SINAI DISORDER: INTERMITTENCY OF RANDOM MAPS?

CHRISTIAN VAN DEN BROECK
Department of Chemistry, UCSD, La Jolla, CA 92093-0340, USA
and LUC, B-3590 Diepenbeek, Belgium

and

TAMÁS TÉL*
IFF, KFA Jülich, D-5170 Jülich
and RWTH, Aachen, D-5100 Aachen, Germany

ABSTRACT

The properties of the escape probability for a random walk on a one-dimensional lattice with disorder is discussed in terms of random maps. For the case of Sinai disorder, we observe geometric and dynamic behavior similar to that found in the case of intermittent deterministic chaos.

1. Introduction

Over the past two decades, a great effort has been devoted to the study of deterministic equations of motion that exhibit random-like behavior. In particular, much progress has been made in the study of one-dimensional maps that give rise to chaotic dynamics. Such maps also occur in contexts not directly related to dynamical systems, renormalization equations from the theory of critical phenomena being a well known example. A comparison of results obtained from different physical points of view may then be very illuminating.

In this paper we consider another type of phenomenon from the field of nonequilibrium processes, namely the case of a random walk with disorder[1] when the jump-probability of a random walker can take on a number of possible values according to a certain probability distribution. Such a model is used, e.g., to describe the motion of thermally agitated particles in random systems where the actual jump-probabilities depend on the local environment of a given lattice point. Here we restrict our attention to one-dimensional random walks with binary disorder and show that they can be discussed in the context of one-dimensional random maps. The situation is somewhat similar to that encountered previously in the study of the partition function for

*Present address: Institute for Theoretical Physics, Eötvös University, Budapest the random field Ising chain[2-8], but the physics of the problem and the details of the maps are very different.

Special attention will be paid to the case of Sinai disorder which corresponds to a disorder with local bias, but with overall average bias equal to zero[9]. We point out that the corresponding maps are characterized by geometric and dynamic properties similar to those observed in the case of intermittent deterministic chaos, even though now these properties are not the outcome of a single map, but rather of a set of random maps.

2. Escape Probability in a One-dimensional Random Walk

Let us consider a continuous time random walk, including trapping, on a one-dimensional lattice. The walk is characterized by the site dependent probability densities $\psi_i^{\pm}(\tau)$ and $\varphi_i(\tau)$. Here $\psi_i^+(\tau)$ and $\psi_i^-(\tau)$ stand for the waiting time densities to jump from site i to site $i+1$ and $i-1$, respectively, after sending time τ at site i, without first jumping to other neighbors. Furthermore, $\varphi_i(\tau)$ is the probability density that the particle is (permanently) trapped after time τ on site i.

To calculate escape probabilities, let us consider the first passage time density $F_i(t)$ to go from site i to $i+1$ for the first time at time t irrespective of any type of excursions to the left of i. Such a first passage from site i to site $i+1$ can be realized by a number n of excursions to $i-1$, each followed by a first passage back to i, and finally by a jump from i to $i+1$. Thus, one finds different contributions to $F_i(t)$. The simplest possibility is a direct jump to $i+1$ yielding $\psi_i^+(t)$. One can also have a jump to $i-1$ at some time τ_1, a first passage to i at some later time τ_2 and a jump back to $i+1$ at $t-\tau_1-\tau_2$. The probability of such excursions is $\int_0^t d\tau_1 \psi_i^-(\tau_1) \int_0^{t-\tau_1} d\tau_2 F_{i-1}(\tau_2)\psi_i^+(t-\tau_1-\tau_2)$. The next more complicated possibility differs from the previous one in that after returning to site i the particle again jumps back to its left neighbor which is followed by a first passage to i and a final jump to $i+1$. This leads to a term containing the convolution of five factors. The contribution of higher order excursions can be computed in an analogous way. Summing them all up, one obtains an integral equation relating F_i to F_{i-1} which can be converted into an algebraic relation by means of a Laplace transform[10].

