

On the Validity of the β -Plane Approximation in the Dynamics and the Chaotic Advection of a Point Vortex Pair Model on a Rotating Sphere

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ABSTRACT

The dynamics of modulated point vortex pairs is investigated on a rotating sphere, where modulation is chosen to reflect the conservation of angular momentum (potential vorticity). In this setting the authors point out a qualitative difference between the full spherical dynamics and the one obtained in a β -plane approximation. In particular, dipole trajectories starting at the same location evolve to completely different directions under these two treatments, despite the fact that the deviations from the initial latitude remain small. This is a strong indication for the mathematical inconsistency of the traditional β -plane approximation. At the same time, a consistently linearized set of equations of motion leads to trajectories agreeing with those obtained under the full spherical treatment. The β -plane advection patterns due to chaotic advection in the velocity field of finite-sized vortex pairs are also found to considerably deviate from those of the full spherical treatment, and quantities characterizing transport properties (e.g., the escape rate from a given region) strongly differ.

1. Introduction

In geophysical fluid dynamics the β -plane approximation plays a central role as the basis of large-scale quasigeostrophic dynamics. The idea goes back to a heuristic reasoning of Rossby (1939), who found that for a shallow fluid on a rotating sphere only the local normal component of the angular velocity vector is relevant. Furthermore, if the scale of the motion is sufficiently small in the north–south direction, then the only effect of the sphericity is the variation of the normal component of the angular velocity with latitude, which made a proper and convenient description of planetary (Rossby) waves possible. By now, it became a widely spread approach to use a locally flat Cartesian coordinate system in the quasigeostrophic context (e.g., Pedlosky 1979; Cushman-Roisin 1994; Vallis 2006; Kundu et al. 2012).

The theory is based on the vorticity equation, and a precise scale analysis of the problem, including curvature effects, carried out by Pedlosky (1979, section 6.3), shows that the assumption of a flat geometry with

a linearly latitude-dependent angular velocity component is indeed a consistent approximation for quasigeostrophic motion. Pedlosky also shows, however, that the same scale analysis of the fluid dynamical momentum equation (Euler equation), restricted to small meridional displacements, leads to the appearance of a term arising from the curvature of the sphere. This new term is negligible only around the equator. This indicates that the β -plane idealization, in which only the variation of the angular momentum is taken into account but the spherical geometry is not, is not entirely consistent. Nevertheless, this mathematical inconsistency turns out to be of marginal importance in some applications (Pedlosky 1979). At the same time, criticism appeared about the universal applicability of the β -plane approach in the aforementioned widely used sense (e.g., Gill 1982; Verkley 1990; Ripa 1997; Harlander 2005; Paldor and Sigalov 2006; Paldor 2007). Our results, focusing on the effects of the mathematical inconsistency, might provide a further contribution to these considerations. Such curvature-related effects are particularly relevant in problems related to material transport, since in this context the use of the momentum equation is unavoidable.

Our aim in this paper is to get insight into the nature of the β -plane approximation via a simple point vortex

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model. An advantage of this is that the problem arises in a dynamics that can be described by an ordinary differential equation, which makes an easy treatment possible. Nevertheless, many features—rotation, spherical geometry, and vorticity—are common with the hydrodynamical settings studied in the literature mentioned so far, which concentrated on Eulerian properties. To our knowledge, the effect of the β -plane approximation on Lagrangian properties, related to material transport, has not yet been investigated. Our model enables us to study also the problem of advection in a relatively simply way and leads to the conclusion that the results of the β -plane approximation for transport properties might considerably differ from the correct ones even if the corresponding velocity fields do not differ too much.

There is a current interest in the dynamics of point vortices on a rotating sphere (Newton 2001; Newton and Ross 2006; Jamalooden and Newton 2006; Newton and Shokraneh 2006; Newton and Sakajo 2007; Newton and Shokraneh 2008; DiBattista and Polvani 1998) because of an increasing focus on environmental and climatic aspects of hydrodynamical flows. An exact equation of motion is known in the form of a system of integro-differential equations only (Bogomolov 1977). In the form of differential equations, approximate forms are available. To take into account the conservation of the fluid's angular momentum, the potential vorticity, a phenomenological approach in the shallow-water setup is the modulation of the vortex circulations.

Motivated by vortices moving over sloping bottoms, there is extended literature (Makino et al. 1981; Zabusky and McWilliams 1982; Hobson 1991; Velasco Fuentes and van Heijst 1994; Benczik et al. 2007) on vortex dynamics in which the circulation Γ_j of any vortex is made linearly location dependent in the spirit of the β -plane approximation (Velasco Fuentes and van Heijst 1994). Although such point vortices are not exact solutions of the hydrodynamical equations (in particular, of the shallow water equations), they have been shown to be useful in understanding several features, for example, the existence of modonlike excitations (Makino et al. 1981; Hobson 1991). In a number of experiments, laboratory generated vortices on a topographic β plane (sloping bottom) could be approximated quite well by the modulated point vortex model over a considerable time span (Kloosterziel et al. 1993; Velasco Fuentes and van Heijst 1994, 1995; Velasco Fuentes et al. 1995; Velasco Fuentes and Velázquez Muñoz 2003).

Recently, we generalized the principle of modulation for vortex motions on the entire rotating sphere by making the vortex circulation nonlinearly dependent on the latitudinal angle φ_j of vortex j (Drótos et al. 2013). Under the assumption that a point vortex represents

a small patch of vorticity of an area $a^2\pi$, the circulation of vortex j with coordinate φ_j is written as

$$\Gamma_j(\varphi_j) = \Gamma_{j_r} - 2\Omega a^2\pi(\sin\varphi_j - \sin\varphi_r), \quad (1)$$

where Γ_{j_r} is the circulation at a reference latitude φ_r . We call Γ_{j_r} the vortex strength (at the reference latitude) and a the vortex radius, the latter being assumed to be the same for all point vortices (similarly for φ_r). Equation (1) sets the modulation of circulation on a sphere rotating with angular velocity Ω .

In this paper we compare different aspects of vortex pair dynamics in the full spherical picture to those in the β -plane approximation, and we point out considerable alterations. Our approach is similar in spirit to that of Ripa (1997), who carried out analogous comparisons with different forms of the equation of motion for the free motion of a particle (geodesic motion) on a rotating sphere. He also concludes that the usual β -plane approximation is an inconsistent one and also provides a consistent form. In our point vortex model, near the reference latitude of the β -plane approximation, we also carry out Taylor expansions, but, in contrast to Ripa, we expand the equations of motion themselves. We thus arrive at a minimal model in which one sees those errors alone that are purely due to mathematically inconsistent approximations. It is worth mentioning that the geodesic trajectories found by Ripa are exactly the same as those described by Paldor and Killworth (1988) a few years earlier. Two basically different types of trajectories are found: a small-amplitude meandering motion extending to both sides of the equator and traveling to the east, called wobbling, and a self-intersecting large-amplitude motion traveling to the west, called tumbling. In our earlier paper (Drótos et al. 2013) we pointed out that the vortex pair trajectories are also of wobbling or tumbling type—just the role of the equator is taken over by the reference latitude φ_r , defined in (1). This can take on any value, allowing us to focus our attention to the vicinity of an arbitrarily chosen latitude. In this sense our study on the validity of the β -plane approximation is a generalization of the part of Ripa's work that deals with the geodesic motion. Ripa briefly extends his results to the shallow-water equation, too, but does not study any solution of this equation. In our model, we go a step further and also investigate the advection of passive tracers in the velocity field of the vortices to see if there is a difference between the results obtained in the usual β -plane approximation and in the full spherical treatment.

The advection of passive tracers (i.e., fluid elements) in geophysically relevant velocity fields is a topic subject to intensive investigations (e.g., Coulliette et al. 2007; Sandulescu et al. 2008; Pattantyús-Ábrahám et al. 2008;

Tew Kai et al. 2009; Peacock and Haller 2013), which is due to its basic character in the investigation of spreading phenomena. There are numerous applications (e.g., the atmospheric spreading of gases) in which the dynamics of the particles to be followed is well approximated by that of passive tracers. This motivated us to investigate the advection of a large number of passive tracers in the field of the vortex pairs, following both the full spherical treatment and the β -plane approximation. Considerable differences can be found in this context, too.

