$

The condition for this measure to be *invariant* is the Frobenius-Perron equation[4,33]

$$\rho(y) = \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{|f'(x)|} \tag{1.3.2}$$

as we shall se in detail in section (2.2). One can also regard the Frobenius-Perron equation as an iterative equation

$$\rho_{k+1}(y) = \sum_{x \in f^{-1}(y)} \frac{\rho_k(x)}{|f'(x)|} \tag{1.3.3}$$

and for an attractor the natural measure can be found by iterating the uniform density $\rho_0(x) = const.$ For a repellor the result of iterating (1.3.3) would be $\rho(x) = 0$: all measure eventually escapes so the probability of being in the interval, I, becomes zero. One can, however, compensate for this by inserting a factor, R, into (1.3.3):

$$\rho_{k+1}(y) = R \sum_{x \in f^{-1}(y)} \frac{\rho_k(x)}{|f'(x)|}$$

For one and only one value of R, iteration of this equation, starting from some smooth ρ_0, converges to a finite, positive $\rho(x)$, and this value turns out to be related[40] to the escape rate: $R = e^{\kappa}$.

Independently of the the problem of how to find the natural measure for repellers also other ways have been found to generalize the Frobenius-Perron equation (1.3.3). It is particularly appealing, from a thermodynamical view point, to introduce a parameter β and for each value of β try to solve

$$Q_{k+1,\beta}(y) = e^{\beta F(\beta)} \sum_{x \in f^{-1}(y)} \frac{Q_{k,\beta}(x)}{|f'(x)|^{\beta}} \tag{1.3.4}$$

again starting from a smooth, positive initial function. This process turns out to define a unique $F(\beta)$ such that the limiting $Q_\beta(x) = \lim_{n\to\infty} Q_{n,\beta}(x)$ becomes finite and positive and independent of the initial condition for both attractors and repellers[18]. An important special case is obtained when $\beta = D_0$, the Hausdorff dimension of the set. It is shown that $F(D_0) = 0$, which provides a useful method for calculating the dimension[41]. In general $e^{-\beta F(\beta)}$ is the largest *eigenvalue* of (1.3.4) and the generalized density Q_β is the corresponding eigenfunction.

In order to see the connection between this formalism and statistical mechanics, let us consider the n ($\gg 1$) fold iterated density $Q_{n,\beta}$. Since the limiting value is unique we can start with the uniform density and we find

$$Q_{n,\beta}(y) = e^{n\beta F(\beta)} \sum_{x \in f^{-n}(y)} e^{-\beta \log|f^{n\prime}(x)|} \tag{1.3.5}$$

The sum plays the role of a partition function over the ensemble of all *preimages* of y under f^n. Since $Q_{n,\beta}$ remains finite, the growth rate of the partition function is exactly $-\beta F(\beta)$. Indeed the function $F(\beta)$ turns out to be closely connected to the pressure defined in section 1.1. In fact we shall show in the next chapter that

$$\beta F(\beta) = -P(\beta) \tag{1.3.6}$$

for hyperbolic sytems and in the last chapter that this holds at least in some non-hyperbolic system -in particular the socalled "completely chaotic cases" studied in the last two chapters. Note that the ensemble (1.3.5) depends on y, whereas the growth rate does not. Further, it is interesting to note that in this formalism a new quantity, the generalized measure $Q_\beta(x)$, appears which characterizes the chaotic system. Its interpretation in terms of *Gibbs measures* will be given in the next chapter.

2. THERMODYNAMICS OF HYPERBOLIC SYSTEMS.

A one-dimensional map, $f(x)$, is called *hyperbolic* if the slope is everywhere finite and larger than one. More generally it is sufficient to require that $1<|f^{k\prime}(x)|<\infty$ for sufficiently large k on a partition of the attractor, or, for a repellor, over the cylinders. The "Cookie cutters" of section 1.1 satisfy this condition with certain assumption about the map, f [33], and in this chapter we shall always assume those conditions to be fulfilled and explore the strong consequences of hyperbolicity basically following refs.4 and 17. In the next chapters simple non-hyperbolic cases will be discussed.

2.1. Gibbs Measures.

For hyperbolic systems there exists a particularly attractive class of invarant measures, the so-called Gibbs measures. The *invariance* of a measure ν means that

$$\int g(x)d\nu = \int g(f(x))d\nu \tag{2.1.1}$$

for any real function $g(x)$ on I. One should note that $\int g(x)d\nu$ is a short hand notation for the limit as $n\to\infty$ of $\sum_j g(x_j)\nu(I_j^{(n)})$, where x_j is any point inside the cylinder $I_j^{(n)}$.

A Gibbs measure can be defined uniquely for any (nice) real function ϕ on I. In fact we shall be interested in particular choices of ϕ, but the derivations become clearer if we don't specify ϕ until necessary. If we introduce the notation

$$S_n\phi(x) = \phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x)) \tag{2.1.2}$$

then, for any n, any cylinder, $I_j^{(n)}$, and any $x\in I_j^{(n)}$ the Gibbs measure satisfies

$$\nu_\phi(I_j^{(n)}) = c_j^{(n)}e^{-nP_\phi+S_n\phi(x)} \tag{2.1.3}$$

the *Gibbs condition*. Here the prefactor is bounded between positive, n-independent numbers, say $c_j^{(n)}\in[c_1,c_2]$ and P_ϕ normalizes the measure:

$$P_\phi = \lim_{n\to\infty}\frac{1}{n}\log\sum_j e^{S_n\phi(x_j)} \tag{2.1.4}$$

where $x_j\in I_j^{(n)}$ is chosen such that $S_n\phi(x)$ is maximal.

The "principle of bounded variation" plays an important rôle: In a hyperbolic system all cylinders become exponentially small and hence the variation of ϕ (which, for our purposes can be assumed smooth) is also exponentially small. Thus, if we assume the existence of a positive number b less than 1 and a positive c such that $|\phi(x)-\phi(y)|<cb^n$ for all n and x,y in any $I_j^{(n)}$, then, since $\phi(f^{n-m}(x))$ and $\phi(f^{n-m}(y))$, $m=1,\dots,n$ belong to the same m-cylinder we get

$$|S_n\phi(x)-S_n\phi(y)|\le|\phi(x)-\phi(y)|+\dots+|\phi(f^{n-1}(x))-\phi(f^{n-1}(y))| \tag{2.1.5}$$

which is bounded *independently* of n, namely by $d=c/(1-b)$. This means that growth rates like (1.1.3) are independent of the precise choice of x_j within each cylinder - any point can be chosen with the same result. The corrections are of order $\frac{1}{n}$ and vanish in the thermodynamical limit.

We shall now prove an important *variational principle* for the Gibbs measures[2,4,42]. Let P_ϕ be defined by (2.1.4) and $h(\nu)=h_1(\nu)$ be the metric entropy (1.2.5) for an invariant measure ν. Then, in general

$$h(\nu)+\int\phi\, d\nu\le P_\phi \tag{2.1.6}$$

and the equality is attained precisely for $\nu=\nu_\phi$, the Gibbs measure. We first show that (2.1.6) is true for any invariant measure ν: From the invariance property (2.1.1) follows, in particular, that

$$\int S_n\phi(x)d\nu=n\int\phi(x)d\nu \tag{2.1.7}$$

Combining this with (1.2.5) we can write the left hand side as

$$h(\nu)+\int\phi\, d\nu=\lim_{n\to\infty}\frac{1}{n}\sum_j\nu(I_j^{(n)})\left[-\log(\nu(I_j^{(n)})+S_n\phi(x_j)\right] \tag{2.1.8}$$

where $x_j\in I_j^{(n)}$. Now denote $\nu(I_j^{(n)})$ as p_j. Then we can certainly bound (2.1.7) by maximizing over all, positive p_j (not necessarily coming from an invariant measure) with the condition $\sum_j p_j=1$. The result is

$$p_j=e^{S_n\phi(x_j)-nP_\phi} \tag{2.1.9}$$

Note the similarity between this procedure and the derivation of the canonical distribution in standard statistical mechanics. When this expression is inserted into (2.1.8) we obtain precisely P_ϕ.