This method need not be used here since we restrict our attention to the global escape probability P_i at site i obtained as the integral of $F_i(t)$ over all times: $P_i = \int_0^\infty dt F_i(t)$. The above consideration then yields for the escape probabilities :

$$P_i = \sum_{k=0}^{\infty} (w_i^- P_{i-1})^k w_i^+ \tag{1}$$

where $w_i^{\pm} = \int_0^\infty dt \psi_i^{\pm}(t)$ is the probability to jump from site i to $i+1$ (sign $+$) or to $i-1$ (sign $-$) at any time. The trapping probability at i is $\int_0^\infty dt \varphi_i(t) \equiv r_i = 1 - w_i^+ - w_i^-$.

Summing up this geometrical series leads to a recursion relation :

$$P_i = \frac{w_i^+}{1 - w_i^- P_{i-1}} \tag{2}$$

connecting the escape probabilities at sites i and $i-1$.

We shall be interested in the case of uncorrelated binary disorder, i.e., the jump rates at any site i can be of two types. The jump probabilities are either $w_i^+ = p_1$ and $w_i^- = q_1$ or $w_i^+ = p_2$ and $w_i^- = q_2$, both situations arising with equal probability, and independently of the actual position of the site. Moreover, the disorder at different sites is supposed to be uncorrelated. In this situation the recursion relation, Eq. 2, is represented by a random map[10]

$$P_{i+1} = h_1(P_i) \quad \text{or} \quad h_2(P_i) \tag{3}$$

where the branches

$$h_\epsilon(P) = \frac{p_\epsilon}{1 - q_\epsilon P} \tag{4}$$

$\epsilon = 1, 2$ are taken with equal probabilities. The form of this map is schematically given in Fig. 1. The maps h_1 and h_2 each possess a stable fixed point $\in (0,1)$, which we call P_1^* and P_2^*, respectively.

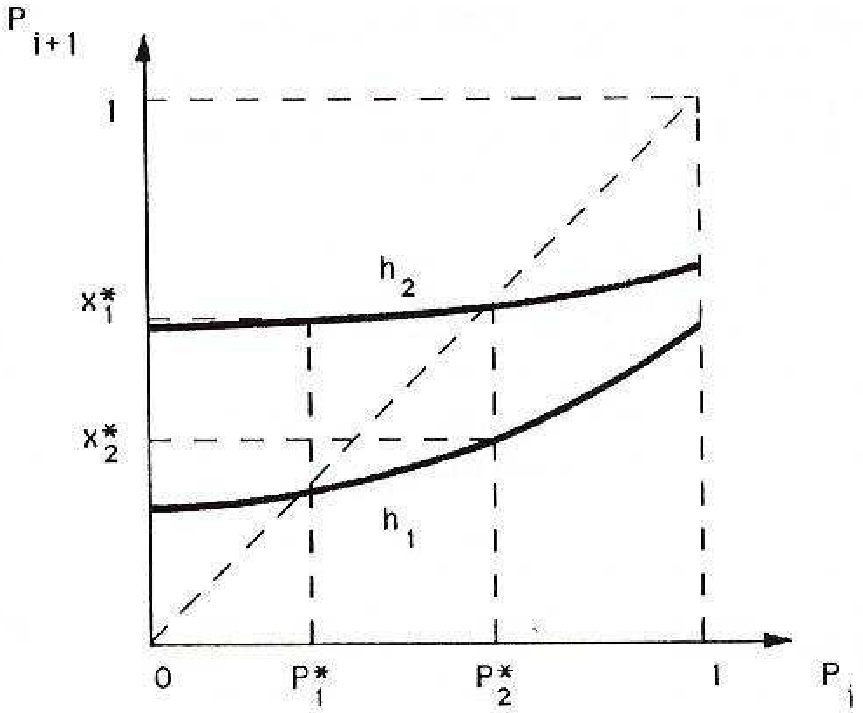

Figure 1: A schematic representation of the random map defined by Eqs. 3, 4.

The question concerning the value of the escape probability in an infinite system is answered by the random iteration of Eq. 3 for $i \to \infty$. Although both branches are, in general, separately contracting, the dynamics is irregular due to the jumps caused by the randomness, and possesses a strange attractor. Consequently, the escape probability becomes a random variable the asymptotic behavior of which is characterized by a stationary distribution, the invariant measure of the random map, with a support located between P_1^* and P_2^*. The strange attractor is a Cantor-set-like object as long as there is a gap of size $\Delta = x_1^* - x_2^* > 0$, and becomes the total interval (P_1^*, P_2^*) for vanishing gap size, i.e. when $x_1^* \leq x_2^*$. Note the analogy between this systems and the map generating the local magnetic field in a random field Ising chain (see also the contribution by U. Behn and A. Lange in this volume).