We note that our model for the vortex dynamics, including the time evolution of the velocity field of the vortex pair, does not aim to mimic real atmospheric or oceanic dynamics. First, owing to neglecting dissipation and the generation of potential vorticity in the background flow, our model may only be reasonable for short time intervals. Second, both the background flow and the vortex configurations, and also the interactions between these elements, are much more complex even in rather simple approaches, leading to a much wider range of phenomena as pointed out by Egger (1992), who attempted to combine a barotropic and quasigeostrophic model with point vortices corresponding to subgrid eddy motion. Therefore, we do not claim that the results presented in this paper describe faithfully any real geophysical phenomena. Instead, a hydrodynamically motivated dynamics is declared on a rotating sphere, the β -plane approximation of which can be analyzed rather straightforwardly and provides insight into the nature of this kind of approximation.

The paper is organized as follows. In the next section we introduce the vortex pair model under the full spherical treatment. The β -plane approximation is derived from this in section 3. In section 4 we present a consistently linearized approximation of the full spherical dynamics and analyze in detail how the β -plane approximation deviates from it. We point out the dynamical importance of the special latitudes φ_{\pm} , corresponding to a uniform eastward or westward propagation, in section 5. These special latitudes under the full spherical and the consistently linearized treatments are shown to qualitatively differ from the corresponding latitude of the β -plane dynamics where both special latitudes simply coincide with φ_r . In section 6, numerical examples are shown for the alterations of the β -plane trajectories from those obtained in the full spherical and the consistently linearized dynamics. Section 7 is devoted to the comparison of the advection dynamics of passive tracers obtained under the β plane and the full spherical treatments. We summarize our findings in section 8.

2. The vortex pair model

As a starting point, we briefly summarize the results of Drótos et al. (2013). We consider two point vortices whose

location is specified by angles λ , φ in geographical coordinates on the surface of a sphere (λ being the longitude, φ being the latitude). The length and time scales, L and T , are naturally chosen as the radius R of the sphere and as $1/(2\Omega)$, where Ω is the rotational frequency, respectively [i.e., $L = R$ and $T = 1/(2\Omega)$]. The vortex circulations, according to (1), are then given by $\Gamma'_j(\varphi_j) = \Gamma'_{jr} - a'^2\pi(\sin\varphi_j - \sin\varphi_r)$, where $j = 1, 2$ and both the reference circulation Γ'_{jr} and the vortex radius a' are dimensionless. The vortex pair is defined by having opposite vorticities $\Gamma'_{1r} = -\Gamma'_{2r} \equiv \Gamma'$ at the reference latitude φ_r .

The chord distance D' of the elements of the pair turns out to be constant during the motion (Drótos et al. 2013). The dimensionless equations for the two elements of the pair are found to be

$$\frac{d\lambda_i}{dt} = \frac{1}{\cos\varphi_i} \frac{1}{2\pi D'^2} [\Gamma'_{jr} - a'^2\pi(\sin\varphi_j - \sin\varphi_r)] \times [\cos\varphi_i \sin\varphi_j - \sin\varphi_i \cos\varphi_j \cos(\lambda_i - \lambda_j)] \quad \text{and} \quad (2a)$$

$$\frac{d\varphi_i}{dt} = \frac{[\Gamma'_{jr} - a'^2\pi(\sin\varphi_j - \sin\varphi_r)] \cos\varphi_j \sin(\lambda_i - \lambda_j)}{2\pi D'^2}. \quad (2b)$$

Here $i, j = 1, 2$, $i \neq j$, and $\Gamma'_{1r} = -\Gamma'_{2r} = \Gamma'$. The initial conditions are given by the initial values λ_{i0} , φ_{i0} , $i = 1, 2$. Alternatively, we can use the initial center-of-mass coordinates λ_0 , φ_0 , the chord distance, and the initial orientation of the line connecting the elements of the pair. This line, owing to the functional form of the velocity field of any point vortex, is always perpendicular to the velocities of the elements of the pair and, hence, to the center-of-mass velocity \mathbf{u} of the pair. As pointed out in Drótos et al. (2013), we can restrict our investigations without the loss of generality to cases with an initially strictly zonal center-of-mass velocity. We thus always choose the initial meridional component v_0 of the center-of-mass velocity to be 0.

A systematic exploration of the possible trajectory shapes, depending on the initial conditions, is found in Drótos et al. (2013). Here we only repeat the most important observation—namely, the regularity of any trajectory. The motion of a vortex pair is almost always characterized by a periodical dynamics in the φ coordinate of the vortex pair's center of mass. During each period, the λ coordinate changes by a particular value. Depending on the sign of this value, the net motion of the vortex pair is either an eastward or a westward drift restricted to a band in φ .

Dipoles, very close and very weak vortex pairs whose velocity is finite, are obtained in the limit Γ' , a' , $D' \rightarrow 0$. We write the vortex coordinates as $\varphi_{1,2} = \varphi \pm d\varphi$ and $\lambda_{1,2} = \lambda \pm d\lambda$, where φ and λ are the center-of-mass

coordinates in the dipole limit. From the equations of motion [(2)] of a vortex pair, we obtain the equations of motion for φ , $d\varphi$, λ , and $d\lambda$. From these, a closed dynamics follows for the center-of-mass coordinates φ and λ :

$$\frac{d}{dt}(\cos\varphi\dot{\lambda}) = \dot{\varphi} \sin\varphi[\gamma\delta(\varphi) + \dot{\lambda}] \quad \text{and} \quad (3a)$$

$$\frac{d}{dt}\dot{\varphi} = -\dot{\lambda} \cos\varphi \sin\varphi[\gamma\delta(\varphi) + \dot{\lambda}], \quad (3b)$$

where

$$\gamma = \frac{a'^2}{D'^2} \quad (4)$$

is a parameter of the geometry in the dipole limit and

$$\delta(\varphi) = 1 - \frac{\sin\varphi_r}{\sin\varphi}. \quad (5)$$

Using the zonal and meridional components of \mathbf{u} ,

$$u \equiv \dot{\lambda} \cos\varphi, \quad v \equiv \dot{\varphi}, \quad (6)$$

the dynamics of the center of mass is written as

$$\dot{u} = \gamma\delta(\varphi)v\sin\varphi + uv\tan\varphi, \quad \dot{v} = -\gamma\delta(\varphi)u\sin\varphi - u^2\tan\varphi. \quad (7)$$

These are the equations of motion for a dipole for a particular reference latitude φ_r and ratio γ .

By definition, the velocity modulus of the dipole is determined by

$$|\mathbf{u}|^2 = u^2 + v^2 = \dot{\lambda}^2 \cos^2\varphi + \dot{\varphi}^2. \quad (8)$$

From this and from the equations of motion for φ and λ [see (B2a) and (B2c) in Drótos et al. (2013)], the velocity modulus can be derived to be

$$|\mathbf{u}| = \frac{\Gamma'}{2\pi D'}. \quad (9)$$

It is interesting to note that the dipole dynamics with $\gamma = 1$, $\varphi_r = 0$ was found in Drótos et al. (2013) to be equivalent to that of the free motion of point particles on the same rotating sphere (Paldor and Killworth 1988; Paldor and Boss 1992; Ripa 1997).

3. The β -plane approximation in the vortex pair model

In any traditional β -plane approximation (in brief, “ β -plane approximation”), one has to choose a latitude where the origin of a local Cartesian reference frame is put. This choice should be made so that the fluid element trajectories under investigation stay close to this latitude

when compared to the radius of the sphere. The latter being unity, ± 0.15 is traditionally taken as the limit for the deviation from the chosen latitude in the variable φ , or, equivalently, for the absolute value of the coordinate y (locally coaligned with latitude φ) in the Cartesian reference frame (Pedlosky 1979). In our particular case, we naturally choose φ_r to be the origin of the β -plane y axis since we are interested in the dynamics taking place in the vicinity of the reference latitude φ_r . As for the origin of the x axis, any longitude λ is equivalent, so we choose $\lambda = 0$.

