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TRANSIENT CHAOTIC MIXING IN OPEN HYDRODYNAMICAL FLOWS

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We investigate particle motion and mixing in time-dependent open flows with uniform inflow and outflow velocities. The dynamics is typically chaotic, of the transient type that can be observed on finite time scales only. There exists an underlying chaotic set consisting of an infinity of unstable tracer orbits, restricted to a finite domain of the flow. Transient dynamics is accompanied by persistent fractality. The latter is reflected by tracer patterns and by the singularity distribution of the particles' time delay function. As an illustrative example, we consider a planar incompressible flow modeling the leapfrogging motion of two vortex rings.

The advection of particles by hydrodynamical flows is an ubiquitous phenomenon. By particle we mean, e.g. a granule of dye when mixing different colors, a piece of foam on the surface of a flow, a tiny cell advected by the blood, or a balloon in the atmosphere, i.e. light granules of small extension. If the particle takes on the velocity of the flow very rapidly, i.e. inertial effects are negligible, we call the advection passive, and the particle a passive tracer. It became clear in the last decade that passive advection even in simple time-dependent flows is typically *chaotic* [Aref, 1986; Chaiken *et al.*, 1986; Ottino, 1989; Muzzio *et al.*, 1992; Wiggins, 1992; Solomon & Gollub, 1988, Solomon *et al.*, 1994; Sommerer & Ott, 1993, Sommerer, 1994; Jones *et al.*, 1989; Young & Jones, 1991]. In all chaotic phenomena, there is a well defined abstract geometrical structure underlying the dynamics. It is a unique feature of chaotic advection in planar flows that these structures then become observable by the naked eye in the form of *spatial patterns.* The investigations so far were mainly concentrated on chaotic advection in confined geometries which are of great importance for understanding mixing in closed containers [Ottino, 1989].

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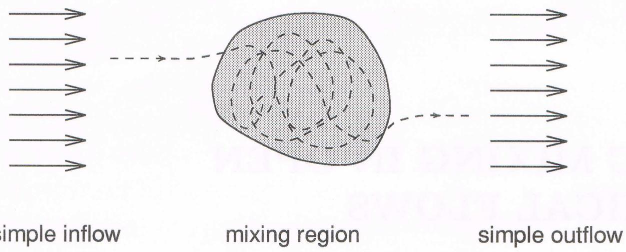

Fig. 1. Schematic diagram of open flows investigated and a corresponding tracer trajectory. Time dependence of the flow is restricted to a finite region where mixing of passive tracers can take place due to chaotic motion. The inflow and outflow regions are asymptotically simple. The shape of a typical tracer trajectory (dotted line) exhibiting chaotic motion in the mixing region is shown.

Here we address the question of how advection manifests itself in *open* flows. We consider cases similar to that of a flow around an obstacle when the inflow and outflow far away from this object (background flow) is *simple* and *time independent* [Rom-Kedar *et al.*, 1990; Shariff *et al.*, 1991; Jung & Ziemniak, 1992; Jung *et al.*, 1993; Ziemniak *et al.*, 1994; Péntek *et al.*, 1995a, 1995b, 1995c]. More generally, the interesting, time-dependent behavior of the flow is assumed to be restricted to a finite region (see. Fig. 1) that corresponds to the wake of the obstacle in the previous example. This will be called the *mixing region* where the advection can be highly nontrivial.

It is worth emphasizing that a complicated flow pattern is not at all required for the latter. Even simple forms of time dependence are sufficient. In this paper we suppose that the velocity field is periodic in time.

Tracer trajectories are typically complicated but this complex behavior is restricted to the mixing region only. Outside of this, the trajectories are simple curves (Fig. 1). Dealing with tracers injected into the flow far away upstream, one can regard the advection problem as a scattering process, with the advected particles being "scattered" on the finite region of nontrivial mixing. Therefore, chaos is necessarily restricted to a finite region both in space and time. We claim that this *transient chaos* [Tél, 1990] is the only form of chaos which can appear in the situation studied. The interpretation of certain advection phenomena, observable also in experiments, is thus possible in this framework only. Concepts taken over from the theory of transient chaos and chaotic scattering [Ott & Tél, 1993] shed new light on these phenomena, see [Jung & Ziemniak, 1992; Jung *et al.*, 1993; Ziemniak *et al.*, 1994; Péntek *et al.*, 1995a, 1995b, 1995c].