To illustrate the behavior of the random map, we study a one-parameter family determined by the set

$$p_1 = 0.8(1-r), \quad q_1 = 0.2(1-r), \quad p_2 = 0.2(1-r), \quad q_2 = 0.8(1-r), \qquad (5)$$

where r is the trapping probability, which is now constant throughout the system. Examples of the stationary distribution are given in Fig. 2 for $r = 5 \cdot 10^{-2}$ (finite gap), $r = 10^{-2}$ (no gap), and $r = 10^{-4}$ which is very close to the case of Sinai's disorder ($r = 0$), discussed in Section 4.

3. Multifractal Properties of the Stationary Distribution

As illustrated by Fig. 2a, the stationary distribution of the escape probability in the case of a Cantor-set-like attractor is clearly a multifractal. In order to study its properties one can use the observation, first applied to the random field Ising model[4], that the attractor of a random iteration like Eq. 3 is the *repeller* of the *inverted* map. More precisely, consider the *deterministic* map $f(x)$ defined by the inverses of the two branches h_1 and h_2 of the random iteration in the interval $I \equiv (P_1^*, P_2^*)$. Subsequent preimages of I taken with respect to $f(x)$ provide a refining coverage of the repeller which coincides with the attractor of the random map. This opens the possibility to apply the knowledge accumulated in the field of deterministic chaos to analyzing random systems. It is worth emphasizing that the connection mentioned above holds for the *geometric* properties of the attractor and repeller only. When studying metric properties, one has to take into account that the invariant measure on the attractor is *not* the natural measure for the repeller. Here we also shall make use of the simplifying fact that both branches of the random map are taken with equal probabilities. More general cases can be studied along the lines of Refs. 6,7.

The inverse of Eq. 3 is shown in Fig. 3 for a case with a gap of size Δ. Interval I has two preimages with respect to map $f(x)$. The preimages of these intervals are 4 smaller intervals. In general, the nth preimage of I consists of 2^n short pieces which provide for $n \to \infty$ a refining partition of the invariant set. Let us consider the lengths $\{l_i^{(n)}, i = 1, ..., 2^n\}$ of these intervals and the sum taken over these length scales raised to some power β at fixed n. When increasing n, one expects an exponential behavior