First, we consider finite-sized vortex pairs (a' , D' , and Γ' being finite). In the spirit of geophysical fluid dynamics (Pedlosky 1979), the β -plane equations of motion for a vortex pair are obtained by assuming planar geometry with the coordinates x and y defined as

$$x = \lambda \cos\varphi_r, \quad y = \varphi - \varphi_r. \quad (10)$$

In the x - y plane simply the planar equations of motion for a vortex pair are considered. The only modification is the modulation of the vortex circulation as a function of the coordinate y , as formulated in Velasco Fuentes and van Heijst (1994), Benczik et al. (2007), Kloosterziel et al. (1993), Velasco Fuentes and van Heijst (1995), Velasco Fuentes et al. (1995), and Velasco Fuentes and Velázquez Muñoz (2003). When referring to the form (1) of the modulation, only the leading-order term is kept in the Taylor expansion of $\sin\varphi_i - \sin\varphi_r$ around φ_r :

$$\sin\varphi_i - \sin\varphi_r \simeq \cos\varphi_r y_i. \quad (11)$$

The dimensionless Coriolis parameter (the local vertical component of the angular velocity of the sphere) is, in our case, $\sin\varphi_i$, thus the dimensionless β parameter (β'), the derivative of the Coriolis parameter at φ_r , is $\beta' = \cos\varphi_r$. The β -plane equations of motion for a vortex pair are thus (e.g., Velasco Fuentes and van Heijst 1994)

$$\frac{dx_i}{dt} = -\frac{1}{2\pi D'^2} (\Gamma'_{jr} - \beta' a'^2 \pi y_j) (y_i - y_j) \quad \text{and} \quad (12a)$$

$$\frac{dy_i}{dt} = \frac{1}{2\pi D'^2} (\Gamma'_{jr} - \beta' a'^2 \pi y_j) (x_i - x_j) \quad (12b)$$

for $i, j = 1, 2$, $i \neq j$, and $\Gamma'_{1r} = -\Gamma'_{2r} = \Gamma'$.¹

¹ A natural question is how (12) can be obtained from (2). The metric factors (next to the circulations) in (2) turn out to be convertible to the differences of the Cartesian coordinates appearing in (12), but only by assuming the longitude difference $\lambda_i - \lambda_j$ to be of the same order as $\varphi_i - \varphi_r$. Furthermore, taking the product of the linearly approximated metric factors with the linearly approximated circulations proves to handle quadratic terms inconsistently. Without any quadratic terms, however, the variation of the Coriolis parameter with latitude cannot be incorporated into the model at all.

For a dipole, we write $x_{1,2} = x \pm dx$, $y_{1,2} = y \pm dy$, where (x, y) is the position of the center of mass in the β plane, and dx, dy, a', D' , and Γ' are considered to be infinitesimally small and to be of the same order. A consistent Taylor expansion of the equations of motion [(12)] leads to

$$\ddot{x} = \frac{a'^2}{D'^2} \beta' y \dot{y} = \gamma \cos \varphi_r y \dot{y} \quad \text{and} \quad (13a)$$

$$\ddot{y} = -\frac{a'^2}{D'^2} \beta' y \dot{x} = -\gamma \cos \varphi_r y \dot{x}. \quad (13b)$$

These are the β -plane equations of motion for a dipole moving near φ_r .

The velocity of the dipole is the derivative of the position vector (x, y) as defined by the planar geometry:

$$\mathbf{u} = (\dot{x}, \dot{y}). \quad (14)$$

To see this, one can also refer to definition (6) and substitute $\cos \varphi$ by $\cos \varphi_r$ in the spirit of the ‘‘planarization’’ of the dynamics:

$$\mathbf{u} = (\dot{\lambda} \cos \varphi_r, \dot{\varphi}) = (\dot{x}, \dot{y}). \quad (15)$$

It is clear then that this is the only reasonable velocity vector associated to a β -plane dipole.

4. The inconsistency of the β -plane dipole equations

As a more precise approach than the β -plane approximation, we now derive the consistently linearized dipole equations around φ_r . Any expressions depending explicitly on the variable φ , including the metric factors, are thus taken into account up to the same order. Although it turns out to be useful to define new variables, denoted again by x and y , for a more convenient representation of the dynamics, these new variables are not derived from any planar geometry, in contrast to the β -plane approximation.

The Taylor expansion in $\Delta\varphi \equiv \varphi - \varphi_r$ of the full spherical dipole equations [see (3)] results in leading order in

$$\begin{aligned} \cos \varphi_r \ddot{\lambda} &= 2 \sin \varphi_r \dot{\Delta\varphi} \dot{\lambda} + (2/\cos \varphi_r) \dot{\Delta\varphi} \dot{\lambda} \Delta\varphi \\ &+ \gamma \cos \varphi_r \dot{\Delta\varphi} \Delta\varphi \quad \text{and} \end{aligned} \quad (16a)$$

$$\begin{aligned} \ddot{\Delta\varphi} &= -\sin \varphi_r \cos \varphi_r \dot{\lambda}^2 - \cos(2\varphi_r) \dot{\lambda}^2 \Delta\varphi \\ &- \gamma \cos^2 \varphi_r \dot{\lambda} \Delta\varphi. \end{aligned} \quad (16b)$$

Defining

$$x = \cos \varphi_r \lambda, \quad y = \Delta\varphi, \quad (17)$$

we obtain

$$\ddot{x} = \gamma \cos \varphi_r y \dot{y} + 2 \tan \varphi_r y \dot{x} + (2/\cos^2 \varphi_r) y \dot{x} y \quad \text{and} \quad (18a)$$

$$\ddot{y} = -\gamma \cos \varphi_r x \dot{y} - \tan \varphi_r x^2 + (\tan^2 \varphi_r - 1) x^2 y. \quad (18b)$$

These are the correct equations for dipoles moving close to φ_r . We emphasize that x and y are not Cartesian coordinates but are arbitrarily defined variables. The conversion, however, between the (λ, φ) and the (x, y) variable pairs is exactly the same as in the β -plane approximation. As only the (λ, φ) coordinates are of physical relevance, the coincidence of the conversion ensures a direct comparability of the current equations with those of the β -plane approximation.

Observe that the second and the third terms on the right-hand side of both (18a) and (18b) are missing in the β -plane equations of motion [(13)]. The origin of this is in the metric factors of the equations of motion of a vortex pair [(2)], which are inconsistently omitted in the β -plane approximation.

Let us now express the velocity of the dipole in terms of x, y , and their derivatives, applying consistent linearizations. Definition (6) yields

$$\mathbf{u} = (\dot{\lambda} \cos\{\varphi_r + \Delta\varphi\}, \dot{\Delta\varphi}), \quad (19)$$

from which, by taking the linear Taylor expansion of the cosine function around φ_r and substituting the definition (17) of x and y , we obtain

$$\mathbf{u} = (\{1 - \tan \varphi_r y\} \dot{x}, \dot{y}). \quad (20)$$

It is important that the zonal (i.e., the λ) component u of the velocity is not equal to \dot{x} .

In the spirit of the β -plane approximation it is worth taking into account that the components of \mathbf{u} should be small in comparison with the equatorial velocity of the surface of the sphere. In a geophysical context, the characteristic dimensional velocity modulus U of the fluid should be much smaller than the equatorial velocity $R\Omega$ of the sphere. The global Rossby number $\text{Ro}_g = U/(2\Omega R)$ is then small. This means that the dimensionless $|\mathbf{u}|$ should also be restricted to small values. As a consequence of (20), \dot{x} (regardless of the particular value of the small variable y) and \dot{y} are considered to be on the order of the small Ro_g .

Ro_g is chosen to be around 0.025 or 0.0025 in our numerical simulations, and y , as a small quantity, with an absolute value of 0.15 at the very most, is considered to be of the same order as Ro_g (see Pedlosky 1979). Then the first and the second terms in system (18) are of order Ro_g^2 , and the third terms are of order Ro_g^3 , the

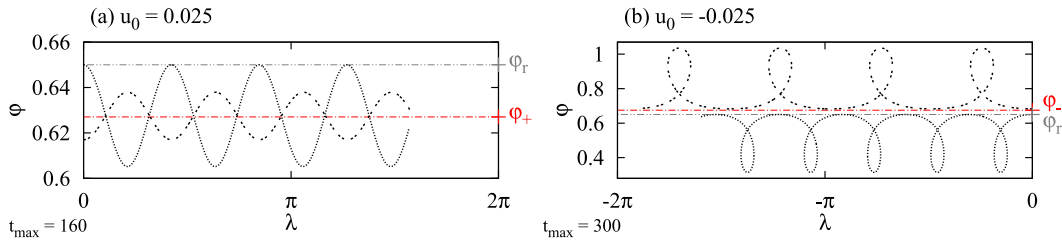


FIG. 1. (a) Eastward- and (b) westward-initiated (wobbling and tumbling) dipole trajectories in a λ - ϕ representation, compared to the relevant special latitude [red dotted-dashed line: (a) $\phi_+ = 0.627$ and (b) $\phi_- = 0.675$]. Initial latitudes for the dipole trajectories: dotted: $\phi_0 = \phi_r = 0.65$ in (a) and (b), intermittently dashed: (a) $\phi_0 = 0.617 < \phi_+$ and (b) $\phi_0 = 0.683 > \phi_-$. Further initial conditions: $\lambda_0 = 0$, $v_0 = 0$, and u_0 is indicated in the panels. The length t_{\max} of the trajectories is also indicated in dimensionless time units. Parameters: $\gamma = 1$, and $\phi_r = 0.65$ is marked by a gray double-dotted-dashed line. Note the different scales in (a) and (b).

latter ones then being not relevant. Thus, as long as $\tan\phi_r$ is of order unity, omitting the second terms, as done in the β -plane approximation, leads to a considerable deviation of the trajectories from the correct ones. [A close agreement is expected around the equator (i.e., for $\tan\phi_r \ll 1$) only, in agreement with Pedlosky's argumentation for the momentum equation in Pedlosky (1979).]