We now show that the Gibbs measure satisfies

$$h(\nu_\phi)+\int\phi\, d\nu_\phi=P_\phi \tag{2.1.10}$$

We look at

$$\nu(I_j^{(n)})(-\log\nu(I_j^{(n)})+S_n\phi(x_j))$$

for one cylinder $I_j^{(n)}$ containing x_j and $\nu=\nu_\phi$. Now using the bounded variation (2.1.5) with constant d, and (2.1.4) we can bound this below by $\nu(I_j^{(n)})(nP_\phi-\log c_1-d)$. We now sum over cylinders

$$h(\nu)+\int\phi\, d\nu=\lim_{n\to\infty}\frac{1}{n}\sum_j\nu(I_j^{(n)})(-\log(\nu(I_j^{(n)})+S_n\phi(x_j))$$

$$\ge\lim_{n\to\infty}\frac{1}{n}\sum_j\nu(I_j^{(n)})(nP_\phi-\log c_1-d)=P_\phi$$

Combining this with (2.1.6) gives (2.1.10). For any given ϕ we can create a family of Gibbs measures by introducing a multiplicative parameter β and looking at the Gibbs measures $\nu_{\beta\phi}$ for the function $\beta\phi(x)$. We then obtain a function $P_\phi(\beta)=P_{\beta\phi}$. By inserting into the variational principle (2.1.10) we see that the derivative of $P(\beta)$ is

$$P_\phi'(\beta)=\int\phi\, d\nu_{\beta\phi} \tag{2.1.11}$$

which we shall use later. Now we have a complete analogy with statistical physics for arbitrary ϕ. The quantities $-nP_\phi(\beta)$ and $-nP'_\phi(\beta)$ play the role of a free energy and of internal energy, respectively. The Boltzmann factor (2.1.3) shows that the energy of a microstate is $S_n\phi(x_j)$, where x_j is a point in the cylinder $I_j^{(n)}$. Equation (2.1.11) tells us that the internal energy is in fact the average over the ensemble. The total entropy is given by $nh(\nu_{\beta\phi})$ and (2.1.6) then shows that other invariant measures do not correspond to equilibrium states.

2.2. Thermodynamics on the Sinai-Bowen-Ruelle (SRB) Measure.

The physically most interesting Gibbs measure is found when

$$\phi=-\log|f'| \tag{2.2.1}$$

For hyperbolic attractors and repellers the Gibbs measure $\mu = \nu_\phi$ for this particular ϕ is the *natural measure* and is named the Sinai-Bowen-Ruelle (SRB) -measure after its discoverers. It is *ergodic* i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g(f^i(x)) = \int g(x)d\mu \qquad (2.2.2)$$

for any continuos function g on I and almost all points x on the invariant set. Further, any *smooth* invariant measure must be the SRB measure. From now on we shall always assume that ϕ is chosen according to (2.2.1). Then (2.1.2) aquires a special meaning:

$$S_n\phi(x) = -\log|f'(x)| - \log|f'(f(x))| - \cdots - \log|f^{n-1\prime}(x))| \qquad (2.2.3)$$

$$= -log\ |f'(x)\cdot\ldots\cdot f'(f^{n-1}(x))| = -\log|f^{n\prime}(x)|$$

which means that $-\frac{1}{n}S_n\phi$ converges to the characteristic exponent, λ. Thus the same reasoning that led to (1.1.1) now gives

$$\Delta(I_j^{(n)}) = e^{-n\lambda} = e^{S_n\phi(x_j)} \qquad (2.2.4)$$

where x_j lies in cylinder $I_j^{(n)}$ with length $\Delta(I_j^{(n)})$. The Gibbs condition (2.1.3) can now be written

$$\nu(I_j^{(n)}) = c_j^{(n)} e^{-nP_\phi}\Delta(I_j^{(n)}) \qquad (2.2.5)$$

As in last section we now look at the Gibbs measures $\nu_{\beta\phi}$ built on (2.2.1) denoted shortly as ν_β. Here

$$\nu_\beta(I_j^{(n)}) = c_j^{(n)}(\beta) e^{-nP_\phi(\beta)}(\Delta(I_j^{(n)}))^\beta \qquad (2.2.6)$$

where the c's are bounded in some interval $[c_1,c_2]$.

If we now sum (2.2.6) over all cyliders, the normalization condition is

$$1 = \sum_j \nu_\beta(I_j^{(n)}) = e^{-nP_\phi(\beta)}\sum_j c_j^{(n)}(\beta)(\Delta(I_j^{(n)}))^\beta$$

The sum is bounded between $c_1\sum(\Delta(I_j^{(n)}))^\beta$ and $c_2\sum(\Delta(I_j^{(n)}))^\beta$ both of which have *growth rates* $P(\beta)$, defined by (1.1.2-3) and therefore the sum itself must also grow as

$\exp(nP(\beta))$. This shows that $P_\phi(\beta)$ with ϕ given by (2.2.1) is equal to the *pressure* $P(\beta)$ defined in section 1.1, i.e.[17]

$$P_\phi(\beta) = P(\beta) \qquad (2.2.7)$$

The Legendre transform of $P(\beta)$ is $S(\lambda)$ - the growth rate of cylinders with characteristic exponent λ. Thus

$$S(\lambda) = P(\beta) + \lambda\beta \qquad (2.2.8)$$

where

$$\lambda(\beta) = -P'(\beta) \qquad (2.2.9)$$

Now, using (2.1.11)

$$\lambda(\beta) = -\int \phi d\nu_\beta \qquad (2.2.10)$$

and from the variational principle (2.1.10) we get

$$h(\nu_\beta) = P(\beta) - \beta\int\phi d\nu_\beta = P(\beta) - \beta P'(\beta) = S(\lambda) \qquad (2.2.11)$$

Thus[17] $S(\lambda)$ is the *metric entropy of the Gibbs measure* ν_β.

The entropy spectrum based on ν_β is given by (1.2.1-2). Using the Gibbs condition (2.2.6) and the same reasoning that led to (2.2.7) we get

$$Z_{\nu_\beta,n}(q) = e^{-nqP(\beta)}\sum_j(\Delta(I_j^{(n)}))^{\beta q} = e^{-nqP(\beta)}e^{nP(\beta q)} \qquad (2.2.12)$$

which shows that

$$h_q(\nu_\beta) = \frac{P(\beta q) - qP(\beta)}{1-q} \qquad (2.2.13)$$

Note that in (2.2.12) we have equated expression with the same growth rate being always interested in the limit $n\to\infty$. In the limit $q\to 1$ we recover (2.2.11) and the generalized entropies for the SRB measure appear for $\beta = 1$.

The $f(\alpha)$ spectrum can also be found. To stress the fact that it is built on the measure ν_β we shall use tha notation $f_\beta(\alpha)$. As before we set $p_j = \nu_\beta(I_j^{(n)})$ and substitute into (1.2.7) using the Gibbs condition (2.2.6). Again we neglect the constants $c_j^{(n)}$ since we only want growth rates and we find

$$\Gamma_{n,\beta}(q,\tau) = e^{-nqP(\beta)}\sum_j \frac{\Delta(I_j^{(n)})^{q\beta}}{\Delta(I_j^{(n)})^{\tau}} = e^{-nqP(\beta)}e^{nP(q\beta-\tau)} \tag{2.2.14}$$

The condition that Γ_β remains finite selects a $\tau_\beta(q)$ which, via (1.2.17) determines the spectrum of generalized dimensions. This $\tau_\beta(q)$ is determined implicitely by

$$qP(\beta) = P(q\beta-\tau) \tag{2.2.15}$$

A more explicit relation follows by defining the pointwise dimension, α, through $p_j \sim \Delta(I_j^{(n)})^{\alpha}$. From (2.2.4-6) we then immediately obtain the relation

$$\alpha = \beta + \frac{P(\beta)}{\lambda} \tag{2.2.16}$$

between α and λ and finally

$$f_\beta(\alpha) = \frac{S(\lambda)}{\lambda} = D(\lambda) \tag{2.2.17}$$

Therefore, as (1.2.11) based on equal measure of all cylinders, $f(\alpha)$ is the same as $D(\lambda)$, but the relation between α and λ is different and depends on β.

In particular we find for the SRB measure

$$\alpha = 1 + \frac{P(1)}{\lambda} = 1 - \frac{\kappa}{\lambda} \tag{2.2.18}$$

where κ is the *escape rate*. The most probable value of λ is the Liapunov exponent $\bar{\lambda}$, and the corresponding α is the information dimension, D_I. Hence

$$\kappa = \bar{\lambda}(1-D_I) \tag{2.2.19}$$

which relates escape rate, Liapunov exponent and information dimension for a hyperbolic system[31,32]. For an attractor $\kappa = 0$ so $\alpha = 1$, which shows that a one-dimensional, hyperbolic strange attractor is really not so strange.

