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Potential for the Complex Ginzburg-Landau Equation.

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Abstract. – A generalized thermodynamic potential is derived for spatially extended pattern-forming nonequilibrium systems whose order parameter obeys the complex Ginzburg-Landau equation in one spatial dimension. The real potential, generalizing the Ginzburg-Landau free energy, is shown to describe typical nonequilibrium phenomena like the Newell-Kuramoto and the Eckhaus-Benjamin-Feir instabilities. It is pointed out that the extremizing order parameter field may exhibit chaotic behaviour. Potential barriers between coexisting plane-wave attractors are calculated.

Spatially extended pattern-forming systems far from thermodynamic equilibrium are currently under intense investigation, with the sustained hope that one might be able to establish some general principles governing processes of «self-organization» in nonequilibrium systems [1]. A major obstacle, so far frustrating such hopes, has been the absence, or lack of knowledge, of something like a thermodynamic potential which would govern pattern-formation in nonequilibrium systems. It is the purpose of the present letter to construct such a potential within certain limits of validity for a general class of nonequilibrium systems and to apply it to a discussion of multistability and secondary instabilities typical for such systems.

There are special cases of pattern-forming systems where a generalized thermodynamic potential exists, in particular those whose order parameter dynamics close to instability are described by the Ginzburg-Landau equation with real coefficients. The Bénard instability in Boussinesq approximation is a well-known example [2]. However, such cases remain exceptions. Far from thermodynamic equilibrium the Ginzburg-Landau equation with real coefficients is a nongeneric special case of the generic Ginzburg-Landau equation with complex coefficients

$$\dot{\psi} = (a - b|\psi|^2)\psi + D\nabla^2\psi . \tag{1}$$

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Equation (1) is the normal form of a Hopf bifurcation from a fixed point ($\psi = 0$) to a limit cycle ($|\psi| = (a/b_r)^{1/2}$) in a spatially extended system. $\psi(x, t)$ is the complex order parameter, a is the control parameter, which we choose real without restriction of generality, $b = b_r + ib_i$ and $D = D_r + iD_i$ are the complex nonlinearity and diffusion coefficients, respectively. For simplicity we shall only consider one-dimensional systems in the following, and we shall assume that $b_r > 0$, $D_r > 0$. Equation (1) holds in the long-wavelength limit sufficiently close to the bifurcation point $a = 0$. If b_i and (or) D_i are nonvanishing, eq. (1) does not possess a potential in the sense that its right-hand side is not proportional to the derivative of a functional with respect to ψ^*.

On the other hand, in the context of discrete dynamical systems we have argued in previous work, and demonstrated by numerous examples [3-5], that a generalized thermodynamic potential exists under rather general conditions. It is defined by splitting the right-hand side of the equation of motion into two terms, so that one of them is proportional to the derivative of the potential, while the second is orthogonal to the derivative. Applied to eq. (1), one may thus state that a potential ϕ exists if eq. (1) can be written as

$$\dot{\psi}(x, t) = -\frac{1}{2} Q \frac{\delta\phi}{\delta\psi^*(x)} + R(x) \tag{2}$$

with

$$\int \mathrm{d}x \left\{ R(x) \frac{\delta\phi}{\delta\psi(x)} + R^*(x) \frac{\delta\phi}{\delta\psi^*(x)} \right\} = 0\,, \tag{3}$$

where Q is a positive constant, playing the role of a generalized transport parameter, ϕ and $R(x)$ are functions of ψ, ψ^*. Combining eqs. (2), (3), one finds that $\mathrm{d}\phi/\mathrm{d}t = -Q\int \mathrm{d}x|\delta\phi/\delta\psi|^2 \leq 0$ along trajectories of eq. (1). This implies that ϕ is minimal in the attractors of (1) and, more generally, that ϕ is a Lyapunov functional of the complex Ginzburg-Landau equation.