The numerical examples shown in section 6 exhibit striking differences for certain cases. To understand these plots, however, a further observation is needed first, leading to the definition of the special latitudes.

5. Special latitudes in the spherical dipole dynamics

Now we consider a fixed $\phi_r \neq 0$ and ask if there exists any latitude that is a dynamical analog of the origin of the β -plane approximation, a latitude corresponding to an eastward or westward uniform propagation. Intuitively, one might think that this special latitude is ϕ_r . Here we show that the special latitudes ϕ_{\pm} , describing eastward and westward uniform propagation of a general dipole, respectively, differ from ϕ_r and also from each other.

We are looking for solutions of the dipole equations of motion [(3)] of the form

$$\phi(t) = \phi_{\pm}, \quad \dot{\phi}(t) = 0. \quad (21)$$

Substituting this into (3a), we obtain

$$\ddot{\lambda} = 0, \quad \lambda(t) = \lambda_0 + \omega t; \quad (22)$$

that is, this propagation is indeed uniform. The constant ω is the angular velocity and is thus geometrically related to the zonal velocity u as

$$\omega = \frac{u}{\cos\phi_{\pm}}. \quad (23)$$

Since both ω and ϕ_{\pm} are constants, u is constant in time and is equal to its initial value u_0 . Equation (3b) then yields

$$\begin{aligned} 0 &= -\omega \cos\phi_{\pm} (\gamma \sin\phi_{\pm} - \gamma \sin\phi_r + \omega \sin\phi_{\pm}) \\ &= -u_0 \gamma \sin\phi_{\pm} + u_0 \gamma \sin\phi_r - u_0^2 \tan\phi_{\pm}, \end{aligned} \quad (24)$$

from which, for $\phi_{\pm} \neq \pm\pi/2$,

$$\sin\phi_{\pm} = \frac{\sin\phi_r}{1 + u_0/(\gamma \cos\phi_{\pm})} \quad (25)$$

holds for the special latitudes ϕ_{\pm} . According to (9), $|u_0|$ is obtained from the infinitesimally small vortex strength Γ' and vortex distance D' (remember that $v = \dot{\phi} = 0$):

$$u_0 = \pm \frac{\Gamma'}{2\pi D'}, \quad (26)$$

where the upper (lower) sign corresponds to the eastward (westward) propagation. The above equations implicitly determine ϕ_{\pm} as a function of γ , ϕ_r , and Γ'/D'^2 .

From (25), one sees that the latitude of the uniform propagation does not coincide with ϕ_r and is different for the eastward and the westward direction of the propagation. Relative to ϕ_r , this latitude is shifted toward and away from the equator in these two cases, respectively.

Numerical results in Fig. 1 highlight the role of the special latitudes in the full spherical dipole dynamics. In Fig. 1a, we can see that a dipole initiated close to ϕ_+

²For small (geostrophic) velocities and latitudes not too close to the pole, an appropriate estimation for the location of the special latitudes is $\sin\phi_{\pm} \simeq \sin\phi_r/[1 + u_0/(\gamma \cos\phi_r)]$. The difference $|\sin\phi_- - \sin\phi_+| \simeq 2 \tan\phi_r |u_0|/\gamma$ is then always small, on the order of Ro_g , for a given velocity modulus $|u_0|$.

with an eastward velocity moves to the east on average, meandering in the vicinity of φ_+ . This trajectory type is called an eastward wobbling (Newton and Shokraneh 2006). As illustrated in the plot, small-amplitude eastward-wobbling trajectories have inflexion points close to φ_+ . Thus φ_+ serves as a “center” for such trajectories and, latitudinally, it “attracts” them. Eastward-wobbling trajectories initiated on the northern (southern) side of φ_+ bend initially to the south (north). As Fig. 1b illustrates, φ_- plays an opposite role for dipoles initiated with a westward velocity: they are “repelled” from φ_- , resulting in circlelike trajectories slowly drifting to the west. These are called westward-tumbling trajectories (Newton and Shokraneh 2006). It is clear that the dipole moves along in a negative (positive) rotational direction on the northern (southern) side of φ_- . We can thus say that φ_- is a separatrix for northern and southern type westward tumblings.

Based on the previous paragraph, the difference between φ_+ and φ_- has the following implication: for any initial latitude $\varphi_0 \in (\varphi_+, \varphi_-)$ there exist both southern-type westward-tumbling trajectories (for westerly initial velocities) and eastward-wobbling trajectories (for easterly initial velocities) that initially exhibit a northern-type behavior (i.e., they initially bend to the south). For example, $\varphi_0 = \varphi_r = 0.65 \in (\varphi_+, \varphi_-)$ behaves qualitatively differently for $u_0 > 0$ and $u_0 < 0$, as Figs. 1a and 1b illustrate, respectively.

The difference of the respective special latitudes from φ_r and from each other is also present in the correctly linearized form of the dipole equations. From (16), substituting $\dot{\lambda} = u/\cos\varphi \simeq u/\cos\varphi_r(1 + \tan\varphi_r y)$ for the angular velocity, one obtains

$$y_{\pm} = -\frac{\sin\varphi_r}{(1/u_0)\gamma \cos^2\varphi_r + 1/\cos\varphi_r} \quad (27)$$

for the positions of the uniform eastward and the uniform westward propagation with constant latitudinal velocity u_0 .

The existence of the special latitudes $\varphi_{\pm} \neq \varphi_r$ under the full spherical treatment (and that of $y_{\pm} \neq 0$ under the correctly linearized treatment) implies in itself a qualitative breakdown of the β -plane approximation via the clear spatial separation of the eastward and the westward uniform propagation, which both belong to $\varphi_0 = \varphi_r$ under the β -plane treatment. Although this separation is small, it leads to the strongest consequences exactly for trajectories moving near φ_r (e.g., initiated in between φ_+ and φ_-), where we would naively expect the β -plane approximation to work the best.

6. Differences in the trajectories

In this section we compare dipole trajectories (i.e., trajectories corresponding to the $\Gamma', a', D' \rightarrow 0$ limit) obtained numerically from the exact spherical, the correctly linearized, and the β -plane equations [(3), (18), and (13), respectively]. Note that these equations represent three different dynamics, one of which [(3)] is considered to be “true” with the other two approximating it. The question is how the latter ones perform in different situations. To obtain the answer, we fix the initial meridional velocity component to be $v_0 = 0$, choose an approximately geostrophic velocity modulus $|u_0| \ll 1$ (see the discussion at the end of section 4), and vary systematically the initial latitude φ_0 . We discuss eastward- and westward-initialized velocities separately. We choose $\gamma = 1$ throughout our investigations without loss of generality; see Drótos et al. (2013).