Next we show that the pressure in a hyperbolic system can be found via the *periodic points* as in (1.1.9-11). This can now be seen as a simple consequence of the bounded variation (2.1.5). For a Cookie cutter f^n is monotonic in each cylinder and maps each of them to the entire interval, I. Thus each cylinder contains precisely one fixed point of f^n, and the exponent of (1.1.10) is precisely $-\beta S_n\phi(x)$ evaluated at those points. Now (2.1.5) ensures that growth rates are unchanged if we choose different points in the cylinders, so the growth rates of $Z_{fix,n}(\beta)$ and $Z_n(\beta)$ must be identical i.e.

$$\beta F_{fix}(\beta) = -P(\beta) \tag{2.2.20}$$

The invariance property (2.1.1) of the Gibbs measures leads to a generalized Frobenius-Perron equation - in fact that is how they are usually constructed (see ref.4). Let us look for Gibbs measure with a smooth *density*. By the density we mean that the Gibbs condition (2.2.6) can be written

$$\nu_\beta(I_j^{(n)}) = \rho_\beta(x)e^{-nP(\beta)}(\Delta(I_j^{(n)}))^\beta = \rho_\beta(x)e^{-nP(\beta)}|f^{n\prime}(x)|^\beta \tag{2.2.21}$$

where $\rho_\beta(x)$ is a smooth function. Note that $\beta = 1$ leads to a normal density, which shows that a smooth measure must necessarily be SRB. That implies also that the spectrum of characteristic exponents given by $S(\lambda)$ will also tell us the probability of finding given "finite time" Liapunov exponents[10] in, say, a time series from an experiment (which of course probes the natural measure). Now, invariance means that the two cylinders on level $n+1$, say $I_k^{(n+1)}$ and $I_l^{(n+1)}$, that are mapped by f onto $I_j^{(n)}$ must have the same measure as the latter: $\nu_\beta(I_k^{(n+1)}) + \nu_\beta(I_l^{(n+1)}) = \nu_\beta(I_j^{(n)})$. When we insert (2.2.19) we get

$$\rho_\beta(y) = e^{-P(\beta)}\left[\frac{\rho_\beta(x_1)}{|f'(x_1)|^\beta} + \frac{\rho_\beta(x_2)}{|f'(x_2)|^\beta}\right] \tag{2.2.22}$$

where $y \in I_j^{(n)}$ and $y=f(x_1)=f(x_2)$. This is precisely the "generalized Frobenius-Perron equation" introduced in section (1.3) and the derivation shows that the function F of (1.3.4) must be related to the pressure by (1.3.5) and that the solution, $Q_\beta(x)$, is precisely the density of the Gibbs measure. Thus we have, finally,

$$P(\beta) = -\beta F_{fix}(\beta) = -\beta F(\beta) \tag{2.2.23}$$

and all the formalisms of chapter 1 are equivalent.

3. THERMODYNAMICS FOR THE LOGISTIC MAP AT THE "ULAM - VON NEUMANN POINT."

In this chapter we shall treat a simple example in detail: the limiting "Cookie cutter" Cantor set generated by the logistic map $f(x)=\mu x(1-x)$ with $\mu=4$ (the "Ulam - von Neumann" point[33]). This is the simplest example of a *non-hyperbolic chaotic attractor*. It is exactly soluble due to the conjugacy of f to a trivial map, but since the conjugating function has zero slope in the ends of the interval the results are not trivial. The thermodynamic functions display a phase transition[17,20-22,43] and the $Q_\beta(x)$ densities, which can be calculated explicitly, show interesting change of behaviour.

3.1. The Thermodynamic Functions.

To construct the "Cookie cutter" for the map

$$f(x)=4x(1-x) \tag{3.1.1}$$

we use the fact that f is conjugate to the "tent map"

$$t(w)=1-|2w-1| \tag{3.1.2}$$

as

$$f(x)=4x(1-x)=h\circ t\circ h^{-1}(x) \tag{3.1.3}$$

where

$$h(w)=\sin^2\frac{\pi}{2}w \tag{3.1.4}$$

The cylinders of the tent map at a given level, n, are simply the 2^n intervals $C_j=[\frac{j-1}{2^n},\frac{j}{2^n}]$, where $j=1,\dots,2^n$, and since f^n is conjugate to t^n, i.e. satisfies

$$f^n(x)=h\circ t^n\circ h^{-1}(x) \tag{3.1.5}$$

the cylinders, D_j, of the logistic map, are the images

$$D_j=[h(\frac{j-1}{2^n}),h(\frac{j}{2^n})]=[\sin^2(\frac{\pi}{2}\frac{j-1}{2^n}),\sin^2(\frac{\pi}{2}\frac{j}{2^n})] \tag{3.1.6}$$

with lengths

$$\Delta_j=\sin^2\left[\frac{\pi}{2}\frac{j}{2^n}\right]-\sin^2\left[\frac{\pi}{2}\frac{j-1}{2^n}\right] \tag{3.1.7}$$

If j is large the difference can be approximated by the derivative and $\Delta\sim 2^{-n}$. That is the case for most j (almost all, in fact). If, however,

$$j\sim 2^{\sigma n} \tag{3.1.8}$$

with $0<\sigma<1$, we find

$$\Delta_j\approx 2^{\sigma n-2n} \tag{3.1.9}$$

Now, using (1.1.1): $\Delta\sim e^{-\lambda n}$, we can find the spectrum of characteristic exponents $S(\lambda)$. First, (3.1.9) and (1.1.1) gives

$$\lambda(\sigma)=(2-\sigma)log2 \tag{3.1.10}$$

which lies in the interval $[log2, 2log2]$. Second, the number of cylinders in an interval $d\sigma$ around σ is proportional to $n2^{n\sigma}d\sigma$ so

$$S(\lambda)=\sigma log2=2log2-\lambda \tag{3.1.11}$$

The function $S(\lambda)$ is shown in fig 3.1a and has a shape which is very different from the shape of $S(\lambda)$ for a hyperbolic system. Since $S'(\lambda)=-1$ for all λ the pressure cannot be found from (1.1.5) and (1.1.6). We must go back to the definitions (1.1.2-4). When we insert the known $S(\lambda)$ into (1.1.5) we get

$$Z_n(\beta)=\int e^{(2log2-\lambda-\lambda\beta)n}d\lambda \tag{3.1.12}$$

which means that

$$P(\beta)=2log2+\lim_{n\to\infty}\frac{1}{n}\log\int_{log2}^{2log2}e^{-\lambda(1+\beta)n}d\lambda \tag{3.1.13}$$

In this case the saddle-point method doesn't work: the maximal value of the exponential is obtained at one of the two *endpoints* of the interval depending on the value of β. Thus[17]

$$P(\beta)=\begin{cases}(1-\beta)log2 & \beta>-1\\ -2\beta log2 & \beta<-1\end{cases} \tag{3.1.14}$$

which is shown in fig3.1b and we see that the system undergoes a first order phase transition at $\beta = -1$. The "disordered" phase is $\beta > -1$ and here almost all cylinders contribute equivalently. In the "condensed" phase $\beta < -1$ only the cylinders very close to the edge contribute. It is easy to see that $\beta F_{fix}(\beta) = -P(\beta)$ also in this case. In fact the slope at all fixed points of f^n is 2^n except at the origin where the slope is 4^n (see (3.2.4) below) and, again, for $\beta < -1$ that one cylinder dominates the partition function.

Because of the conjugacy to the tent map the natural measure can simply be carried over and this shows that all cylinders have equal measure. That is very different from the Gibbs condition (2.2.6), but in fact it simply means that the constant $c_2 \to \infty$, i.e. that the density in (2.2.21) *diverges*. The logistic map is very special in that the divergence in the density $\rho(x)$, which is determined by the order of the critical point precisely cancels the "extra compression" of the edge cylinders, which is determined by the slope of f at the origin. Thus the generalized entropies (1.2.2) are trivial i.e.

$$K_q = \log 2 \qquad (3.1.15)$$

for *all* q. Further in this special case $f(\alpha)$ can be found from (1.2.12) giving a straight line from (1/2,0) to (1,1) as it should in any system with a square root divergence of the invariant density[8].

3.2. Solution of the Generalized Frobenius-Perron Equation.

We shall now calculate the $Q_\beta(x)$ functions by solving the generalized Fronbenius-Perron equation (1.3.4). We shall explicitely see that (1.3.6) holds, i.e. $\beta F(\beta) = -P(\beta)$. By inserting this value of $F(\beta)$ into (1.3.4) we shall se that $Q_\beta(x)$ can be found iteratively and is positive and finite (except for possible divergences at the ends.)

