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LETTER TO THE EDITOR

Geometrical multifractality of growing structures

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Abstract. We study the mass distribution of fractal objects growing by subsequent addition of units to the structure. It is demonstrated that in the limit of very large sizes such structures generally exhibit multifractal properties without defining a singular measure on the fractal support. Using the mass as a measure on the fractal, for a few simple deterministic processes we give the spectrum of corresponding fractal dimensions exactly. According to our results such geometrical multifractals should be very common and consist of parts which have a varying local fractal scaling of mass different from that of the whole object. We demonstrate the variation of multifractal properties as a function of the increasing size, in this way establishing for the calculation of multifractality an approach common in the studies of fractal properties of growing clusters.

Growing clusters and interfaces typically have a complex structure which in many cases can successfully be described in terms of fractal geometry [1]. This means that the mass as a function of size of such objects scales according to a non-integer exponent called fractal dimension. Recently it has been shown that in addition to the single fractal dimension of the cluster itself, singular distributions of various physical quantities defined on the fractal determines an infinite set of fractal dimensions each corresponding to the distribution of a given kind of singularity of the measure [2–11]. Different distributions define different multifractal spectra on the same fractal object. One may thus conclude that multifractality in the present approaches is a concept which is manifested through non-geometrical properties. In this letter we address the question of whether the growing structures themselves (with the mass as a measure defined on them) are multifractals.

Concentrating on the definition of fractal dimension through the amount of mass within a box of linear size L we shall show that the simple growing fractal structures introduced in this letter are geometrical multifractals. According to our results the scaling of the mass within such objects can be expressed in terms of an infinite hierarchy of exponents corresponding to different scalings of the mass distribution. Analogous phenomena follow implicitly from certain fractal measure constructions treated in [8], and can also be seen in examples of dynamical systems [12, 13] when the stationary distribution has a constant density. In this letter the approach of Hentschel and Procaccia [3], Grassberger [4] and Halsey *et al* [8] is extended to describe the geometrical multifractality of growing structures.

Let us consider a structure on a lattice built up by identical particles and let L and M denote the actual linear size and mass of the growing cluster, respectively. This structure is to be covered by boxes of size l such that

$$a \ll l \ll L \tag{1}$$

where a is the lattice constant. One can then determine the mass M_i of the ith non-empty box. The mass index α_i of this box is defined by

$$M_i \sim M(l/L)^{\alpha_i} \tag{2}$$

for $l/L \ll 1$. Those boxes which correspond to the same mass index α form a subset of all boxes. The centres of these boxes are on a fractal set of dimension $f(\alpha)$. Their number $N(\alpha)$ is, therefore, related to $f(\alpha)$ via

$$N(\alpha) \sim (l/L)^{-f(\alpha)}. \tag{3}$$

If there exists a set of different mass indices the growing structure will be called a geometrical multifractal, since the measure generating the spectrum is the uniform mass distribution (the Lebesque measure), and thus $f(\alpha)$ characterises the pure geometry of the system.

The generalised dimensions D_q [3, 4] can be extracted in such systems from the scaling relation

$$\chi_q(M, l, L) \equiv \sum M_i^q \sim M^q(l/L)^{(q-1)D_q}. \tag{4}$$

Following the arguments of [8], the relation between $f(\alpha)$ and D_q is found to be

$$(q-1)D_q = q\alpha(q) - f(\alpha(q)) \tag{5}$$

with $f'(\alpha(q)) = q$. It is to be noted that there are several ways to work with equations (2)-(4) depending on whether one changes l, L or l/L to see scaling behaviour.