In Fig. 2 we explore the behavior of trajectories initiated with an eastward velocity (i.e., with $u_0 > 0$). In Fig. 2a with an initial latitude $\varphi_0 \in (\varphi_+, \varphi_r)$ the exact and the correctly approximated trajectories bend initially to the south, toward φ_+ , in accordance with the previous section. Meanwhile, the β -plane trajectory bends initially to the north, toward φ_r . This indicates that the β -plane approximation does not reflect the relevance of φ_+ , this special latitude being more relevant for the correct dynamical description than φ_r . Although the trajectory is very close to φ_r during the entire motion under the exact and either of the approximated treatments, the β -plane approximation leads to a qualitatively incorrect behavior in this situation. The message of Fig. 2b, with $\varphi_0 > \varphi_r$ but $|\varphi_0 - \varphi_r| \ll 1$, is similar. Although any trajectory bends initially to the south here, the amplitude of the eastward wobbling is an order of magnitude smaller in the β -plane approximation than under the exact and the correctly approximated treatments. Figure 2c corresponds to an initial latitude being relatively far away from φ_r but inside the validity range of any linear approximation in the variable φ . In this case all three trajectories are qualitatively similar. This leads to the counterintuitive conclusion that the investigated motion should have a portion that lies considerably far away from φ_r for a qualitative applicability of the β -plane approximation. In Fig. 2d we demonstrate that the breakdown of the β -plane approximation is present also for a different value of φ_r and for trajectories initiated on the southern side of φ_+ but not far away from it.

Figure 3 is similar to the previous one but shows trajectories initiated to the west. Figure 3a provides an example for an initial latitude $\varphi_0 \in (\varphi_r, \varphi_-)$, leading to

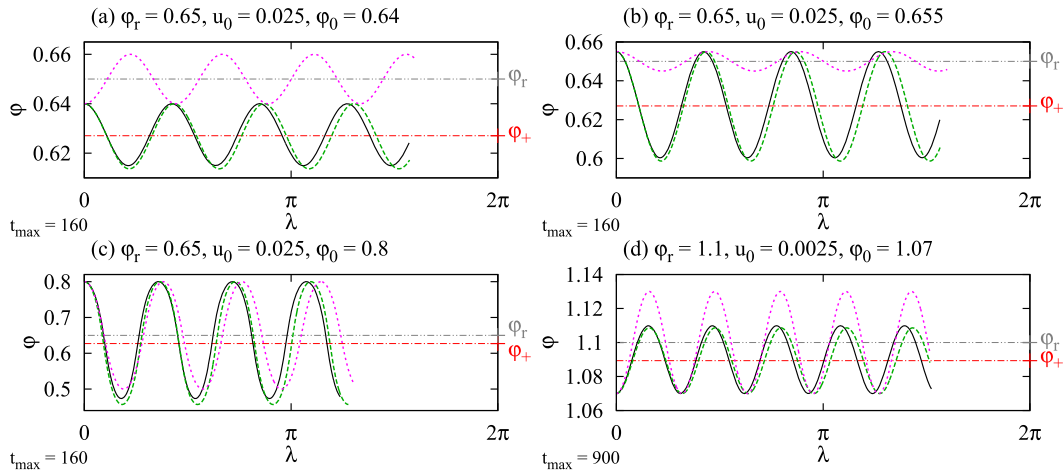


FIG. 2. Eastward-initiated (wobbling) dipole trajectories in a λ - ϕ representation, under full spherical (black solid line), correctly linearized (green dashed line), and β -plane (magenta short dashed line) treatments. The special latitude ϕ_+ is marked by a red dotted-dashed line. Initial conditions: $\lambda_0 = 0, v_0 = 0$, and u_0 and ϕ_0 are indicated above each panel. The length t_{\max} of the trajectories in dimensionless time units and the value of ϕ_r (marked by a gray double-dotted-dashed line) are also indicated; $\gamma = 1$.

a completely different direction of evolution in the β -plane approximation compared to the other two trajectories. This is due only to the fact that the uniform westward propagation (taking place along ϕ_r and ϕ_- in the β -plane approximation and in the full spherical description, respectively) is a separatrix between southern-type and northern-type westward tumblings; see Fig. 1b and the related discussion. We note, however, that trajectories initiated close to ϕ_r , as in the current setting, should be described correctly at least in the beginning when treated under any consistent first-order approximation around ϕ_r , and this is not the case for the β -plane approximation. Figure 3b exhibits an initial condition with $\phi_0 < \phi_r$ but $|\phi_0 - \phi_r| \ll 1$. Based on the initial condition alone, we could expect a large error for the β -plane approximation, in view of our experience for trajectories moving near ϕ_r (see Fig. 2). Now, however, the trajectories spend a long time far away from ϕ_r (but still in the

traditional validity range of linearized approximations), as in Fig. 2c. As a result, the β -plane treatment does not perform much worse than the correctly linearized one, although there are severe quantitative differences.

To gain a global view, we also show phase-space portraits in Fig. 4 for the full spherical and for the β -plane treatment in one particular setting. We focus on the ϕ dynamics and introduce a new variable α , defined in the figure caption, for characterizing the orientation of the dipole. The special latitude ϕ_+ (ϕ_-) corresponds to a stable (unstable) fixed point of this dynamics [denoted by a filled (an open) circle]. Westward-tumbling trajectories are seen as lines stretching through all values of α (representing “rotation”), and wobbling trajectories are seen as closed lines around the stable fixed point (representing “oscillation”). When monitored in time, all trajectories are traced out in a counterclockwise direction. Initiation is eastward (i.e., with $\alpha = 0$), and solid

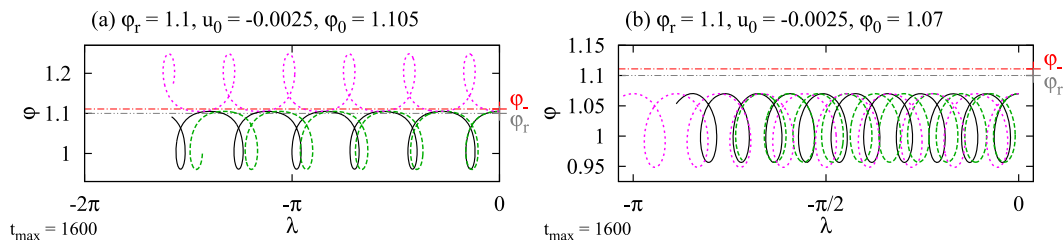


FIG. 3. Westward-initiated (tumbling) dipole trajectories in a λ - ϕ representation, under full spherical (black solid line), correctly linearized (green dashed line), and β -plane (magenta short dashed line) treatments. The special latitude ϕ_- is marked by a red dotted-dashed line. Initial conditions: $\lambda_0 = 0, v_0 = 0$, and u_0 and ϕ_0 are indicated above each panel. The length t_{\max} of the trajectories in dimensionless time units and the value of ϕ_r (marked by a gray double-dotted-dashed line) are also indicated; $\gamma = 1$.

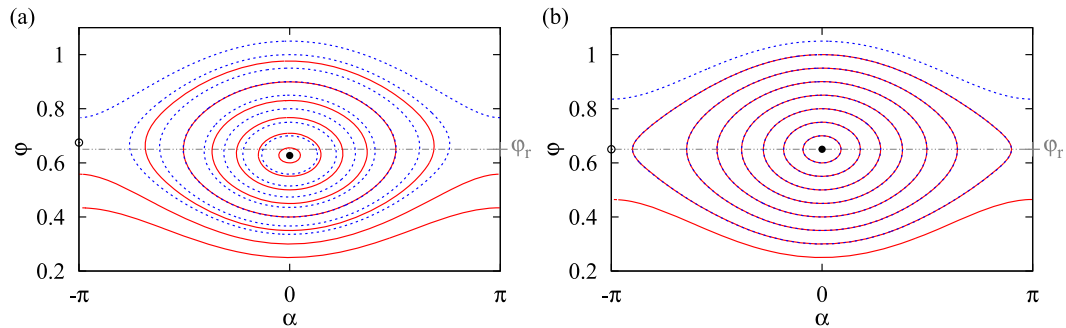


FIG. 4. Phase-space portraits of the dipole dynamics under (a) full spherical and (b) β -plane treatments. $\alpha = \arctan(v/u)$ if $u > 0$, $\alpha = \arctan(v/u) + \pi$ if $u < 0$ and $v > 0$, and $\alpha = \arctan(v/u) - \pi$ if $u < 0$ and $v \leq 0$. Trajectories are shown in the α - φ plane corresponding to several initial latitudes: $\varphi_0 \in \{0.25, 0.3, \dots, 0.6\}$ in red (solid lines) and $\varphi_0 \in \{0.7, 0.75, \dots, 1.05\}$ in blue (dashed lines). Initial conditions in the other variables are $u_0 = 0.025$, $v_0 = 0$, and $\lambda_0 = 0$. $\varphi_r = 0.65$ is indicated by a gray double-dotted-dashed line. Uniform eastward (westward) propagation is marked by a filled (an open) black circle at the position $(0, \varphi_+)$ [$(-\pi, \varphi_-)$] in (a) and $(0, \varphi_r)$ [$(-\pi, \varphi_r)$] in (b). Note the symmetry breaking of (a) compared to (b); $\gamma = 1$.

red (dashed blue) trajectories correspond to an initial value of φ below (above) φ_+ (φ_-). We observe that the phase space of the β -plane treatment (in Fig. 4b) is symmetric to the lines $\alpha = 0$ and $\varphi = \varphi_r$, which formally coincides in this case with $\varphi_+ = \varphi_-$. As a consequence of the latter symmetry, fixed points are found on the latitude $\varphi = \varphi_r$, and trajectories initiated symmetrically in φ on the two sides of this latitude coincide. The symmetry to the line $\varphi = \varphi_r$ breaks down under the full spherical treatment (in Fig. 4a): even the fixed points happen to be on different sides of the latitude $\varphi = \varphi_r$, and the coincidence of trajectories initiated symmetrically to the this latitude does not hold any more. This figure illustrates in a pictorial way how different the two dynamics are.