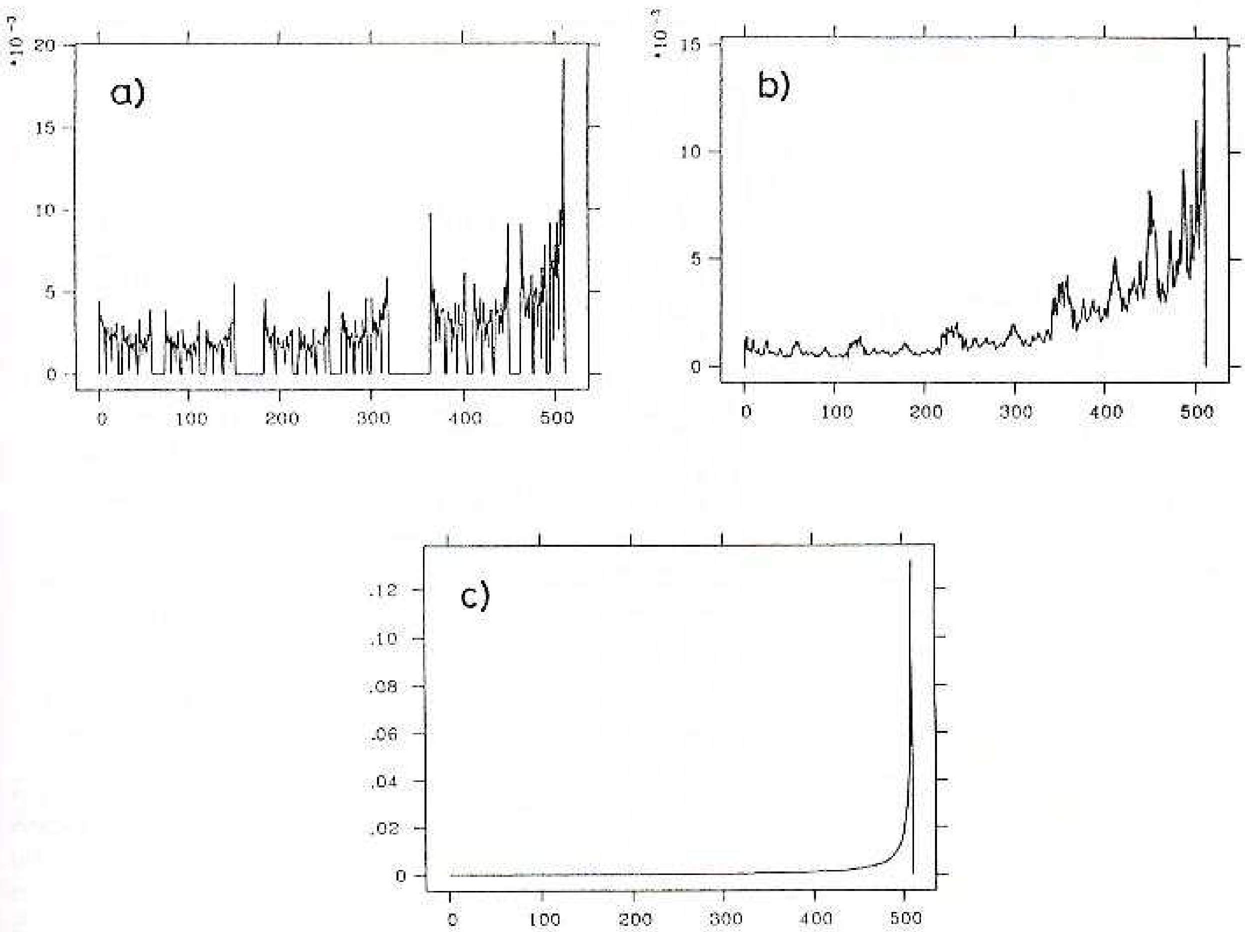

Figure 2: The invariant density of Eqs. 3, 4 for different values of the trapping probability: a) $r = 5 \cdot 10^{-2}$, b) $r = 10^{-2}$, c) $r = 10^{-4}$. The interval (P_1^*, P_2^*) has been divided into identical bins and the occupation probability of bin $j, j = 0, ..., 511$ has been plotted.

in this variable, i.e.,

$$\sum_i l_i^{(n)^\beta} \sim \exp\left(-\beta F(\beta) n\right). \tag{6}$$

The quantity $F(\beta)$ appearing in the exponent is the so-called free energy which is thus an important characteristics[11] of the length scale distribution in the coverage of the strange set, i.e., the repeller of $f(x)$ or, equivalently, the attractor of the random map.

An easy and very accurate way for computing the free energy is from the solution of an iteration scheme (generalized Frobenius-Perron equation[12–17])

$$\lambda(\beta) Q_{n+1}(x) = \sum_{x' \in f^{-1}(x)} \frac{Q_n(x')}{\mid f'(x') \mid^\beta}. \tag{7}$$

Starting with any smooth initial function $Q_0(x)$, one finds at any fixed β one single prefactor $\lambda(\beta)$ (the largest eigenvalue of the equation) so that for $n \to \infty$ a finite limiting $Q(x)$ exists, and this prefactor was shown[13,17] to be related to the free energy