We then have to solve

$$Q_\beta(y) = e^{-P(\beta)} \sum_{x \in f^{-1}(y)} \frac{Q_\beta(x)}{|f'(x)|^\beta} \qquad (3.2.1)$$

with $f(x)$ given by (3.1.1) and $P(\beta)$ given by (3.1.14). We solve this by iteration, as described in section 1.3, starting from a uniform density. Thus

$$Q_\beta(y) = \lim_{n\to\infty} e^{-nP(\beta)} \sum_{x \in f^{-n}(y)} \frac{1}{|f^{n\prime}(x)|^\beta} \qquad (3.2.2)$$

Using (3.1.5) we can now express f^n as

$$f^n(x) = \sin^2(2^n(\arcsin\sqrt{x} - \frac{\pi}{2}\frac{j}{2^n})) \qquad (3.2.3)$$

which, for any even j, is valid when $x \in D_j \cup D_{j+1}$. The derivative which appears in (3.2.2) can now be expressed as

$$|f^{n\prime}(x)| = 2^n \left\{\frac{y(1-y)}{x(1-x)}\right\}^{1/2} \qquad (3.2.4)$$

where $y = f^n(x)$. In each cylinder D_j there is one $x = x_j(y) \in f^{-n}(y)$ found by inverting (3.2.3). That is, for j even,

$$x_j(y) = \sin^2(\frac{\pi j - 2z(y)}{2^{n+1}}) \qquad (3.2.5)$$

and

$$x_{j+1}(y) = \sin^2(\frac{\pi j + 2z(y)}{2^{n+1}}) \qquad (3.2.6)$$

where we have defined $z = \arcsin\sqrt{y}$. The sum (3.2.2) can now be rewritten as

$$Q_\beta(y) = \lim_{n\to\infty} e^{-nP(\beta)}\, 2^{1-(n+1)\beta}\, (y(1-y))^{-\beta/2} F_{N,\beta}(z) \qquad (3.2.7)$$

where $N = 2^{n-1}$ and

$$F_{N,\beta}(z) = \sin^\beta\frac{z}{N} + \sum_{k=1}^{\frac{N}{2}-1}\left[\sin^\beta(\frac{\pi k + z}{N}) + \sin^\beta(\frac{\pi k - z}{N})\right] + \cos^\beta\frac{z}{N} \qquad (3.2.8)$$

When the explicit form of the pressure function $P(\beta)$ ((3.1.14)) is inserted this becomes

$$Q_\beta(y) = \lim_{N\to\infty}\begin{cases}(y(1-y))^{-\beta/2}\, 2^{-\beta} N^{-1} F_{N,\beta}(z) & \beta > -1\\ (y(1-y))^{-\beta/2}\, 2 N^\beta F_{N,\beta}(z) & \beta < -1\end{cases} \qquad (3.2.9)$$

The expression $F_{N,\beta}(z)$ looks like a Riemann sum. In fact

$$N^{-1}F_{N,\beta}(z) \to \frac{2}{\pi}\int_0^{\pi/2} \sin^\beta w \; dw \tag{3.2.10}$$

independently of z provided the integral converges. This is the case when $\beta>-1$ and then

$$Q_\beta(y) = (y(1-y))^{-\beta/2}c(\beta) \tag{3.2.11}$$

where $c(\beta)$ is a constant.

For $\beta<-1$, $N^{-1}F_{N,\beta}$ diverges for $N \to \infty$ and to obtain $Q_\beta(y)$ we must look closer at $N^{-|\beta|}F_{N,\beta}(z)$. We do this by subtracting the divergent part. That is, we look at

$$G_{N,\beta}(z) = N^{-|\beta|}\left[F_{N,\beta}(z)-(\frac{N}{z})^{|\beta|}-\sum_{k=1}^{N/2}(\frac{N}{\pi k+z})^{|\beta|}+(\frac{N}{\pi k-z})^{|\beta|}\right] \tag{3.2.12}$$

which can be rearranged as

$$G_{N,\beta}(z) = N^{-|\beta|}\sum_{k=1}^{N/2}\sin^{-|\beta|}(\frac{\pi k+z}{N})+\sin^{-|\beta|}(\frac{\pi k-z}{N})-(\frac{N}{\pi k+z})^{|\beta|}-(\frac{N}{\pi k-z})^{|\beta|}$$

Now we have a Riemann sum for a *convergent* integral:

$$G_{N,\beta}(z) \to N^{-|\beta|+1}\,\frac{2}{\pi}\int_0^{\pi/2}(sin^{-|\beta|}(w)-w^{-|\beta|})\; dw \tag{3.2.13}$$

which approaches 0 for $N\to\infty$ when $\beta<-1$. Thus

$$Q_\beta(y) = c(\beta)\,(y(1-y))^{|\beta|/2}\,T_\beta(z) \tag{3.2.14}$$

where

$$T_\beta(z) = \frac{1}{z^{|\beta|}} + \sum_{k=1}^{\infty}(\frac{1}{\pi k+z})^{|\beta|}+(\frac{1}{\pi k-z})^{|\beta|} \tag{3.2.15}$$

Here, again, $c(\beta)$ is a constant and $z=arcsin\,\sqrt{y}$.

For $\beta=-2$ the result is particularly simple. Using

$$(\sin(z))^{-2} = \sum_{k=-\infty}^{\infty}(\frac{1}{\pi k+z})^2 \tag{3.2.16}$$

we find

$$Q_{-2}(y) \propto 1-y \tag{3.2.17}$$

We can check that $Q_\beta(x)$ defined by (3.2.14) really do satisfy the Frobenius-Perron equation (3.2.1) for $\beta<-1$. Inserting (3.2.14) into (3.2.1) gives

$$T_\beta(\frac{z}{2})+T_\beta(\frac{\pi-z}{2}) = 2^{|\beta|}\,T_\beta(z) \tag{3.2.18}$$

which is satisfied by $T_\beta(x)$ defined by (3.2.15)

It is interesting to note that the functions $Q_\beta(x)$ are symmetric in the "disordered" phase, whereas this symmetry is broken in the ordered phase. Fig 3.2 shows $Q_\beta(x)$ for different β. Notice that Q_β given by (3.2.9) does not remain finite as $\beta\to-1^-$, which may also be interpreted as a sign of a phase transition.

4. THE THERMODYNAMICS OF "COMPLETE CHAOS".

In the last chapter we treated the special case of the logistic map at the completely chaotic point. We shall now look into the more general case of completely chaotic, smooth maps of the interval. The main issues will be to determine which parts of the theory of chapter 2 that remain valid, i.e. the relation between different thermodynamical formalisms, and how close the general case is to the logistic one - in particular whether the phase transitions found in that case (chapter 3) will remain. The answers to those questions will be incomplete and partly conjectural, reflecting our present understanding and emphasizing the need of further work.

The fractals to be studied in this chapter are limiting cookie cutters generated by a unimodal map that maps the interval, I, *onto* itself. If the map is smooth such a system is *non-hyperbolic* since the attractor (the whole of I) contains the critical point x_c, where $f'(x_c)=0$. The cylinders are still well-defined since every point in I has two preimages inside I. In the case of the logistic map (3.1.1) the existence of a function (3.1.4) conjugating it to a map with trivial thermodynamics made it possible to solve everything exactly. In the general case this is of course not possible and one has to resort to numerical computations, using techniques borrowed from statistical mechanics. These techniques will not be discussed in any detail here, but relevant references will be given.

One simplifying aspect should be noted from the outset. All the maps discussed here have nice, smooth natural measures. That is, the density is smooth everywhere except around the edges where singularities develop. In general any forward image of the critical point must have a singularity (as will be discussed in section 4.2 below), but in these special cases the only images are the two edges since the left edge is a fixed point. The nature of these singularities depend only on the *order of the critical point* and therefore the $f(\alpha)$ spectrum (with respect to the natural measure) are all identical.This is, as we shall see, not true for the entropy function or the pressure of section 1, and this difference goes beyond simply reflecting that sizes of characteristic exponents must change when the maps change slopes, since indeed the existence of phase transitions depends on the particular map. The differences comes from the fact that the cylinders carry different measures. Consequently the K_q spectrum is not q-independent. Furthermore the distribution of characteristic exponents given by $S(\lambda)$ is not the same as the distribution with respect to the natural measure[10,44,45]. It can be shown, however, that the thermodynamics based on the cylinders, periodic orbits and preimages are equivalent for completely chaotic maps.

4.1. A Family of Quadratic Maps.

To analyze a more general case of "completely chaotic" maps the limiting cookie cutters for a family of maps, f_γ, was analyzed in ref.24. They chose the maps

$$f_\gamma(x) = Ax(1-x)(1+\gamma x) \qquad (4.1.1)$$

where A is adjusted so the maximal value of f_γ is 1. The maps can be thought of as non-symmetric "perturbations" of the logistic map (γ=0), they all map the unit interval onto itself and all have quadratic maxima. The cookie cutters of the maps (4.1.1) for a sequence of γ-values were studied and an attempt was made to determine the structure of the entropy functions, whether they showed phase transitions and how the different phases could be described.

To do this one has to overcome two difficulties. First, one has to compute pressure functions with high accuracy (without, of course, being able to take the limit $n \to \infty$) and second, one has to find a good way of locating phase transitions and describing the structure of a given phase. Although we shall not describe any details here we shall briefly indicate the relevant concepts and refer the reader to the literature.