So far we have only included examples for dipoles, having had mathematical discussions only for this limiting case of vortex pairs. Nevertheless, the basic findings hold for finite-sized vortex pairs, as illustrated in Figs. 5a and 5b, exhibiting β -plane and full spherical trajectories with initial conditions corresponding to those in Figs. 2a and 3a, respectively.³ In both plots for finite-sized vortex pairs (of distance $D' = 0.1$) one can observe exactly the same qualitative behavior for the center-of-mass trajectories as that in the corresponding plots for dipoles.

Based on the results of this section, we can say that there is a strong difference, on the order of the whole

latitudinal extension of the trajectories, between β -plane and full spherical treatments for any vortex pair staying in a range $|\varphi - \varphi_r| < 0.05$ during the motion (this is not the case under a consistently linearized treatment). The difference is much weaker farther away where the β -plane approximation appears to be reasonable, in a qualitative sense at least, regardless of the fact that linearizations have larger errors farther from the reference latitude. As the β -plane approximation can be considered to be valid within a range of φ about 0.15 at most, we can say that the β -plane approximation gives qualitatively incorrect results in the middle one-third of its validity range around the investigated φ_r .

7. Advection in the field of vortex pairs

So far, we have investigated the dynamics of the vortex centers, which is a kind of dynamics obtained in an Eulerian spirit by observing the conservation of potential vorticity. Here we turn to the study of a Lagrangian feature—the advection dynamics in the velocity field determined by the moving vortices. To this end, it is necessary to deal with finite-sized pairs instead of dipoles. We now only compare the full spherical and the β -plane dynamics for the sake of simplicity. The finite-sized vortex pair dynamics is described by the equations of motion under the full spherical and under the β -plane treatments, (2) and (12), respectively. The equations of motion for the position (λ, φ) or (x, y) of a passive tracer is obtained by the simple summation of the velocity field of the two point vortices, which, under full spherical treatment, leads to (Drótos et al. 2013)

³The derivation of any consistently approximated equations of motion around φ_r for a finite-sized vortex pair with linear modulation is ill defined and is therefore beyond the scope of the present paper. The reason is the unnecessary but traditional restriction to small longitudinal coordinate differences; see footnote 1.

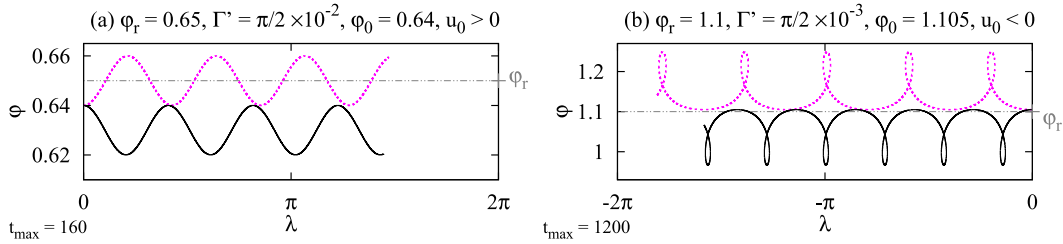


FIG. 5. Center-of-mass trajectories of finite-sized vortex pairs in a λ - ϕ representation, under full spherical (black solid line) and β -plane (magenta short-dashed line) treatments. Initial conditions and parameters: $\lambda_0 = 0$, $v_0 = 0$; $D' = a' = 0.1$; ϕ_0 , the sign of u_0 , and Γ' are indicated above the panels. The length t_{\max} of the trajectories in dimensionless time units and the value of ϕ_r (marked by a gray double-dotted-dashed line) are also indicated. $|u_0| \approx$ (a) 0.025 and (b) 0.0025.

$$\frac{d\lambda}{dt} = \frac{1}{\cos\phi} \frac{1}{4\pi} \sum_{j=1}^2 \frac{[\Gamma' - a'^2\pi(\sin\phi_j - \sin\phi_r)][\cos\phi \sin\phi_j - \sin\phi \cos\phi_j \cos(\lambda - \lambda_j)]}{1 - \cos\gamma_j} \quad \text{and} \quad (28a)$$

$$\frac{d\phi}{dt} = \frac{1}{4\pi} \sum_{j=1}^2 \frac{[\Gamma' - a'^2\pi(\sin\phi_j - \sin\phi_r)]\cos\phi_j \sin(\lambda - \lambda_j)}{1 - \cos\gamma_j} \quad (28b)$$

in dimensionless form, where

$$\cos\gamma_j = \sin\phi \sin\phi_j + \cos\phi \cos\phi_j, \quad \cos(\lambda - \lambda_j) \quad (29)$$

and the chord distance between the tracer and vortex j is $2(1 - \cos\gamma_j)$.

Under the β -plane treatment of the modulation of the vortex circulations (Benczik et al. 2007) the equations of motions read as

$$\frac{dx}{dt} = -\frac{1}{2\pi} \sum_{j=1}^2 \frac{(\Gamma' - \beta' a'^2 \pi y_j)(y - y_j)}{\sqrt{(x - x_j)^2 + (y - y_j)^2}} \quad \text{and} \quad (30a)$$

$$\frac{dy}{dt} = \frac{1}{2\pi} \sum_{j=1}^2 \frac{(\Gamma' - \beta' a'^2 \pi x_j)(x - x_j)}{\sqrt{(x - x_j)^2 + (y - y_j)^2}}. \quad (30b)$$

Note that both dynamics (28) and (30) of a tracer can be considered as a dynamical system of two variables driven by the vortex pair dynamics. Therefore, this dynamics is typically chaotic (Aref 1984; Sommerer et al. 1996). In our setting, two different types of advective chaos can arise: one is the open type (Jung et al. 1993; Péntek et al. 1995; Sommerer et al. 1996; Neufeld and Tél 1998; Daitche and Tél 2009), and the other is the closed type (Aref 1984, 2002; Ottino 1989; Provenzale 1999). In open chaotic advection, particles eventually leave the observation region forever, while they are confined to this region in the closed case.

In the vortex pair model the observation region is a neighborhood of the center of mass of the vortex pair, taken with a reasonable radius ρ , called the escape radius. We initiate particles in this region and concentrate on those particles that exhibit a chaotic behavior. We are interested in the associated (space filling or fractal) patterns and also ask if the particles leave the observation region. In the close vicinity of the position of any point vortex, where the velocity field of the particular vortex dominates any other velocity contribution, the particles exhibit a regular circulatory motion around that vortex (Kuznetsov and Zaslavsky 1998, 2000; Leoncini et al. 2001). These subregions are called the vortex cores and are not interesting from our current point of view. Farther away from the vortices, but still inside the escape radius, there is a subregion that is weakly affected by the vortices' velocity field. Particles in this subregion escape the observation region soon and, without exhibiting any fractal pattern, they are thus also not relevant. Regular and confined motion occurring outside the vortex cores could be relevant, but this phenomenon is rather rare in our experience. What remains of interest are only the particles subject to chaoticity.