The complicated form of trajectories implies a long time spent in the mixing region. In other words, tracers can be temporarily trapped there. Due to the incompressibility of the fluid, which can be assumed in real-world flows, there cannot be attractors in this region. *All* particles escape the mixing region sooner or later. Trapped particles have some *time delay* τ relative to the background flow: the more complicated is the trajectory the longer is τ. Observation of several trajectories defines the *time delay distribution*: $P(\tau)$ [Ott & Tél, 1993]. The initial points of these trajectories can be taken from a closed domain of the flow or along a straight line in the inflow region. The quantity $P(\tau)d\tau$ is the probability to find a particle with time delay in the interval $(\tau, \tau + d\tau)$. $P(\tau)$ must tend to zero for large times. It typically decays exponentially,

$$P(\tau) \sim \exp(-\tau/\overline{\tau}), \qquad (1)$$

where $\overline{\tau}$ is the average time delay. The quantity $\overline{\tau}$ can then also be considered as the *average lifetime of chaos* and generally is on the order of a few times the flow period. The fact that chaos can be observed on times scale $\overline{\tau}$ only is a novel feature compared with chaotic advection in closed containers where $\overline{\tau}$ is formally infinite.

A more detailed characterization of the trapping process is based on the observation of individual trajectories. Let us inject tracer particles into the flow along a straight line perpendicular to the inflow velocity and determine the time τ they spend in the mixing region. This defines a function $\tau(y)$ where y denotes the initial position along the line of injection. A clear-cut sign of chaotic tracer scattering is the rather *irregular* form of the time delay function. An example is shown in Fig. 2. The different time delays of neighboring points reflect high sensitivity to initial conditions that is usually considered as a criterion of chaotic behavior. An irregular scattering function is thus characteristic of chaotic advection in open flows, which can be regarded to be as typical as stationary chaos is in closed containers. The time delay distribution $P(\tau)$ is a global characteristic of the scattering process and can easily be derived from the time delay $\tau(y)$ of the initial trajectories: $P(\tau)d\tau$ is proportional to the number of tracers whose time delay falls into the interval $(\tau, \tau + d\tau)$.

Because of the strong fluctuation of the delay times and the divergence of nearby trajectories, the

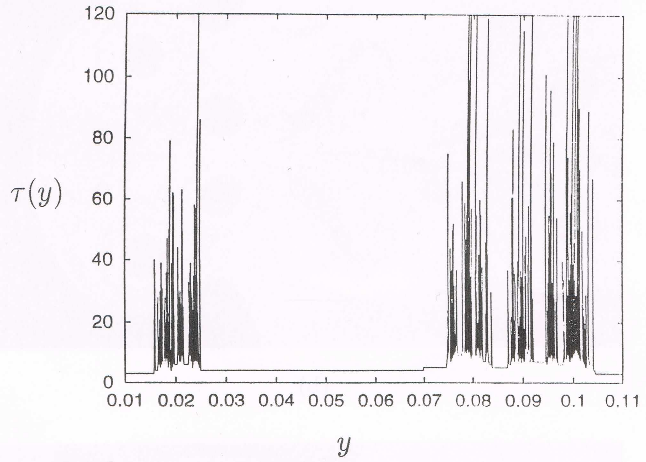

Fig. 2. Time delay function $\tau(y)$ versus initial coordinate y along a straight line perpendicular to the flow in the incoming asymptotic region. The large time delay values mark initial conditions with a long trapping in the mixing region. Note the sensitive dependence of τ on y, a criterion for chaotic behavior. The system used is the one described in the text and also in the caption to Fig. 3.

evolution of tracer *ensembles* provides us with a more natural and experimentally realizable characterization of the advection dynamics than that of individual trajectories. Such an ensemble can be a small droplet or a long band of colored dye. These ensembles always trace out well defined fractal structures in and behind the mixing region.