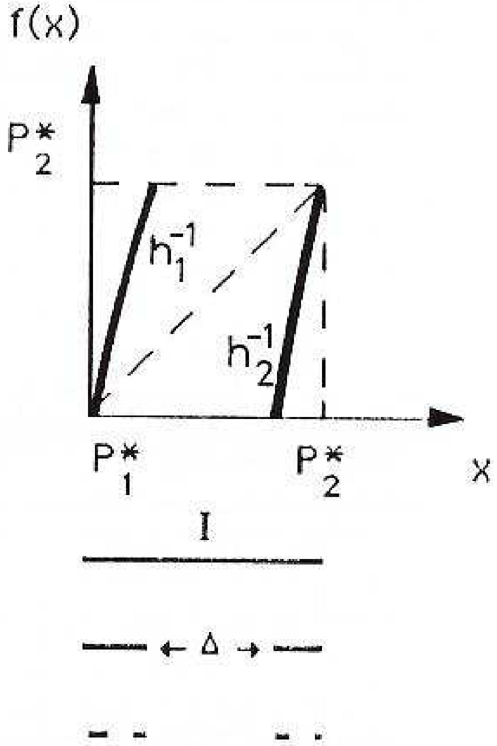

Figure 3: A schematic representation of the inverted map $f(x)$ on the interval (P_1^*, P_2^*) for a case when the invariant set is a fractal. The first preimages of the support are also indicated.

as $\lambda(\beta) = \exp(-\beta F(\beta))$. The numerical determination of λ is based on the n-fold iterate of Eq. 7, for large n. The essential part of it is the evaluation of a sum containing the derivatives of the n-fold iterated map $f^n(x)$ taken at the nth preimages of x. The dotted line of Fig. 4 shows the free energy curve obtained by iterating Eq. 7 up to $n = 18$ for the disorder given in Eq. 5, and corresponding to the invariant measures represented in Fig. 2a.

When the measure of the preimage intervals is a simple function of their lengths, the free energy also contains metric information and, in particular, the spectrum of generalized dimensions D_q. We are interested here in the invariant density of the attractor on which all intervals at level n have the same measure $\mu_i^{(n)} = 2^{-n}$, because of the equal probability in choosing the two branches of Eq. 3. Using the rule $\sum_i \mu_i^{(n)q} l_i^{(n)(1-q)D_q} \sim 1$ (see Ref. 19), one easily derives[18] that D_q fulfills the implicit equation

$$\beta F(\beta)|_{\beta=(1-q)D_q} = -q \ln 2. \tag{8}$$

Consequently, the fractal dimension D_0 is that value of β at which the free energy vanishes. (Since the natural measure of the repeller is proportional to the length scales: $\mu_i^{(n)} \sim l_i^{(n)}$, a different equation holds for the generalized dimensions taken with respect to this measure.)