The first difficulty is most efficiently overcome by the *transfer matrix technique* [12] in (almost) complete analogy with statistical mechanics. The elements[11,14] of the transfer matrix are constructed from the *scaling function* [46], which expresses how the length of cylinders scale when we go from Λ_n to Λ_{n+1}. The reason why the analogy to statistical mechanics is not quite complete is that in that case the transfer matrix relates, say, two columns containing the same number of spins (see e.g. ref.47), whereas in our case Λ_{n+1} contains twice as many cylinders as Λ_n. The advantage of this formalism is that the pressure simply emerges as the largest eigenvalue of the transfer matrix which can be evaluated very accurately.

The second difficulty was overcome in ref.24 by introducing the concept of an *order parameter*. This is done by using the formal analogy between the thermodynamical formalism and statistical mechanics for an Ising spin chain[12,20,25]. Every cylinder has a unique, binary address which simply expresses the symbolic dynamics (with respect to the critical point) of the points inside as the map is iterated. But the "energies" (i.e. $e^{-\beta\log\Delta}$) entering the partition function (1.1.2) can then be thought of as those of an appropriate Ising model, where each spin (binary digit) can be up or down. Out of this analogy an order parameter can be constructed simply from the local spin averages (although care has to be taken in defining the spins appropriately). It should be noted that the ensuing Ising models have very complicated interactions, far from the standard nearest neighbor case. Indeed to have phase transitions in a one-dimensional spin system the interactions must be infinite ranged, which turns out to be intimately connected with the existence of a critical point for the map, f, inside I.

The resulting $S(\lambda)$ spectra for the maps (4.1.1) depend strongly upon γ and particularly upon its sign. Fig.4.1 represents the results for a negative and a positive γ. The figure is not drawn to fit any particular value of γ, but is an idealization showing the essential features as given in ref.24. To understand the results, let us first remind ourselves of the behaviour for the logistic map (γ=0) as given by fig.3.1. There the entropy is positive (and linear) between $\log s_1$ and $\log s_0$, where s_0=4 is the slope at the origin and s_1=2 is the slope at the fixed point inside I (or, in this case, for any cycle away from the origin). There is a phase transition at $\beta_c=-1$, which is the slope of the entropy at the right hand edge.

For the map (4.1.1) with $\gamma<0$, the slope s_0 at the origin is larger than 4 and it still represents the maximal characteristic exponent: $\lambda_{max}=\log s_0$. The minimal λ is still close to $\log s_1$, which, in this case, is less than 2. The phase transition remains: indeed, the whole decreasing part of the entropy seems to be a straight line going from ($log2$,$log2$) to (0,$logs_0$). Thus the phase transition takes place at

$$\beta_c = -\frac{\log 2}{\log s_0 - \log 2} = -\frac{\log 2}{\log s_0/2} \qquad (4.1.2)$$

In fact[48], if the idealized entropy of fig.4.1a really represents the correct asymptotic structure (which can be hard to extract unambiguosly from the data) the discontinuity of the slope at $\lambda=log2$ should imply another transition at large, positive $\beta=\beta_1$ in order of magnitude given by (4.1.2) with s_0 replaced by s_1. According to the Ehrenfest classification the latter transition should be 2.order: Since the the entropy is continuos at β_1, the pressure function should only have a discontinuity in the second derivative. This is not inconsistent with fig.4d of ref.24, where the order parameter seems to grow up continuosly for $\beta>\beta_1$, but further work is need to clarify the situation.

For $\gamma>0$ the situation is different. Here $s_0<4$ and $s_1>2$ so the λ-interval shrinks. Now, for all "complete" cases the total length of cylinders must equal 1 (the whole of I) for any n. Therefore the interval $[\lambda_{min},\lambda_{max}]$ must contain log2 and λ_{min} can no longer be given by $\log s_1$. There is, however, another important scaling due to the critical point. The cylinders on level n+1 are obtained from those on level n by iterating the map (4.1.1) once backwards. In general it is scaled down by the slope of f_γ, but if the cylinder passes close to the critical point, where f' is zero, it is not scaled down by a factor but becomes proportional to its square root. This can only happen if the cylinder is very close to the origin so the new relevant scaling is $\sqrt{s_0}$. Indeed λ_{min}=1/2$\log s_0$, at least within a few percent.

If γ is not too big (up to $\gamma\approx1$) the phase transition at $\beta<0$ remains, this time with $\beta_c>1$. In this case probably the entropy is a straight line only in a finite segment of the decreasing ($\beta<0$) part as shown in fig.4.1b. The transition at positive β has dissappeared. For greater values of γ no transition is observed at all. It disappears around (probably slighly before) the point where s_0 and s_1 become identical and presumably the critical point has $\beta_c=-\infty$, i.e. takes place at zero temperature.

4.2. Analysis Based on the Functional Equation

The study of the eigenfunctions of the generalized Frobenius-Perron equation (1.3.4) provides additional information about the thermodynamics and phase transition of completely chaotic maps[49]. For the sake of simplicity we keep on working with quadratic maps i.e. we assume that the function $f(x)$ behaves as $f(x) = 1-a(x-x_c)^2$ around the critical point x_c. We perform a *singularity* analysis of the eigenfunctions Q_β and study how they arise.

Let us write the equation in the form

$$S_{k+1,\beta}(y) = R(\beta) \sum_{x\in f^{-1}(y)} \frac{S_{k,\beta}(x)}{|f'(x)|^\beta}. \qquad (4.2.1)$$

As initial functions we now allow also ones having singularities at least at one of the endpoints of the interval. $1/R(\beta)$ is the corresponding eigenvalue ensuring convergence to a finite limiting $S_\beta(x)$. We are interested in $S_{k+1,\beta}$ around the two endpoints of the interval. For $y\rightarrow1$

$$S_{k+1,\beta}(y) = \frac{R(\beta)S_{k,\beta}(x_c)}{2^{\beta-1}a^{\beta/2}}(1-y)^{-\beta/2} \qquad (4.2.2)$$

since both preimages of y coincide in this limit. Consequently, $S_{k+1,\beta}$ is always singular at the right corner. For $y \to 0$ the preimages of y can be written as y/s_0 and $1-y/s_1$ where s_0 and s_1 are the slopes of $f(x)$ at the origin and at $x = 1$, respectively. Thus

$$S_{k+1,\beta}(y) = R(\beta)\left[S_{k,\beta}(y/s_0)s_0^{-\beta} + S_{k,\beta}(1-y/s_1)s_1^{-\beta}\right] \tag{4.2.3}$$

The singularity of $S_{k+1,\beta}(y)$ at $y = 0$ depends on the singularity of the previous iterate at the two corners.

Consider first functions $S(x)$ having the same singularity, of order $-\beta/2$, at both ends. An example is $(x(1-x))^{-\beta/2}$. We call these functions of type I. Note that the eigenfunctions $Q_\beta(x)$ for $\beta > -1$ of the logistic case are exactly of this type. When chosing S_0 from this class, $S_{1,\beta}$ and all $S_{k,\beta}$ will be member of the class, too, since both terms on the right hand side of (4.2.3) have the same singularity for $y \to 0$. The requirement of having a finite S_β selects a prefactor

$$R(\beta) = e^{\beta F_1(\beta)}. \tag{4.2.4}$$

The reciprocal value of it is the largest eigenvalue of (4.2.1) for functions of class I. Note that $F_1(\beta)$ is not necessarily $F(\beta)$ of (1.3.4) since the latter is defined for smooth initial functions. (The relation betwee $F_1(\beta)$ and the generalized entropies is discussed in ref.59.) The spectrum $F_1(\beta)$ can easily be obtained numerically by iterating (4.2.1) with $(x(1-x))^{-\beta/2}$ as initial function and requiring a finite limiting value of Q.

Let us now turn to functions of type II, defined by having a singularity of order $-\beta/2$ at the right corner, but being finite at $x = 0$, like e.g. $(1-x)^{-\beta/2}$. The eigenfunctions of the logistic case are of this type for $\beta < -1$. By starting with such an S_0 the first iterate for $\beta > 0$ will be of type I since the first term is then negligible on the right hand side of (4.2.3). For $\beta < 0$, however, the first term dominates, and S_1 and its further iterates will remain inside class II. This class for $\beta < 0$ possesses its own eigenvalue. Applying now (4.2.3) on the limiting function S_β, one immediately sees that the prefactor must be

$$R(\beta) = e^{\beta F_0(\beta)} = s_0^\beta. \tag{4.2.5}$$

We can now turn to the original problem of smooth initial functions Q_0. It follows from (4.2.2) that $Q_{1,\beta} \sim (1-y)^{-\beta/2}$ for $y \to 1$. In the range $\beta > 0$ the same singularity appears also at the left corner and Q_β will be of type I. For $\beta < 0$, however, $Q_{2,\beta}$ will be of type II since $Q_{1,\beta}(1-y/s_1)$ vanishes for $y \to 0$. In this range, therefore, besides the eigenvalues characterizing class I, also $e^{-\beta F_0(\beta)}$ is to be taken into account. The actual fate of $Q_{k,\beta}$ is determined by the *largest* eigenvalue. If $s_0^{-\beta} > e^{-\beta F_1(\beta)}$, $Q_{k,\beta}$ remains in class II, otherwise $Q_{k,\beta}(0)$ decreases in course of the iteration as one sees from (4.2.3). The limiting function will be then of type I.