We found it especially useful to study the advection dynamics in a fluid whose color is artificially changed in the inflow region after certain periods of time. This corresponds to a color coding of the tracers' initial conditions and provides us with more information than just the fractal pattern since coloring encodes time delay properties, too. It also implies the definition of a nontrivial distribution accumulating on the fractal pattern. The motion of domains of different colors then clearly illustrates the transport and mixing process (Fig. 3). The bulk of the colored domains, which do not enter the mixing region, will be smoothly deformed and transported away. A finite part of the domains will, however, be captured and seemingly isolated from the bulk of the same color. In fact, there is always a *narrow filament*, i.e. a range of strong stretching, connecting this part with the bulk. The captured domain will then be split further in other strongly folded subdomains connected by narrow filaments. The larger domains escape the mixing region after a few repetitions of the velocity field's period but the filaments remain longer. The color coding indicates that tracers injected several periods before are still present in the mixing region. We thus conclude that particles with long time delays come close to regions of intense stretching and folding. Filaments of different colors and of ever-decreasing width *accumulate* in such regions. Filaments of finite width are transported away by the flow.

At this point, it is worth presenting an illustrative example in some detail. We emphasize that the features mentioned above are valid for *any* incompressible planar flow irrespectively of its viscosity. Our example will be taken from the realm of ideal inviscid fluids because even the finest fractal features can then be investigated in a numerical simulation. The particular system used to plot Fig. 3 is a model of the so-called 'leapfrogging' motion of smoke rings [Shariff *et al.*, 1988; Shariff & Leonard, 1992; Saffman, 1992]. If the rings have the same sense of rotation and move along the same axis, the rear vortex ring attempts to pass through the front one. The leading ring then widens due to the mutual interaction and travels more slowly. Simultaneously, the other ring shrinks, travels faster and penetrates the first one. This process is then repeated continuously, with some period T. We studied [Péntek *et al.*, 1995a, 1995b] the two-dimensional analogue of this process: advection in the field of two pairs of coaxial ideal point vortices of the same strength, which also exhibit a strictly periodic motion of period T. Figure 3 shows the pattern in a fluid whose color has been changed periodically with the same T. A frame co-moving with the center of mass of the vortex pairs has been used.

The key observation in interpreting patterns like those of Fig. 3 is the existence of a *chaotic set* in the mixing region. It is the union of all *closed orbits*, including periodic ones, that never escape the mixing region but are all unstable. In view of Eq. (1), these must be rather exceptional ones and not even their union covers a finite portion of the plane. They are, however, *infinite* in number and form a *fractal subset* of the mixing region. There are powerful numerical methods to determine the chaotic set [Nusse & Yorke, 1989]. Figure 4 displays the results obtained for the same system and at exactly the same instants as in Fig. 3. Note that certain parts of the set, like the ones denoted by A, A', ..., D, D', are approximately direct products of two Cantor sets. The full set is moving on a stroboscopic map, as illustrated by Fig. 4, and its motion is also periodic.