We next show that the free energy is a well defined quantity even in cases

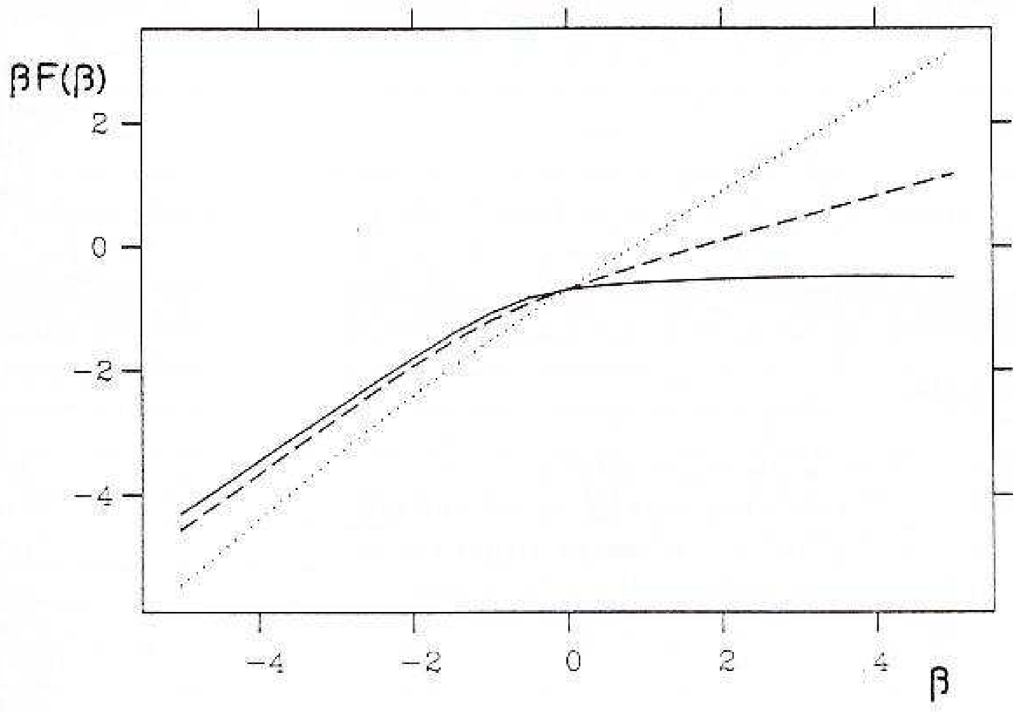

Figure 4: The function $\beta F(\beta)$ vs. β at different values of the trapping probability. Dotted line : $r = 5 \cdot 10^{-2}$, the fractal dimension of the attractor is $D_0 = 0.87$; dashed line : $r = 10^{-2}$, the curve intersects the horizontal axis at $\beta = 1.80$; full line : $r = 10^{-4}$, close to Sinai's disorder when the free energy goes to zero asymptotically from below. All curves are passing through the point $-\ln 2$ at $\beta = 0$ because of the binary character of the disorder.

when the intervals $h_1(I)$ and $h_2(I)$ overlap (no gap exists). A set of length scales $\{l_i^{(n)}, i = 1, ..., 2^n\}$ can then be obtained via the subsequent preimages of I taken with respect to the branches h_1^{-1} and h_2^{-1} . These preimage intervals overlap, but we keep considering them as *separate entities.* Note that even though the map $f(x)$ is not well defined because of its nonuniqueness, the preimages are still unique and organized in a complete binary tree. From their length scales a free energy can be obtained via Eq. 6.

Moreover, one can also find a generalization of Eq. 7 allowing to determine the free energy from an iteration. Let us consider[15,17]

$$\lambda(\beta)Q_{n+1}(x) = \sum_{\epsilon=1,2} \mid h'_\epsilon(x) \mid^\beta Q_n(h_\epsilon(x)) \qquad (9)$$

where h_ϵ denote the two branches defined by Eq. 4. (Note that Eq. 9 reduces to Eq. 7 when the inverses of h_ϵ define a single valued function $f(x)$.) The computation of the largest eigenvalue of Eq. 9 proceeds along similar lines as that of Eq. 7. The results of Fig. 4 were obtained for $r = 10^{-2}$ and $r = 10^{-4}$ in this way. We emphasize that relation (8) specifying the dimensions does not hold since the measure of the overlapping intervals is no longer uniform. They can only be determined by direct numerical methods.

It is worth mentioning, however, that D_q, as obtained from Eq. 8, has, nevertheless, some meaning. It is the order-q dimension of an *auxiliary* fractal embedded in a higher dimensional space that can be covered by circles of radii equal to $l_i^{(n)}$, all of which carry identical weights 2^{-n}. In particular, the fractal dimension of this object fulfills Eq. 8 for $q = 0$, and is found to be larger than 1. The deviation from unity is a measure of overlap.

4. Sinai Disorder

Let us first consider a site characterized by the jump-probabilities p and q to go to its right and left neighbor respectively, in the absence of trapping, i.e. $p + q = 1$, and without disorder. When $p > q$, one says that the random walk is biased to the right hand side, since motion in that direction is more likely. Such a situation could be explained by assuming that the walker corresponds to a thermally agitated particle that has to overcome a higher potential barrier to move to its left hand side than the one it encounters at its right hand side. More precisely, the difference δU in potential height can be evaluated from the well-known Arrhenius relation $\exp(-\delta U) \sim p/q$, or $\delta U \sim \ln(q/p)$. In the absence of bias, the potential difference is of course equal to zero. Returning now to our problem of binary disorder, we recall that the set of jump-probabilities to go from a site i to its right and left neighbor are chosen to be p_1 and q_1 or p_2 and q_2 with equal probability. We restrict our attention to the case of a *trap-free* system, i.e. $p_\epsilon + q_\epsilon = 1, \epsilon = 1, 2$, and note that this type of disorder implies the existence of a random local bias or local potential differences equal to $\ln(q_1/p_1)$ and $\ln(q_2/p_2)$, respectively. Of special interest is the situation, first studied in detail by Sinai[9], in which the walker has no preferred direction, at least on average. This will be the case if the effective potential remains on average at the same level, even though the realizations of the potential may be characterized by random walk excursions around this constant average level. As is intuitively clear from the above discussion, the condition for this situation reads:

$$\langle \ln \frac{q}{p} \rangle = 0. \tag{10}$$

A second condition, limiting the "strength" of the disorder (a very small or large ratio of p over q can only occur with low probablity): $\langle \ln^2(q/p) \rangle < \infty$, has also to be imposed in the general situation. The latter is trivially fulfilled for the case of binary disorder (provided none of the jump probabilities are equal to zero). Taking into account that the two types of sites occur with equal probability, the basic condition for Sinai disorder thus reduces to $\ln(q_1/p_1) + \ln(q_2/p_2) = 0$.