In other words, the free energy $\beta F(\beta)$ defined by (1.3.4) for smooth initial functions appears as a minimum of βF_0 and βF_1:

$$\beta F(\beta) = \min\left[\beta F_0(\beta), \beta F_1(\beta)\right]. \tag{4.2.6}$$

This is in complete analogy with thermodynamics, where a first order phase transition occurs when the minimum of the free energy crosses over from one branch onto another. In our case F_0 does not exist for positive β. A phase transition takes place only if the branch of F_0 intersects that of F_1 at some negative β_c. For $\beta < \beta_c$ $F(\beta) = \log s_0$. The slope at the origin determines the complete thermodynamics in this range, so in these models the "condensed phase" is trivial, like a T=0 phase in thermodynamics, and nothing varies with β.

As an illustrative example we take the map

$$f(x) = 1-(1-\varepsilon)(1-2x)^2 - \varepsilon(1-2x)^4 \tag{4.2.7}$$

with $\varepsilon = -0.263$. On fig4.2 numerical approximations to $\beta F(\beta)$, $\beta F_0(\beta)$ and $\beta F_1(\beta)$ are shown -the latter two as thin lines. One can see clearly how the full free energy emerges as the lowest of those two curves, and that this method gives a rather precise way of locating the phase transition point, which in this case is $\beta_c \approx -0.869$. The asymptotic behaviour is given by $F_1(\infty) = 0.54$ and $F_1(-\infty) = 0.76$ ($F_0 = 1.08$).

As mentioned above, for the completely chaotic cases the pressure, $P(\beta)$ and the free energy, $F(\beta)$, coming from the Frobenius-Perron equation carry the same information and are still related by (1.3.6) although the map is not hyperbolic. The reason for this can be best understood from the representation (1.3.5). As long as the points x (preimages of y) avoid the critical point the "Boltzmann factors" $e^{-\beta \log|f^{n\prime}(x)|}$ are nothing but the corresponding Δ^β of eqn. (1.1.2) just as argued in section 2.2. Now, for x close to the critical point this breaks down, but this happens (for our complete maps) only at y=0 and y=1. In these points $Q_\beta(y)$ can diverge so $F(\beta)$ is no longer related to the growth rate of the partition function in (1.3.5). If a *y-independent* prefactor $e^{\beta F(\beta)}$ exists for (1.3.5) it must therefore be given by $e^{-P(\beta)}$.

It is worth briefly mentioning the case of more general maps with a maximum of order $z \neq 2$ for which $f(x) = 1 - a\,|x - x_c|^z$. The most important change occurs for $z < 1$, i.e, if the maximum is a cusp. Then the critical point and the "condensed phase" lie in the region $\beta > 0$. Another interesting property is that the K_q spectrum may also exhibit phase transition, if the map is intermittent[23]. Thus, the following picture emerges.

For complete nonhyperbolic maps there are three essentially different thermodynamic formalisms providing different spectra:

- The spectrum of characteristic exponents, periodic orbits and preimages (the latter defined by (1.3.5)). This seems to be the most sensitive one. A phase transition may or may not appear in it. This defines the pressure function and the densities $Q_\beta(x)$. The Legendre transform $S(\lambda)$ gives the distribution of characteristic exponents with respect to Lebesgue measure (i.e. usual length) not with respect to the natural measure, which has singularities.

- The spectrum of generalized dimensions or $f(\alpha)$ is a rigid spectrum. A phase transition always occurs in it and the spectrum can be expressed in terms of a single parameter, the order of maximum z.

- The spectrum of generalized entropies is not quite as well understood at present. In the case of intermittent maps a singularity appears which is connected with a real time slowing down. In that case the transition is independent of z.

4.3. Final Remarks

Functions mapping an interval *into* itself define incomplete maps since in such cases not all possible (binary) sequences can be realized by trajectories. The thermodynamic description of incomplete maps is most hopeful at parameter values where *finite* symbol sequences (and their permutations) are forbidden only. This means that a so-called local grammar exists and a kind of topological universality holds. In such cases cylinders are well defined, the convergence of (1.3.1) is established and different spectra can be worked out. Maps at general parameter values can be approached through such cases just like irrational numbers through rationals.

The simplest cases are the so-called Misiurewicz points[33], where the critical point maps onto an unstable periodic orbit. The simplest one is actually the "complete" case studied above, where the critical point after two iterations falls onto the unstable fixed point in the origin. The next simplest possibility is to have the critical point fall onto the unstable fixed point inside the interval. That requires three iterates and happens for the logistic map $f(x) = \mu x(1-x)$ as $\mu = 3.6786$, the situation which is shown in fig.4.3. Fig4.3b shows the second iterate and the boxes show that the dynamics is actually no more complicated than the complete case. Within each box (i.e. between two iterates of the critical point) we have a small, completely chaotic map. The thermodynamic functions in each box must be identical since they map onto each other by f (and the critical points are mapped onto each other). If we look at a sequence of Misiurewicz points, where the critical point falls onto an unstable 2^{n-1}- cycle we get 2^n little boxes and within each we have the same thermodynamical functions. Aside from rescaling the function within the boxes are given by a "universal chaos function"[50], which is well represented by the expansion (4.2.7). Thus fig.4.2 shows the "universal" pressure function (with a minus sign).

It should be noted that the $f(\alpha)$-spectrum is the same for all these cases. Since the map is complete in every little box the $f(\alpha)$-spectrum is still a straight line between (0,1/2) and (1,1). Although any "chaotic point" can be reached by a sequence of (generalized) Misiurewicz points this does *not* imply that they all have that same spectrum. Indeed these $f(\alpha)$ curves represent the asymptotic scaling reached only for distances much less than the distances between images of the critical point. For a general chaotic point these images are ergodic, filling out the whole attractor, so this length-scale can never be reached.

In closing we would like to mention some recent work which goes beyond the scope of this article, but which might interest the reader. A severe limitation of our exposition has come from the constraint to one-dimensional problems. Recently a lot of work has been done in higher dimensional systems, specifically the Hénon map[26-28,51-53] and maps of the annulus[54] to which we would like to draw attention. A different issue, the thermodynamics of mode-locking, (see the article on circle maps by Bak et.al. in this volume) is dealt with in ref.55 and refs.56-57 discuss relations between the different thermodynamical formalisms of the same type as those discussed in the preceeding chapters. Further, in ref.58, the spectra of fractal aggregates have been modelled in terms of Julia sets. There it turns out that the "balanced measure" (1.2.12) precisely gives the "harmonic measure" of the aggregate showing where the growth takes place.

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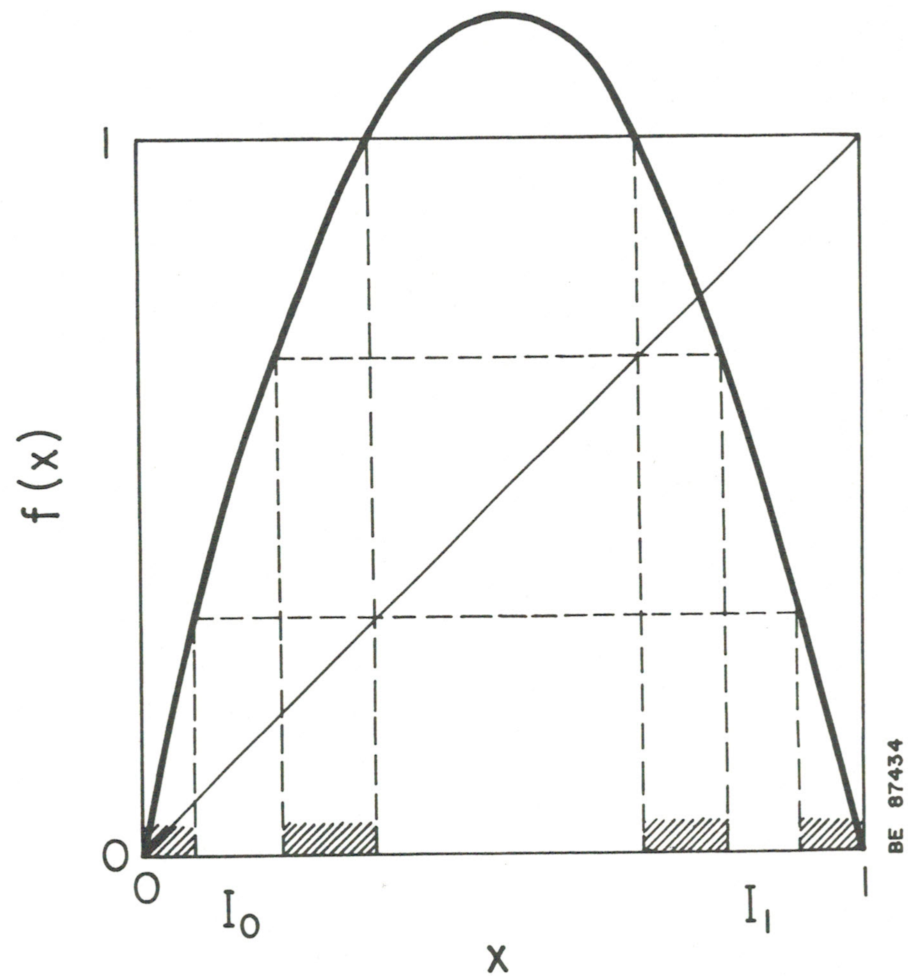


Fig. 1.1
The construction of a "Cookie cutter" Cantor set. The hatched intervals are the cylinder for n=2, i.e. the points that remain inside I for at least 2 iterates.