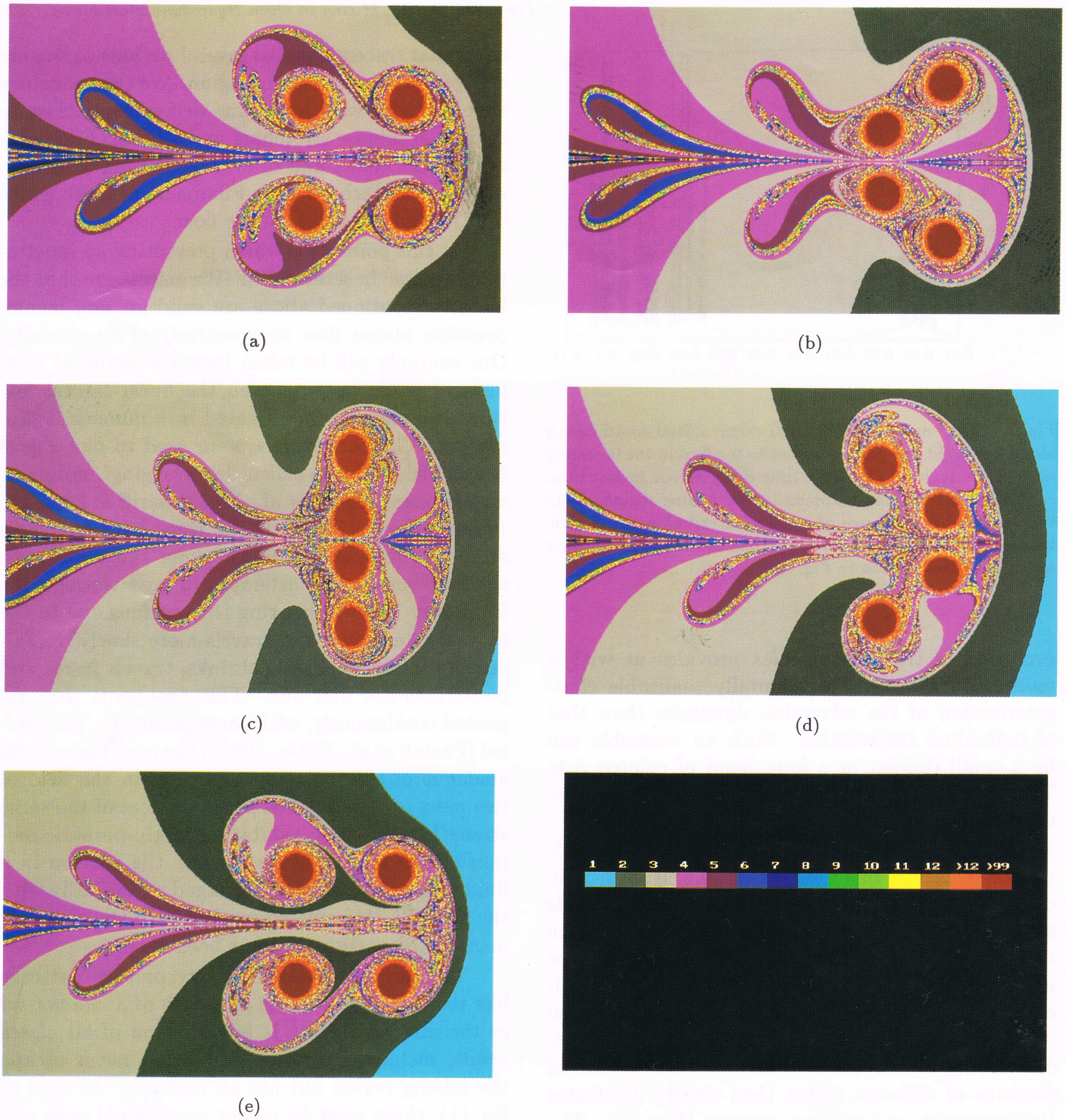

Fig. 3. Leapfrogging motion of two ideal vortex pairs of equal strength in a fluid whose color is changing, displayed in a frame comoving with the center of mass of the pairs. The rectangle shown can be considered as the mixing region. The vortex centers sit in the midpoints of the red domains. Dye is injected upstream along a vertical line on the right hand side of the region shown. The boundary between domains of different colors is thus the image of this line after a certain time. Part (a) displays the distribution of differently colored tracers at $t_0 = 99T$. The index n used in the legend of color coding indicates the color of tracers injected in the period $(n-1)T \leq t < nT$. The dark red regions (marked by > 99) correspond to the vortex cores not reachable by tracers coming from outside, and can formally be considered as regions of infinite lifetimes. Light red tracers were injected more than 12 periods before (marked by > 12) and the color being injected currently is blue. The bulk of a domain of a given color is transported away, but filaments of it remain trapped around the vortices. The thickness of these filaments is strongly decreasing with time: yellow and blue is much less dominant around the vortices than magenta. Despite of their complex forms, each domain of a given color is *single connected.* Notice that the boundary between different colors converges in certain regions to a fractal curve that appears in this numerical plot as a multicolored filamental structure. Figures (b)–(e) show how these material regions evolve in time by taking snapshots at $t - t_0 = 0.356T$, $0.5T$, $0.644T$, T. The large light gray domain below the upper vortices in Fig. 3(a) appears to be cut from the rest of the light gray domain in Fig. 3(b) and transported further to the upper left lobe of Fig. 3(e). Due to the periodicity of the flow, Fig. 3(e) is the same as Fig. 3(a) but the color code has been shifted by 1.