This condition has the following interesting consequence for the random maps discussed above. Map h_1 must possess then, besides P_1^*, an additional fixed point which coincides with $P_2^* = 1$, the fixed point of h_2 (remember that we consider a trap-free situation: $p_\epsilon = 1 - q_\epsilon$; cf. Fig. 1). The slopes of the maps at unity are equal to q_1/p_1 and q_2/p_2, respectively. We thus conclude that in case of Sinai disorder the product of these slopes is equal to 1.

It is instructive to study the dynamics of the individual maps $P_{i+1} = h_\epsilon(P_i), \epsilon = 1, 2$ in the vicinity of the fixed point at $P = 1$. One finds for the logarithm of the deviation

$$\ln(1 - P_{i+1}) = \ln(1 - h_\epsilon(P_i)) = \ln(1 - P_i) + \ln\frac{q_\epsilon}{p_\epsilon} + \text{ higher order terms.} \qquad (11)$$

We conclude that Sinai disorder corresponds to the critical situation at which the drift averaged over the choice of the random map vanishes: $\ln(q_1/p_1) + \ln(q_2/p_2) = 0$. The logarithm of small deviations from the fixed point 1 then undergoes an unbiased random walk. The left fixed point P_1^* is unstable under random iterations because of the possibility to jump over to the upper branch h_2. The global dynamics will therefore always return for long periods to the vicinity of the fixed point at unity, and the natural measure is singular at $P = 1$. In the family defined by Eq. 5, Sinai disorder corresponds to $r = 0$, and the observation of Fig. 2c supports the divergence of the density at unity. The situation is very similar to that found in deterministic maps exhibiting strong intermittency where the natural density is also singular at the intermittent point[20]. Moreover, in our case, the approach of the fixed point at unity can be estimated by invoking the scaling for a diffusion process $\ln^2(1 - P_i) \sim i$ or $P_i = 1 - \exp(-\text{constant}\sqrt{i})$, i.e., the approach is *slower then exponential.* Exponential behavior is restored as soon as the Sinai condition is violated. In this case, an average bias $\langle \ln(q/p) \rangle$ different from zero appears. When $\langle \ln(q/p) \rangle > 0$ the fixed point of the random map at 1 is no longer stable which leads to the existence of a strange attractor.

The analogy with intermittency can also be seen in the free energy function. In intermittent deterministic maps one observes preimage intervals with length scales decreasing slower than exponentially[21,17]. In our case the preimage of $(P_1^*, 1)$ is equal to itself and the length scale does not change at all. Both conditions imply that for large positive β, which tests the scaling of the longest intervals, the free energy must be *zero.* For one-dimensional maps, however, the preimages exactly cover the total interval, therefore, the free energy has to vanish at $\beta = 1$ and stay identically zero above. This is different for random maps with Sinai disorder because of the overlapping of the preimages, and we expect $F(\beta)$ to be negative at any finite value of β (note that the function $\beta F(\beta)$ must be monotonic increasing), and to approach zero for $\beta \to \infty$. This is confirmed by the numerical results for the free energy for a case very close to Sinai disorder, c.f., Fig. 4.

5. Discussion

We have presented a description in terms of random maps for the escape in one-dimensional random walk on a disordered lattice. In the case of Sinai disorder, we observe intermittent behavior and a corresponding anomaly of the free energy. Even though we were led to consider this situation in the specific context of Sinai disorder, it is clear that we can formulate a Sinai condition for a general random map. As seen above, the essential point is that the *average logarithm of the slopes is zero* at the fixed point. Therefore, we conjecture that the intermittent-like behavior observed for Sinai disorder will occur for random maps that have a common fixed point such that their corresponding slopes s_ϵ obey the condition $\langle \ln s \rangle = 0$.

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