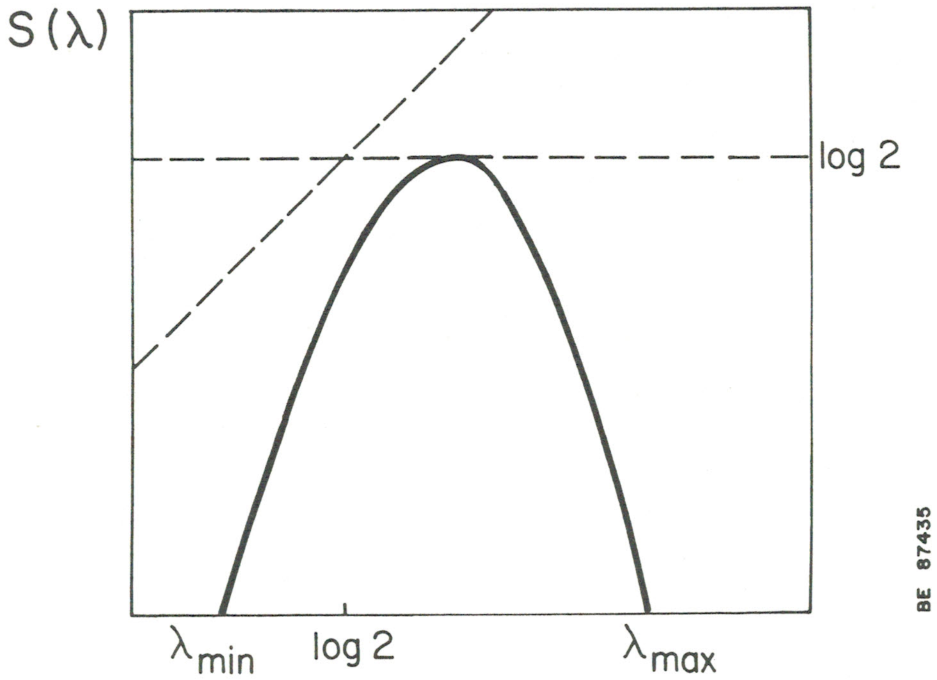


Fig. 1.2

Typical structure for the entropy function. The two dashed lines are $y=log2$ and $y=x$. The fact that the entropy touches $y=log2$ just means that the total number of cylinders is 2^n. The fact that it doesn't touch $y=x$ shows that this system is a repellor, it has a finite escape rate.

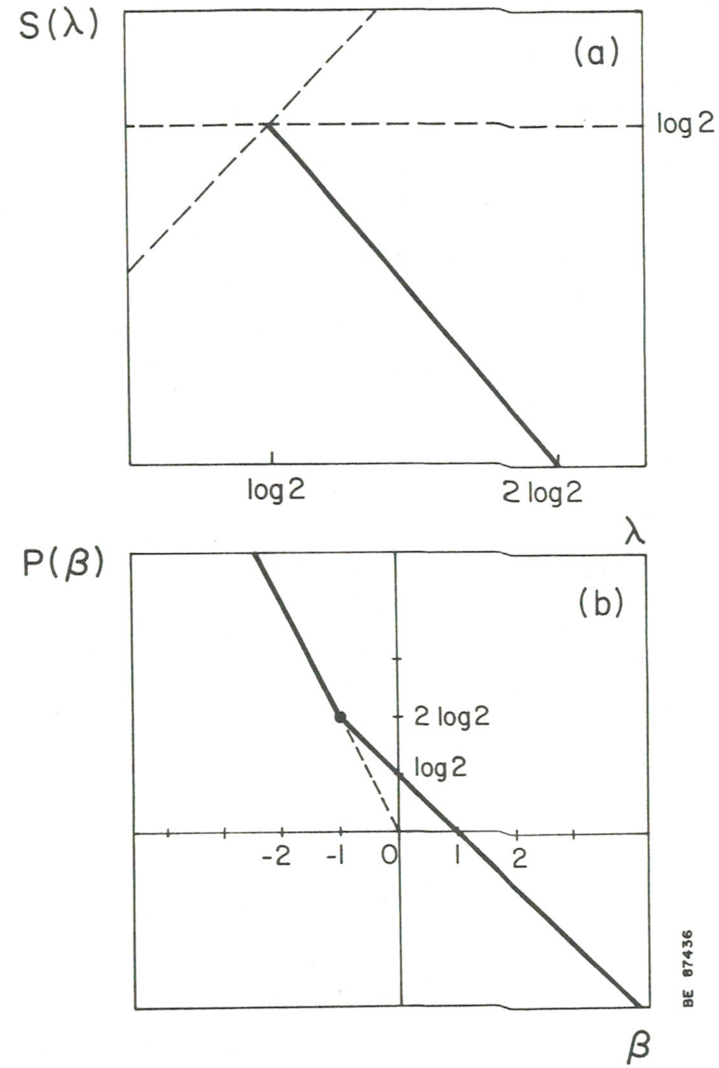


Fig. 3.1

Thermodynamical functions for the logistic map at the "Ulam -von Neumann" point. The function $P(\beta)$ is analog to the free energy so the discontinuity in its slope implies a first order transition.