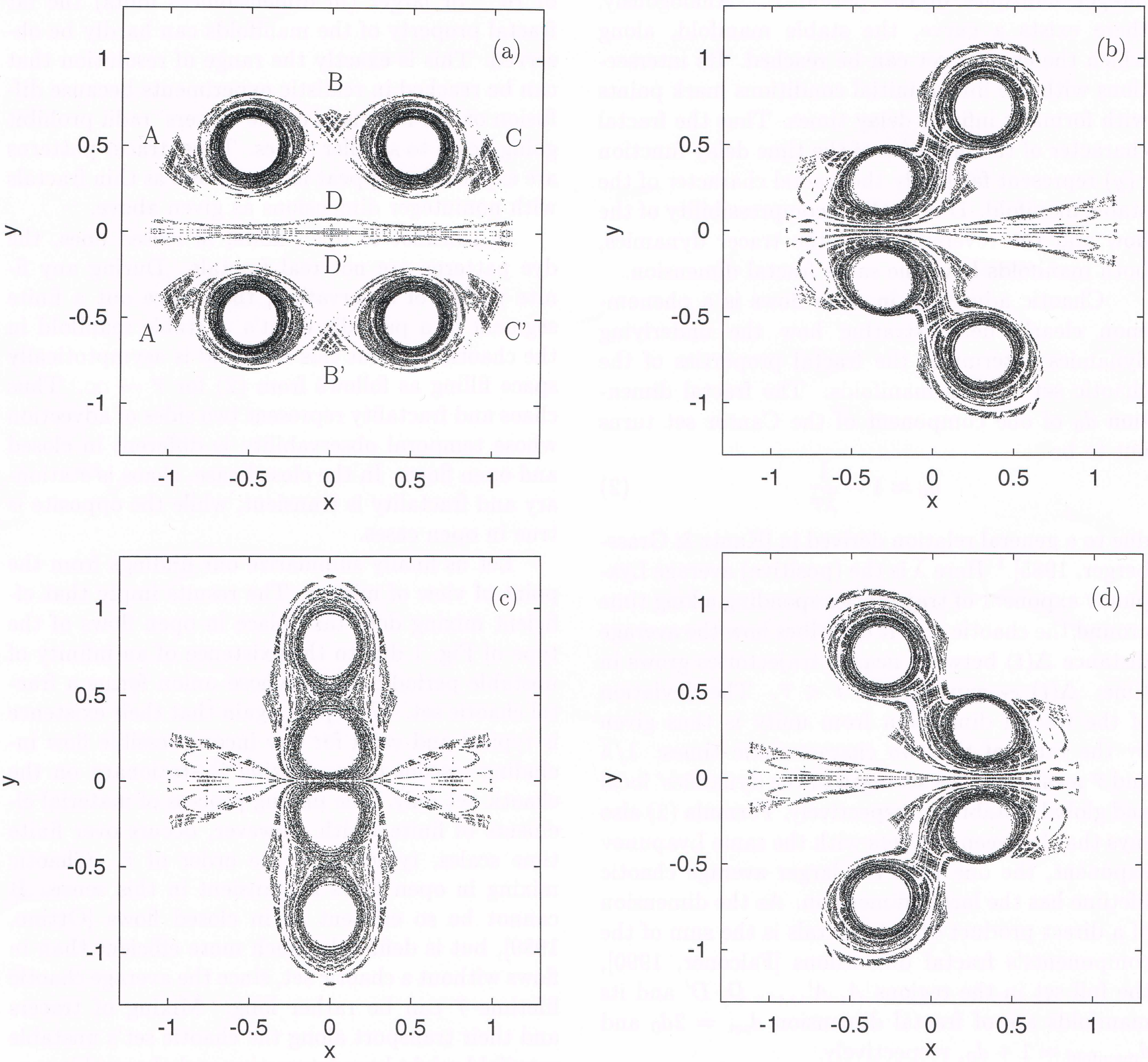

Fig. 4. The chaotic set at times $t = 0$, $0.356T$, $0.5T$, $0.644T$ (mod T). Mixing takes place in the vicinity of this set. Trajectories from its neighborhood are trapped for a long time. Notice the direct product structure of some parts (see regions A, B, C, D and their mirror images $A', \ldots, D'$) and the dense spirals around the vortex cores. The multicolored filamental region of the previous figure corresponds to the unstable manifold of this set.

To find the relation to Fig. 3, we first notice that tracers of long lifetime can only be the ones coming close to the chaotic set. The chance that they exactly hit the set is zero since the latter is a fractal whose area is zero. Just as an unstable hyperbolic fixed point has a curve along which particles leave its neighborhood (see [Ottino, 1989]), the chaotic set also has such a curve that is called its *unstable manifold.* This manifold is a rather complicatedly winding curve extending downstream to infinity, whose intersection with any straight line is a fractal. In view of this, we can say that particles not escaping the mixing region too rapidly accumulate on a close neighborhood of the unstable manifold of the chaotic set. Similarly, the multicolored filamental structure of Fig. 3 traces out the unstable manifold, too. Of course, the shape of the manifold is also changing in time but returns to itself after integer multiples of the period T. Analogously, there exists a curve, the stable manifold, along which the chaotic set can be reached. Its intersections with the line of initial conditions mark points with formally infinite delay times. Thus the fractal character of the infinities of the time delay function $\tau(y)$ represent faithfully the fractal character of the stable manifold. Due to the incompressibility of the flow and the reversibility of the tracer dynamics, both manifolds have the same fractal dimension.

Chaotic advection in open flows is a phenomenon clearly demonstrating how the underlying dynamics determines the fractal properties of the chaotic set and its manifolds. The fractal dimension d_0 of one component of the Cantor set turns out to be