For finding such passive tracers, the best choice is to put them initially in between the vortices. A tracer is then either confined to an easily recognizable vortex core or behaves in a chaotic way, contributing to the interesting patterns. If the latter tracers are observed to be left in the wake of the vortex pair, we consider the chaotic advection open; otherwise, we consider it closed. The patterns and the openness can easily be investigated by simply plotting the positions of the tracers in various time instants. Using this algorithm, a "phase transition" was found earlier in the advection pattern between open and closed "phases" as a function of the initial latitude

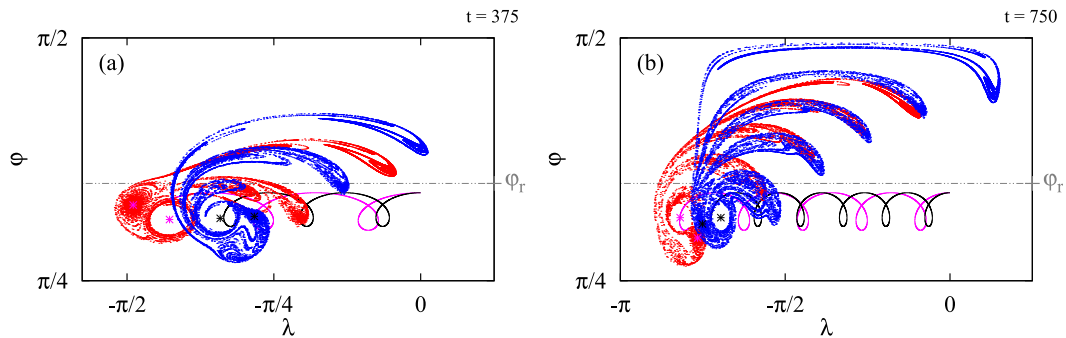


FIG. 6. The positions of $N \approx 17\,500$ tracers (dots) and the two vortices (symbols) at the time instants t indicated. The tracers are initiated at $t = 0$ in between the elements of the vortex pair. Blue (red) tracers correspond to full spherical (β plane) treatment. The center-of-mass trajectory of the vortex pair is also shown up to the time instant t as a black (magenta) solid line, corresponding to full spherical (β plane) treatment. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 1.07$, $v_0 = 0$, and $u_0 < 0$. $\varphi_r = 1.1$ is marked by a gray double-dotted-dashed line. Vortex pair parameters: $D' = a' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-3}$.

φ_0 of the center of mass of the vortex pair, both under β -plane (Benczik et al. 2007) and full spherical (Drótos et al. 2013) treatments.

A more delicate subject is the characterization of the escape in the open advection. For this purpose, it is worth filling a larger subregion of the observation region by tracers. For each tracer, the escape time τ , the time needed to leave the circle of radius ρ (the escape radius), is calculated. Plotting the escape time at the initial position of each particular tracer draws out the stable manifold of a chaotic saddle (Jung et al. 1993; Péntek et al. 1995; Lai and Tél 2011) as ridges with higher values, organized into fractal filaments, in a sea of low values of the escape time. The higher are the values found near the ridges, the longer is the escape process of the tracers. The dominant part of the escape time probability density function is always exponential with an exponent κ called the escape rate (Jung et al. 1993; Péntek et al. 1995; Lai and Tél 2011). The escape rate characterizes the long-term rate of the depletion of the observation region by tracers.

Our aim in this paper is to point out important alterations of the β -plane approximation from the full spherical dynamics in the above aspects of the advection. It is not a surprise to obtain qualitatively different advective properties when the vortex trajectories are also qualitatively different under the two treatments. It adds, however, a new insight into the poorness of the β -plane approximation if we find strong differences in the advective properties when the vortex trajectories are rather similar.

Such a case is shown in Fig. 6, where the vortex initial conditions correspond to those in Fig. 3b. Here the advection pattern is open and consists of lobes, formed after every time period of the φ dynamics, and migrating to the north in the farther wake. In Fig. 6a the positions of the tracers are shown shortly after the initialization, whereas in Fig. 6b they are shown later, when they even

approach the North Pole in the full spherical dynamics. Although the β -plane tracer dynamics is not expected to work far away from φ_r , the way in which the tracers leave the vicinity of the vortices and the beginning of their migration is restricted to a region $|\varphi - \varphi_r| \ll 1$ and, thus, allows for a fair comparison of the two treatments. Our first observation is that the particular positions and geometry of the lobes are rather different. Apart from this, the main difference is a slower migration of the β -plane wake to the north. This is related to the more southern position (which is φ_r) of the vortex motion separatrix in the β -plane approximation compared to that in the full spherical dynamics. Under the β -plane treatment, the vortex center-of-mass trajectory thus stays closer to its separatrix than under the full spherical treatment, which results in a faster movement to the west. This implies that the tracers, left in the wake, get farther away from the vortex pair in a particular time interval, where they are influenced less by the velocity field of the pair. This mechanism, providing a background for our experience, is considered to be general when comparing advection under the β -plane and the full spherical treatment. We emphasize, however, that the similarity of the β -plane and the full spherical vortex trajectories is by far stronger than the similarity of the advection patterns.

In fact, one can find initial conditions for the vortex pair when the similarity in the vortex trajectories is of the same degree as before, but the advection patterns are also similar under the two treatments. This situation is illustrated in Fig. 7, exhibiting closed advection patterns. It is thus hard or maybe even impossible to predict the agreement of the advection pattern between the two treatments via only the visual inspection of the vortex trajectories.

The background mechanism described above leads also to enhanced escape under the β -plane treatment.

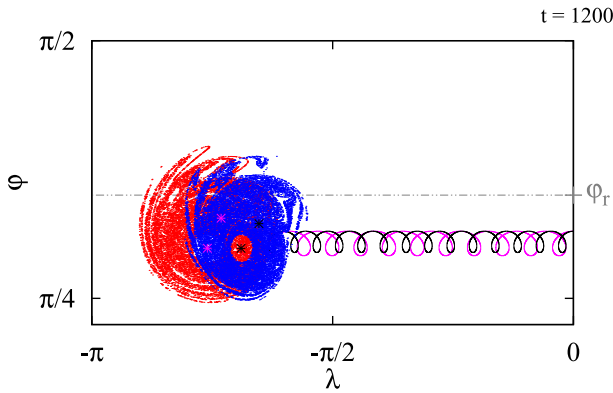


FIG. 7. The positions of $N \approx 17500$ tracers (dots) and the two vortices (symbols) at the time instant $t = 1200$. The tracers are initiated at $t = 0$ in between the elements of the vortex pair. Blue (red) tracers correspond to full spherical (β plane) treatment. The center-of-mass trajectory of the vortex pair is also shown up to the time instant t as a black (magenta) solid line, corresponding to full spherical (β plane) treatment. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 0.99$, $v_0 = 0$, and $u_0 < 0$. $\varphi_r = 1.1$ is marked by a gray double-dotted-dashed line. Vortex pair parameters: $D' = a' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-3}$.

The vortex pair's faster movement to the west under the β -plane treatment involves a smaller importance of self-intersections of its trajectory. Under the full spherical treatment, the close revisiting of the same spatial positions by the vortex pair make the escape of tracers more difficult. (This is also the mechanism responsible for the closure of the advection when initiating the vortex pair trajectory farther away from the separatrices.⁴)

A direct numerical comparison of the escape times of the tracers confirms that the β -plane treatment is characterized by lower escape times than the full spherical one, as seen in Fig. 8 (corresponding to the same vortex pair setting as that of Fig. 6). The difference in the escape times is reflected in the darker colors of Fig. 8b than those of Fig. 8a. The escape rates are found to be $\kappa = 0.0011$ and $\kappa = 0.0036$ for the β -plane approximation and for the full spherical treatment, respectively. We emphasize that the escape properties originate mainly in the dynamics taking place in the vicinity of the vortices (where linearization is expected to be applicable) and are, therefore, not affected by the behavior of the tracers far away from φ_r (where linearization would be incorrect in itself). The reason for this is that the migration of the particles from the vicinity of the vortices

⁴The background mechanism of the enhanced escape is valid only for the southern side of the separatrices in its presented form. A similar mechanism works on the northern side but with opposite result.