Eliminating $\dot{\psi}$, $\dot{\psi}^*$ and R, R^* from eqs. (1)-(3) a nonlinear partial functional differential equation of first order is obtained for ϕ. This equation is of the Hamilton-Jacobi type whose solution can be expressed by the minimum of an action integral. Applying the usual procedure of classical mechanics, we obtain in this infinite-dimensional case

$$\phi(\{\psi, \psi^*\}) = \min \left\{ \int_{-\infty}^{0} \mathrm{d}\tau \frac{1}{Q} \int \mathrm{d}x \, |\dot{\psi} - a\psi + b|\psi|^2\psi - D\nabla^2\psi|^2 = C(A^{(k)}) \right\}. \tag{4}$$

Here the minimum is taken over all paths $\psi(\tau, x)$ starting at $\tau = -\infty$ in the attractors $A^{(k)}$ and ending in $\psi(0, x) \equiv \psi(x)$ at $\tau = 0$, and over k, *i.e.* over all the attractors. The function $C(A^{(k)})$ gives the value of ϕ in the attractor $A^{(k)}$ and has to be determined independently (see remark after (6)).

Equation (4) cannot be evaluated exactly. However, in the long-wavelength limit (length scales large compared to $(D_r/a)^{1/2}$) we may expand eq. (4) in the spatial derivatives of ψ. In the following we describe crucial steps of the calculation for $a > 0$ and give the main results.

Putting the spatial derivatives to zero (*i.e.* formally $D = 0$) a continuum of degenerate attractors $A^{(k)}$ given by $\psi_A(t, x) = (a/b_r)^{1/2} \exp[-iab_i t/b_r + i\varphi_0(x)]$ exists. The path $\psi(\tau, x)$ minimizing the action (4) turns out to be the solution of a simple equation: $\mathrm{d}\psi/\mathrm{d}\tau = (-a + b^*|\psi|^2)\psi$ with boundary conditions $\psi(0, x) = \psi(x)$ and $|\psi(-\infty, x)| = (a/b_r)^{1/2}$. Correspondingly, one finds in this order

$$\phi_0 = \frac{2}{Q} \int \mathrm{d}x \left[-a|\psi|^2 + \frac{1}{2} b_r |\psi|^4 \right] + \text{const}\,. \tag{5}$$

In the next order, terms linear in the Laplacian are to be kept. Since the action is extremal, there is no contribution to the time integral of (4) from the first-order correction of the path. Thus we obtain

$$\phi_1 = \min_{\{\varphi_0\}}\left\{\phi_{A_1}(\{\varphi_0\}) + \frac{2}{Q}\int_{-\infty}^{0}\mathrm{d}\tau\int \mathrm{d}x\,(a - b_r|\psi(\tau,x)|^2)(\mathrm{D}\psi^*(\tau,x)\,\nabla^2\psi(\tau,x) + \text{c.c.})\right\} \tag{6}$$

with $\psi(\tau, x)$ as the minimizing path in zeroth order. Here we have put $C(A^{(k)}) = \phi_{A_1}(\{\varphi_0\})$, where the phase $\varphi_0(x) = (1/2_i)\ln(\psi(-\infty, x)/\psi^*(-\infty, x))$ has to be expressed in terms of $\psi(0, x) \equiv \psi(x)$. $\phi_{A_1}(\{\varphi_0\})$ is evaluated in an independent calculation by a local expansion with respect to the phase fluctuations on the attractor [6]. The restriction to one spatial dimension turns out to be essential for our ability to perform this step. It turns out that φ_0 can be eliminated completely from (6) before taking the minimum and this latter step becomes superfluous in the present case. We obtain to first order

$$\begin{aligned}\phi_1 = \frac{2}{Q}\int \mathrm{d}x \Bigg[&\left(-a + \frac{b_r}{2}|\psi|^2\right)|\psi|^2 + D_r|\nabla\psi|^2 + \\ &+ \frac{D_-}{|\psi|^4}\left(-\frac{ab_i|\psi|^2|\nabla\psi|^2}{b_r|b|^2} + i(\psi^*\nabla\psi - \psi\nabla\psi^*)\nabla|\psi|^2\cdot\right.\\ &\left.\cdot\left(\frac{1}{4b_r^2}(a - b_r|\psi|^2) - \frac{a}{2|b|^2}\right) + (\nabla|\psi|^2)^2\left(-\frac{b_i}{12b_r^3}(a - b_r|\psi|^2) + \frac{ab_i}{2b_r|b|^2}\right)\right)\Bigg]\end{aligned} \tag{7}$$

with the abbreviation $D_- = D_r b_i - D_i b_r$, and omitting surface terms, for simplicity.