$$d_0 \approx 1 - \frac{1}{\overline{\lambda}\overline{\tau}} \qquad (2)$$

due to a general relation derived in [Kantz & Grassberger, 1985].[1] Here $\overline{\lambda}$ is the (positive) average Lyapunov exponent of trajectories spending a long time around the chaotic set. It describes how the average distance $\Delta(t)$ between nearby trajectories grows in time: $\Delta(t) \sim \exp(\overline{\lambda}t)$, for $t < \overline{\tau}$. The deviation of the fractal dimension from unity is thus given by the ratio of the two characteristic times: $1/\overline{\lambda}$ and $\overline{\tau}$ measuring the strength of the dynamics' local and global instability, respectively. Formula (2) also says that between two sets with the same Lyapunov exponent, the one with the larger average chaotic lifetime has the larger dimension. As the dimension of a direct product of two fractals is the sum of the components's fractal dimensions [Falconer, 1990], the full set in the regions $A, A', \ldots, D, D'$ and its manifolds are of fractal dimension $d_{\text{set}} = 2d_0$ and $d_{\text{manifold}} = 1 + d_0$, respectively.

The chaotic sets typically also contain nonhyperbolic components. In the case of the leapfrogging vortex pairs, they are situated around the vortex cores where the point density is much higher than further away. This is an indication of the fact that in these regions, the chaotic set's partial fractal dimension tends to unity in the limit of extremely fine resolution [Lau *et al.*, 1991]. The escape rate and the average Lyapunov exponent are expected to be zero on this component [Christiansen & Grassberger, 1993]. It is, however, interesting to note that on the practically relevant length scale of 10^{-4} or larger (in dimensionless units) the fat fractal property of the manifolds can hardly be observed. This is exactly the range of resolution that can be reached in realistic experiments because diffusion or the finiteness of the tracers' radii prohibit going down to smaller scales. Thus, tracer patterns are expected to appear in open flows as thin fractals with noninteger dimensions as given above.