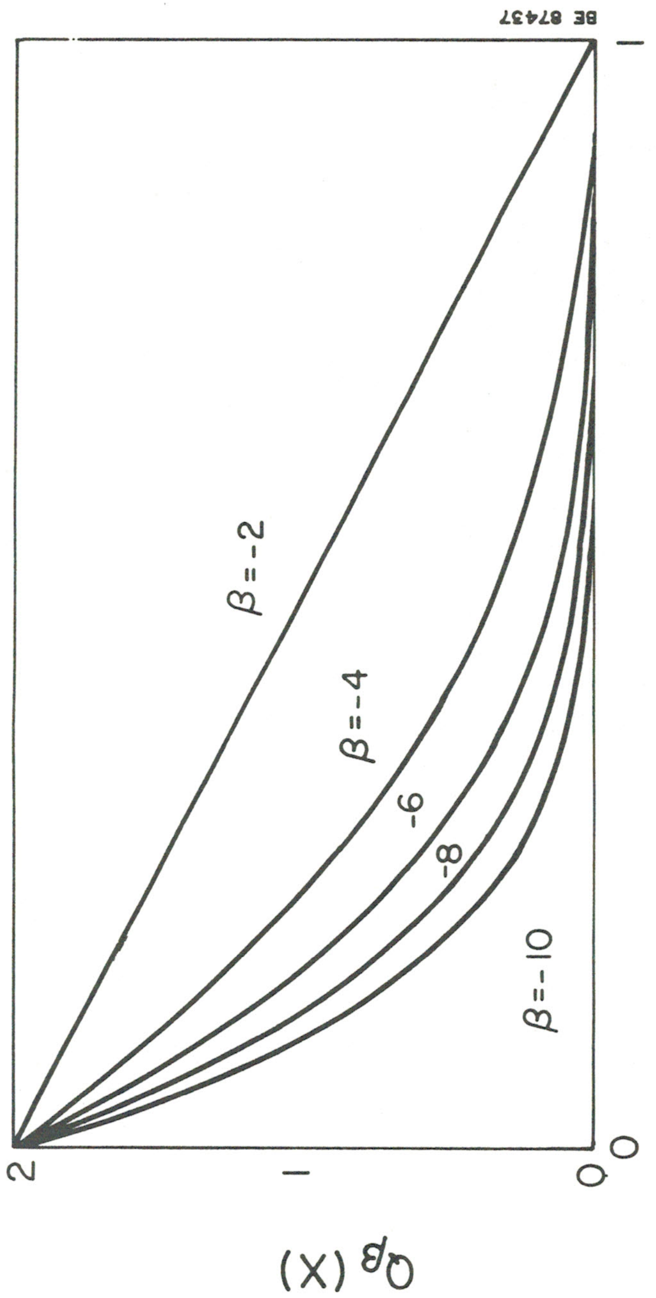
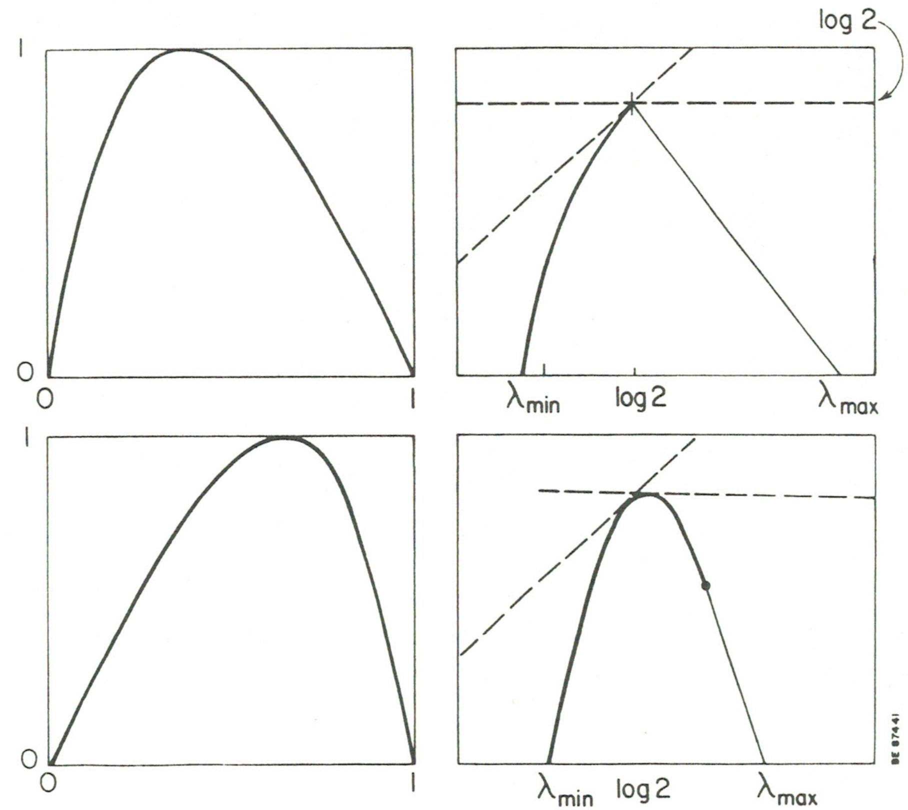


Fig. 3.2
The invariant densities $Q_\beta(x)$ (3.2.14) for different β

Fig. 4.1
Sketch of the map (4.1.1) (left) and the associated entropy curve (right) for a positive (top) and a negative (bottom) value of γ.



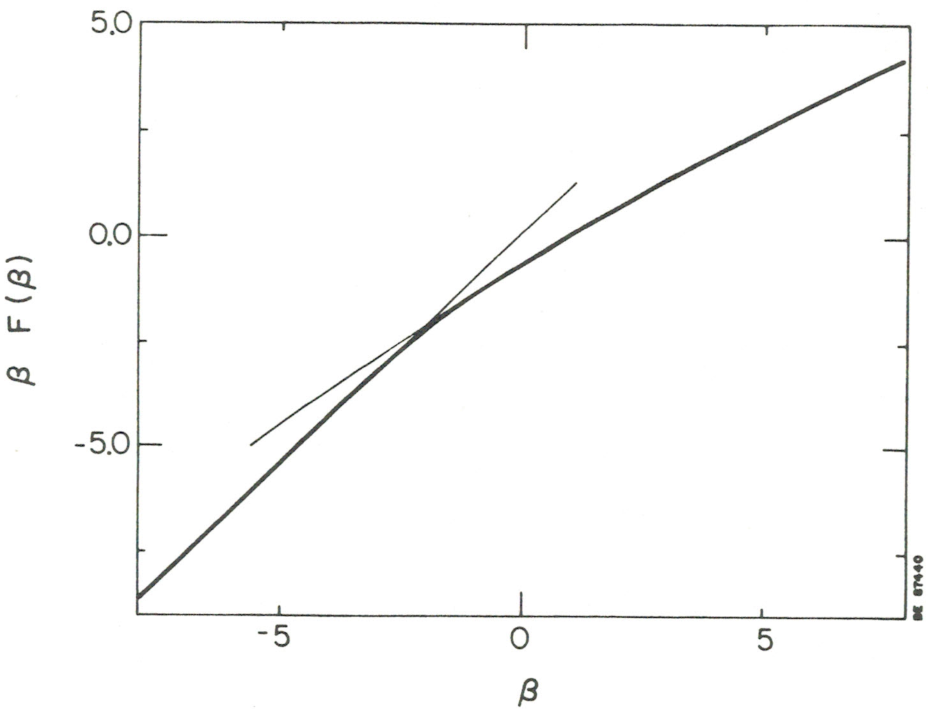


Fig. 4.2

Numerical approximation to the free energy of the "universal map" (4.2.7). The heavy curve shows $\beta F(\beta)$ found by iterating a non-singular initial density. The thin lines are the curves βF_0 (the straight line) and βF_1 which come from iterating singular densities as explained in the text.

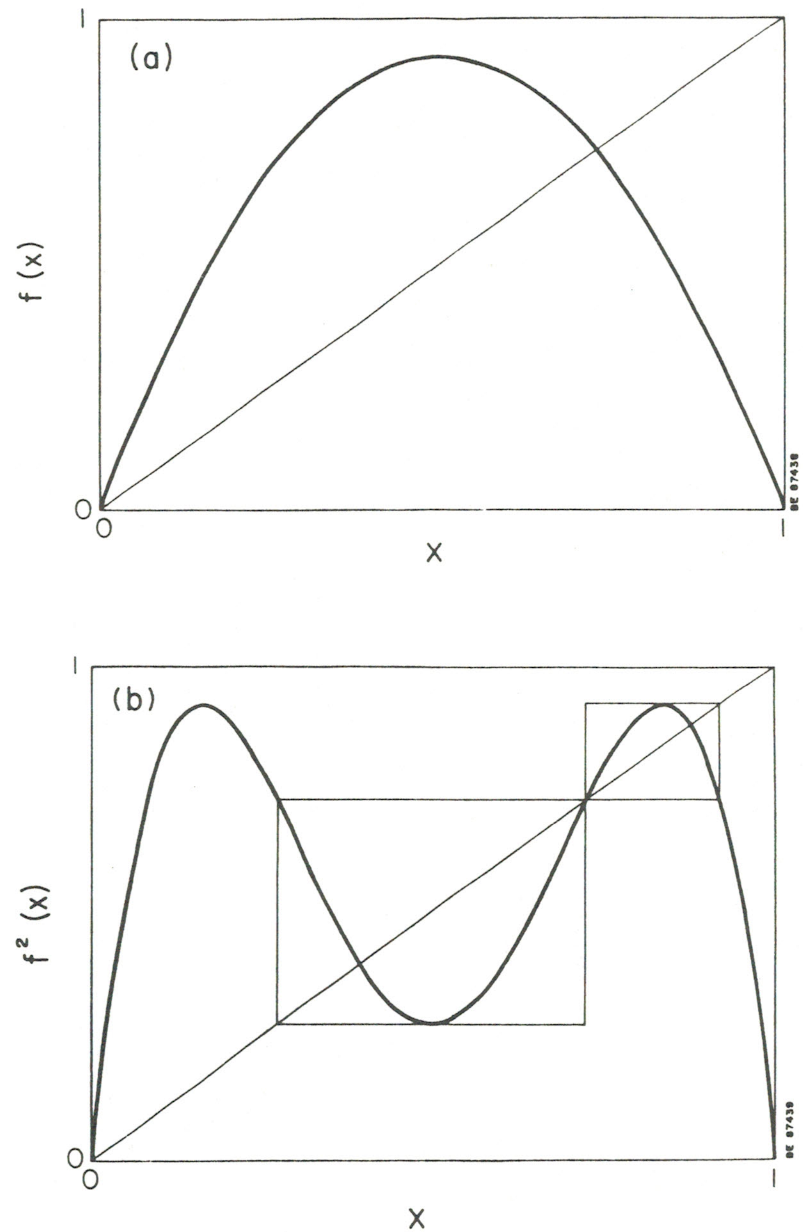


Fig. 4.3

The first Misiurewicz point. Within the boxes f^2 behaves like a "complete map".