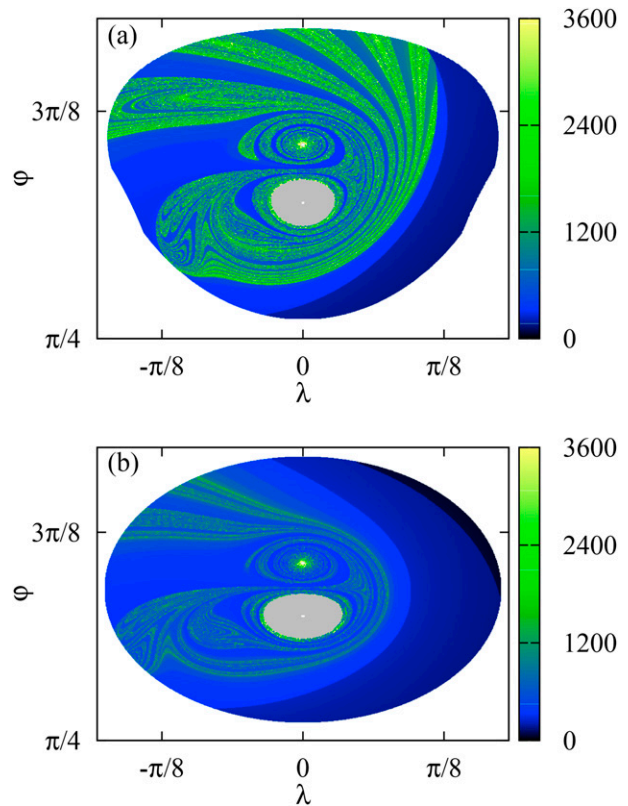


FIG. 8. The escape times under (a) the full spherical and (b) the β -plane treatments of $N \approx 195000$ tracers indicated by the coloring, plotted at the tracers' initial positions. Gray positions outline the vortex cores and correspond to tracers that do not leave the escape circle (with $\rho = 0.4$) within 3600 time units. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 1.07$, $v_0 = 0$, and $u_0 < 0$. Parameters: $\varphi_r = 1.1$, $D' = a' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-3}$.

out to the farther wake is a regular, fast, and mostly uniform process, whereas they enter this process only after their chaotic wandering in the vicinity of the vortex pair, in a rate dictated by the chaotic saddle located in this region. A factor more than 3 appearing in an exponent (the escape rate) characterizing material transport and the rather different patterns of the escape time distributions of Fig. 8 are perhaps the most striking effects to which the use of a β -plane approximation instead of the full spherical treatment can lead. This is even more surprising when taking into account that the time period of the φ dynamics is $T_p = 165.6$ and $T_p = 141.2$ under the β -plane approximation and the full spherical treatment, respectively. These close values also support the observation that the Eulerian properties of the β plane and the full spherical flows are rather similar.

The example in Fig. 9 shows that one can even find cases in which the full spherical treatment already exhibits closed advection when the β -plane advection is

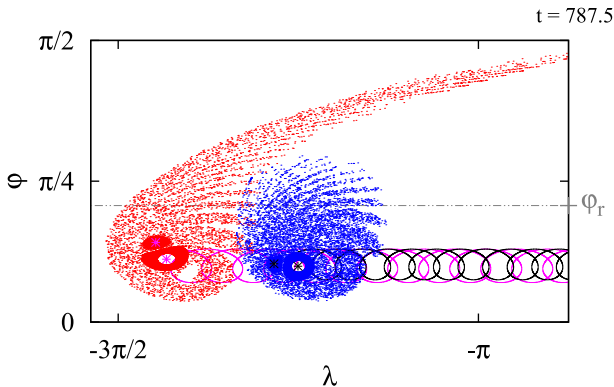


FIG. 9. The positions of $N \approx 17\,500$ tracers (dots) and the two vortices (symbols) at the time instant $t = 787.5$. The tracers are initiated at $t = 0$ in between the elements of the vortex pair. Blue (red) tracers correspond to full spherical (β plane) treatment. The center-of-mass trajectory of the vortex pair is also shown up to the time instant t as a black solid (magenta short dashed) line, corresponding to full spherical (β plane) treatment. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 0.407$, $v_0 = 0$, and $u_0 < 0$. $\varphi_r = 0.65$ is marked by a gray double-dotted-dashed line. Vortex pair parameters: $D' = a' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-2}$.

still open. This observation is particularly surprising given that the trajectories are now far away from the separatrices during the entire motion. Although we are a little bit too far from φ_r in this particular example for the applicability of any linear approximation in φ , our experience shows the possibility of the existence of such a behavior. Our finding is interpreted as a consequence of the following fact: the value of the critical point (in the initial latitude of the vortex pair) describing the previously mentioned phase transition in the advection pattern (between open and closed phases) is estimated to be approximately $\varphi_0 = 0.38$ under the β -plane treatment and $\varphi_0 = 0.41$ under the full spherical treatment. This shift may be regarded as a further indicator for the inappropriateness of the β -plane approximation.

8. Summary

Our aim was the investigation of the validity of the well-known and widely used approximation in geophysical fluid dynamics describing cases with a small-scale latitudinal motion of fluid elements—the so-called β -plane approximation. In the traditional β -plane approximation one replaces the sinusoidal dependence of the Coriolis parameter (the locally vertical component of the angular momentum of the planet) by its linearly approximated form, while the geometry is assumed to be fully planar. We have pointed out that the latter choice neglects even linear metric terms originating from the spherical geometry. The discrepancy in the

order of the approximation for the Coriolis parameter and for the geometry is a mathematical inconsistency of any traditional β -plane approximation.

We considered a point vortex pair model, describing a fluid dynamics on a rotating spherical surface, in which different approximations can easily be carried out. In this model the conservation of potential vorticity (angular momentum) on the rotating sphere is taken into account via the modulation of the circulation associated to any point vortex. As the modulation is proportional to the Coriolis parameter, it is an appropriate model for the study of the β -plane approximation. The model has a natural latitude, the reference latitude φ_r , that can be chosen as the origin of the β -plane approximation.

We have shown that the traditionally derived β -plane equations of motion for a vortex dipole do not contain certain terms that inevitably appear in a consistent first-order approximation of the full spherical equations around φ_r . Interestingly, the effect of these terms proved to be the most important near φ_r .

In the full spherical description, the special latitudes φ_{\pm} , which correspond to eastward and westward uniform propagation, have been found to differ from φ_r for $\varphi_r \neq 0$, in contrast to the case of the β -plane treatment where $\varphi_{\pm} = \varphi_r$. This fact in itself indicates some degree of inconsistency of traditional β -plane approximations. We note that the correctly approximated linear equations are able to reproduce the described difference of φ_{\pm} from φ_r .

Chaotic advection in the velocity field of modulated vortex pairs of finite size has also been studied. We have found that the different position of φ_{-} from φ_r has a latitudinal shifting effect on the advection when compared to the β -plane treatment (in which uniform westward propagation occurs on φ_r). The importance of the shift cannot be estimated from the pure observation of the vortex trajectories. In particular, this shift can be reflected in an enhanced escape of fluid elements from the neighborhood of the vortex pair, including cases when the open character of the advection, obtained under the β -plane treatment, contradicts the closed character of the full spherical treatment.

It is not excluded that one can find an appropriately chosen planar projection of the spherical coordinates and a corresponding nonlinear rescaling of the velocity component [as Gill and others did (Gill 1982; Verkley 1990; Harlander 2005) in the basic hydrodynamical context], in which the β -plane approximation of our vortex problem would appear without any further curvature terms. As we have illustrated, however, in the widely used flat geometry approach, the β -plane approximation is inconsistent and leads to results qualitatively different from those of a consistent local

linear approximation in which curvature terms appear unavoidably.

Although the β -plane approximation is, of course, not used today in the analysis and the simulation of real flows, it is a widespread tool in studying basic phenomena in environmental flows, both in theoretical and experimental works. Our results imply that in theoretical works (Paldor et al. 2007; Oruba et al. 2013) the use of the traditional β -plane approximation should be avoided and be replaced by mathematically consistent approximations. Similarly, the results of experiments with a topographic β plane in a planar geometry (Velasco Fuentes and van Heijst 1994) would have to be analyzed with a certain care when drawing conclusions for situations with a spherical geometry. In experiments applying vessels with a cylindrical geometry [obtaining a β effect either by a sloping topography (Fultz and Murty 1968; Sommeria et al. 1988; Meyers et al. 1989) or by an appropriate combination of the shape and the angular velocity of the vessel (Nezlin and Snezhkin 1993)], the results can be interpreted in a precise way only if a theoretical background is available that treats the cylindrical geometry consistently. Though the observations from such experiments are expected to be closer to the spherical phenomena, the different geometry of the sphere might still lead to significant deviations from these observations. We conjecture that a rotating sphere dynamics can be consistently approximated even in linear order by an experimental setting with a cylindrical geometry only if the parameters are adjusted appropriately; the discussion of this idea is, however, beyond the scope of the present paper.

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