In contrast to open flows, in closed ones, the dye patterns are not real fractals. During any finite period of observation, they trace out a finite segment of a periodic orbit's unstable manifold in the chaotic sea, but this manifold is asymptotically space filling as follows from (2) for $\overline{\tau} \to \infty$. Thus chaos and fractality represent two sides of advection whose temporal observability is different in closed and open flows. In the closed case, chaos is stationary and fractality is transient, while the opposite is true in open cases.

Let us finally summarize our findings from the point of view of mixing. The results imply that efficient mixing does take place in open flows of the type of Fig. 1 due to the existence of an infinity of unstable periodic orbits whose union forms a fractal chaotic set. We repeat again that their existence is typical and valid for any incompressible flow including viscous ones. Mixing is stationary on the chaotic set only. The mixing process of material elements of finite width, however, occurs over finite time scales, typically of the order of $\overline{\tau}$. Chaotic mixing in open flows is transient in this sense. It cannot be so efficient as in closed flows [Ottino, 1989], but is definitely much more efficient than in flows without a chaotic set, since the average chaotic lifetime $\overline{\tau}$ can be rather long. Mixing of tracers and their transport along the chaotic set's unstable manifold might have interesting and observable consequences in open flows, e.g. in autocatalytic chemical reactions, in an analogous way to that which they have in closed containers [Metcalfe & Ottino, 1994; Epstein, 1994].

Our findings have consequences for everyday life phenomena, too. Examples can be fractal patterns of floaters (e.g. pollutants) in the wake of a bridge's pillar or patterns traced out on a satellite picture in a cloud layer lying in a horizontal air current of constant speed behind a tall mountain. Whenever one observes these or similar patterns, he can conclude that they are consequences of the transiently chaotic dynamics and mixing of the advected particles. Conversely, if the monitoring of a few particles in any open flow indicates chaotic

[1]Note, that the Eq. (1) holds exactly for d_1, but since d_0 is typically close to d_1 this is a very good approximation.

motion before a simple asymptotic regime is reached, we can be sure that tracer ensembles generate in that flow fractal patterns on resolutions typically accessible in these phenomena.

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References

Aref, H. [1986] "Chaotic advection in a Stokes flow," *Phys. Fluids* **29**, 3515–3521.

Chaiken, J., Chevray, R., Tabor, M. & Tan, Q. M. [1986] "Experimental study of Lagrangian turbulence in a Stokes flow," *Proc. Roy. Soc. London* **A408**, 165–174.

Christiansen, F. & Grassberger, P. [1993] "Escape and sensitive dependence on initial conditions in a symplectic repeller," *Phys. Lett.* **A181**, 47–53.

Epstein, I. R. [1994] "The consequences of imperfect mixing in autocatalitic chemical and biological systems," *Nature* **374**, 321–327.

Falconer, K. [1990] *Fractal Geometry* (J. Wiley, Chichester).

Jones, S., Thomas, O. & Aref, H. [1989] "Chaotic advection by laminar flow in a twisted pipe," *J. Fluid. Mech.* **209**, 335–357.

Jung, C. & Ziemniak, E. [1992] "Hamiltonian scattering chaos in a hydrodynamical system," *J. Phys.* **A25**, 3929–3943.

Jung, C., Tél, T. & Ziemniak, E. [1993] "Application of chaotic scattering to particle transport in a hydrodynamical flow," *Chaos* **3**, 555–568.

Jung, C. & Ziemniak, E. [1994] "Streaklines in an open hydrodynamical flow," in *Fractals in the Natural and Applied Sciences* ed. Novak, M. M. (North Holland, Amsterdam), pp. 211–219.