Due to the phase symmetry it is desirable to proceed, in this case, to the next order of this expansion, at least with respect to the gradients of φ, with $\psi = r\exp[i\varphi]$. New attractors $A^{(k)}$ given by $\psi_k = ((a - D_r k^2)/b_r)^{1/2}\exp[ikx - it(ab_i + D_- k^2)/b_r]$ appear in this order whose function $C(A^{(k)})$ is again determined by an independent calculation based on a local expansion, using results of ref. [6]. Again, it is possible to eliminate k under the minimum and the minimum over the $A^{(k)}$ becomes superfluous. This is the reason why singularities of the type discussed by us in ref. [5] do not appear in the present case. We obtain

$$\phi_2 = \frac{1}{Q}\int \mathrm{d}x\left\{\alpha(\nabla\varphi)^4 + (\nabla^2\varphi)^2\left[\beta + \frac{D_-^2}{2b_r^3}\left(1 - \frac{a}{b_r r^2}\right) + \frac{D_-(D_r b_i + D_i b_r)}{2b_r^3}\ln\frac{a}{b_r r^2}\right]\right\} \tag{8}$$

with

$$\alpha = \frac{b_i D_r D_-}{3b_r|b|^2}, \qquad \beta = \frac{-2\,\mathrm{Re}\,(iD^* b^2)D_-}{|b|^4}.$$

We note that the terms with α and β in eq. (8) account for ϕ_{A_2}, which is the second-order contribution to $C(A^{(k)})$, while the remainder is the second-order contribution of the action integral equation (6). For global stability to this order we have to require that $\alpha > 0$, *i.e.*

$$D_- b_i > 0\,, \tag{9}$$

which we assume in the following. The desired potential is now

$$\phi = \phi_0 + \phi_1 + \phi_2\,. \tag{10}$$

It contains a Ginzburg-Landau part and a correction proportional to D_-. In fact, for $D_- = 0$, the Ginzburg-Landau part is an exact solution of eq. (4) [6]. The potential becomes singular for $a > 0$ if $|\psi| \to 0$, while $\nabla|\psi| \neq 0$. If we approach this singularity, the potential is increasingly dominated by the ∇r-terms and $\nabla^2\varphi$ (and higher-order derivatives in higher orders of our expansion) and we move out of the region of validity of our expansion. It is quite possible that the exact potential is, in fact, singular for $|\psi| \to 0$, but the existence and nature of this singularity cannot be investigated by our present methods.

We now investigate the extrema of the potential for $a > 0$.

Writing $\phi = \int \mathrm{d}x\, L(\nabla^2\varphi, \nabla\varphi, \nabla r, r)$ the extremizing fields satisfy the Euler-Lagrange equations

$$\nabla J = 0\,, \qquad \nabla\left(\frac{\partial L}{\partial\, \nabla r}\right) - \frac{\partial L}{\partial r} = 0 \tag{11}$$

with the «conserved angular momentum» $J = (\partial L/\partial\, \nabla\varphi) - \nabla(\partial L/\partial\, \nabla^2\varphi)$. Due to the appearance of $\nabla^4\varphi$ eqs. (11) may have spatially chaotic solutions, despite the existence of the conservation law, and in contrast to the usual Ginzburg-Landau potential ($b_\mathrm{i} = D_\mathrm{i} = 0$) and to our first-order result, where only $\nabla^2\varphi$ appears in eq. (11).

In higher orders of the present expansion derivatives of even higher order appear in eq. (11), and it becomes increasingly unlikely that the extremizing solutions $r(x)$, $\varphi(x)$ are nonchaotic. This property might be connected with the well-known turbulent behaviour of eq. (1) [7] and deserves further attention in particular for the case $D_\mathrm{r} b_\mathrm{r} + D_\mathrm{i} b_\mathrm{i} \to 0$, where the typical wavelengths of these chaotic solutions can be expected to become sufficiently long to validate our expansion.