Kantz, H. & Grassberger, P. [1985] "Repellers, semi-attractors, and long-lived chaotic transients," *Physica* **D17**, 75–86.

Lau, Y. T., Finn, J. M. & Ott, E. [1991] "Fractal dimension in nonhyperbolic chaotic scattering," *Phys. Rev. Lett.* **66**, 978–981.

Metcalfe, G. & Ottino, J. M. [1994] "Autocatalytic processes in mixing flows," *Phys. Rev. Lett.* **72**, 2875–2878.

Muzzio, F. J., Swanson, P. D. & Ottino, J. M. [1992] "Mixing distributions produced by multiplicative stretching in chaotic flows," *Int. J. Bifurcation and Chaos* **2**(1), 37–50.

Nusse, H. E. & Yorke, J. [1989] "A procedure for finding numerical trajectories on chaotic saddles," *Physica* **D36**, 137–156.

Ott, E. & Tél, T. [1993] "Chaotic scattering: An introduction," *Chaos* **3**, 417–426.

Ottino, J. M. [1989] *The Kinematics of Mixing: Stretching, Chaos and Transport* (Cambridge University Press, Cambridge).

Péntek, Á., Tél, T. & Toroczkai, Z. [1995a] "Chaotic advection in the velocity field of leapfrogging vortex pairs," *J. Phys.* **A28**, 2191–2216.

Péntek, Á., Tél, T. & Toroczkai, Z. [1995b] "Fractal tracer patterns in open hydrodynamical flows: The case of leapfrogging vortex pairs," *Fractals* **3**, 33–53.

Péntek, Á., Toroczkai, Z., Tél, T., Grebogi, C. & Yorke, J. [1995c] "Fractal boundaries in open hydrodynamical flows: Signatures of chaotic saddles," *Phys. Rev.* **E51**, 4076–4088.

Rom-Kedar, V., Leonard, A. & Wiggins, S. [1990] "An analytical study of transport, mixing and chaos in an unsteady vortical flow," *J. Fluid. Mech.* **214**, 347–394.

Saffman, P. G. [1992] *Vortex Dynamics* (Cambridge Univ. Press, Cambridge).

Shariff, K., Leonard, A., Zabusky, N. J. & Ferziger, J. H. [1988] "Acoustic and dynamics of coaxial interacting vortex rings," *Fluid. Dyn. Res.* **3**, 337–343.

Shariff, K., Pulliam, T. H. & Ottino, J. M. [1991] "A dynamical system analysis of kinematics in the time-periodic wake of a circular cylinder," *Lect. Appl. Math.* **28**, 613–646.

Shariff, K. & Leonard, A. [1992] "Vortex rings," *Ann. Rev. Fluid. Mech.* **24**, 235.

Solomon, T. H. & Gollub, J. P. [1988] "Chaotic particle transport in time-dependent Rayleigh-Benard convection," *Phys. Rev.* **A38**, 6280–6286.

Solomon, T. H., Weeks, E. R. & Swinney, H. L. [1994] "Chaotic advection in a two-dimentional flow: Levy flights and anomalous diffusion," *Physica* **D76**, 70–84.

Sommerer, J. C. & Ott, E. [1993] "Particles floating on a moving fluid: A dynamically comprehensible physical fractal," *Science* **259**, 335–339.

Sommerer, J. C. [1994] "Fractal tracer distributions in complicated surface flows: An application of random maps to fluid dynamics," *Physica* **D76**, 85–98.

Tél, T. [1990] "Transient chaos," in *Directions in Chaos* ed. Hao Bai-Lin (World Scientific, Singapore), Vol. 3, pp. 149–221.

Wiggins, S. [1992] *Chaotic Transport in Dynamical Systems* (Springer).

Young, W. R. & Jones, S. [1991] "Shear dispersion," *Phys. Fluids* **A3**, 1087–1101.

Ziemniak, E., Jung, C. & Tél, T. [1994] "Tracer dynamics in open hydrodynamical flows as chaotic scattering," *Physica* **D76**, 123–146.