In the following discussion, we shall however restrict our attention to cases where $\nabla^4\varphi$ is negligible. Then the phase can be eliminated completely by means of the conservation law and we are left with the «radial» equations described by the conserved energy

$$E = T(r, \nabla r) + U_\mathrm{eff}(r)\,, \qquad U_\mathrm{eff}(r) = 3\alpha f^4(r) + 2f^2(r)\left(D_\mathrm{r} r^2 - D_- \frac{ab_\mathrm{i}}{b_\mathrm{r}|b|^2}\right) + 2ar^2 - b_\mathrm{r} r^4\,, \tag{12}$$

where $f(r) \equiv \nabla\varphi(J, r, \nabla r = 0)$ is defined by the conservation law and the «kinetic energy» T is a function of r, ∇r which vanishes for $\nabla r = 0$. The explicit form of f and T will not be given here. We consider a plane-wave state $r = r_0$, $\nabla r_0 = 0$, $\varphi = kx$. It must be an extremum of $U_\mathrm{eff}(r)$ and must satisfy $f(r_0) = k$, which leads to $r_0 = ((a - D_\mathrm{r} k^2)/b_\mathrm{r})^{1/2}$. This state is a minimum of ϕ (and hence an attractor) if

$$\left.\frac{\partial J}{\partial\, \nabla\varphi}\right|_{r = r_0,\, \nabla\varphi = k} > 0\,, \tag{13}$$

$$\left.\frac{\partial^2 T}{\partial(\nabla r)^2}\right|_{r = r_0} > 0 \tag{14}$$

and

$$\left.\frac{\partial^2 U_\mathrm{eff}}{\partial r^2}\right|_{r = r_0} < 0\,. \tag{15}$$

Equations (13), (14) express stability against the built-up of phase gradients and amplitude gradients. Equation (13) reduces to $D_\mathrm{r} b_\mathrm{r} + D_\mathrm{i} b_\mathrm{i} > 0$. This condition, together with eq. (9), also ensures that the coefficient β in the potential is positive. In eq. (14) we may take $r_0^2 = a/b_r$ to the order we are considering and we obtain the condition

$$D_r + D_i D_- / (D_r b_r + D_i b_i) > 0 \tag{16}$$

for stability against amplitude gradients which seems not to have been considered before. Equation (15) can be evaluated explicitly and yields

$$k^2 < \frac{a(D_r b_r + D_i b_i)}{3(D_r b_r + D_i b_i) b_r + 2D_- b_i} \frac{b_r}{D_r}. \tag{17}$$

It cannot be satisfied, unless the right-hand side is positive, which is satisfied due to (9) and (13). Its vanishing specifies the borderline of the Newell-Kuramoto phase instability [8]. If the right-hand side is positive, the condition is violated nevertheless if k^2 is sufficiently large. This is the Eckhaus-Benjamin-Feir [9, 10] instability. Both instabilities are well-known, and it is gratifying and even somewhat surprising to see that the potential we constructed does correctly account for them, despite the fact that a long-wavelength expansion has been made. The reason is that (17) results from the vanishing of a second-order polynomial in k^2 which is constructed exactly in our expansion.

We close this short discussion of applications by emphasizing that the potential contains information far exceeding a linear stability analysis of attractors. *E.g.* the potential barrier $\Delta\phi$ of saddle separating a given plane-wave attractor from the neighbouring ones, differing (infinitesimally) in wavenumber, may be computed [11] by familiar methods [12]. It takes the form

$$\Delta\phi = F a^{3/2} (1 - k^2/k_c^2)^{5/2}, \tag{18}$$

where F depends on D and b, but the explicit form of F will not be given here. (We remark that, like for the real Ginzburg-Landau equation [12], $|\psi|$ has a node not at the saddle but somewhere in its neighbourhood. Therefore (18) can be calculated even though (7), (8) have a singularity for $|\psi| = 0$.) The potential barrier vanishes as the stability border (17) is approached. The height of the potential barriers furnishes a nonlinear measure of the stability of the attractors which cannot be obtained by other methods. In addition ϕ also determines the probability distribution of fluctuations, if eq. (1) is driven by weak Gaussian white noise. A discussion of this point and a detailed derivation of the results presented here must be left to a future publication [11].

Finally, we note that there have been earlier efforts to find an approximate potential for eq. (1) [4, 6, 13]. They were all based on polynomial expansions of the potential in the field variable. As can be seen from eqs. (7), (8) the potential we obtained here in the long-wavelength limit is a nonpolynomial functional. The general idea of another approach [14], based on the phase dynamics associated with eq. (1) [15], has more similarity with the present work. However the amplitude variations were adiabatically eliminated in [14], which is justified only close to the attractors, while they are treated as independent variables here, which is necessary if the description of saddles like (18) is desired. It is worth emphasizing that the potential turned out to be differentiable in the field variable (for $\psi \neq 0$). This is due to the one-dimensional character of the problem. We obtained indications that nondifferentiability may show up in higher-dimensional cases [11].

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