Since geometrical multifractality as defined above is a consequence of local density fluctuations, it should be most pronounced in inhomogeneous growth processes. A necessary condition for observing this phenomenon is the existence of three well separated length scales (see (1)) which may require the linear size L to be close to the largest cluster size ever produced in numerical simulations. Instead of carrying out very large scale simulations, here we illustrate the general concepts by analytically tractable growth models. The first example is a generalisation of the deterministic fractal treated in [14]. The rule of construction is shown in figure 1(a). In the nth stage the twice enlarged version of the configuration corresponding to the $(n-1)$th stage of the growth is added to the four corners of the $(n-1)$th stage configuration. It is worth noting that, in spite of the fact that large connected plaquettes of size 2^{n-1} appear at the nth stage, their size relative to that of the cluster is $(2/5)^{(n-1)}$, vanishingly small for $n \to \infty$. A large cluster, therefore, can be considered to be a fractal. We calculate its generalised dimensions by applying self-similarity arguments [3, 8]. By reducing a large cluster of size $L \to \infty$ (which consists of a smaller central and four larger units, each having the overall shape of a square) by factors 5 and $\frac{5}{2}$ one obtains two structures, the first corresponding to the central square of size $L/5$ and the second one to the larger squares of size $2L/5$. Since the mass of these different squares is $M/17$ and $4M/17$, this leads to

$$\chi_q(M, l, L) = \chi_q(M/17, 5l, L) + 4\chi_q(4M/17, 5l/2, L). \tag{6}$$

An application of the scaling relation (4) then yields an (implicit) equation for D_q

$$(\tfrac{1}{17})^q 5^{(q-1)D_q} + 4(\tfrac{4}{17})^q(\tfrac{5}{2})^{(q-1)D_q} = 1. \tag{7}$$

These D_q values lies in the range between $D_\infty = \ln(17/4)/\ln(5/2) = 1.579\ldots$ and $D_{-\infty} = \ln(17)/\ln(5) = 1.760\ldots$. Consequently, there exists also a non-trivial $f(\alpha)$ spectrum and the object is a geometrical multifractal.

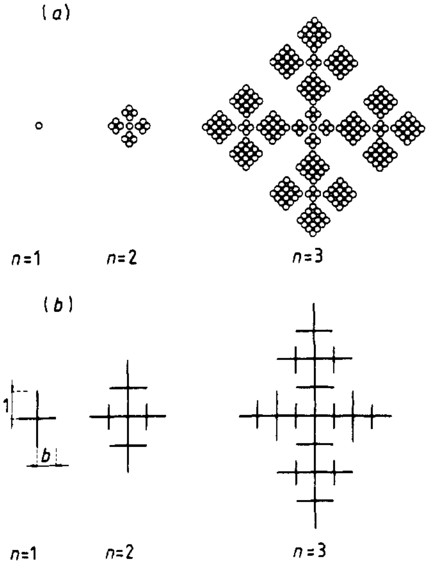

Figure 1. (a) Geometrical multifractal constructed on a lattice by an iteration process. In the nth step the twice enlarged copy of the previous $(n-1)$th stage is added to the corners of the $(n-1)$th configuration. (b) Another example growing by adding to the four principal tips of the $(n-1)$th configuration the structure itself without the lower main stem. This addition has to be done by applying appropriate rotation and shrinking to keep the ratio of the corresponding branches equal to $b<1$. One can make this model grow on a lattice by simultaneous blowing up of the configuration so that the smallest size always remains the same.

Although in this paper we deal with growing structures made of identical particles in figure 1(b) we demonstrate that other types of growing geometrical multifractals are also possible. This branching tree-like object develops by adding to the four main tips appropriate parts of the structure obtained in the previous stage of the growth. An expression analogous to (7) and the generalised dimensions $D_\infty = 1+\ln(1+b)/\ln(2/b)$ and $D_{-\infty}=1+\ln(1+b)/\ln(2)$ can easily be obtained for this fractal.

Our next example is a growth process in one dimension which enables us to determine how the multifractal behaviour emerges as the structure grows. We can keep track of the multifractal characteristics approaching their exactly calculated limiting values. The first unit of this one-dimensional process consists of three particles placed at the first, third and fourth sites. In the next stage the twice enlarged copy of this 'seed' configuration is added between the 9th and 16th sites (leaving out four sites equal to the length of the first configuration) and this procedure is repeated with the new configuration playing the role of the seed. Figure 2 shows the objects obtained in the first three steps of the construction.

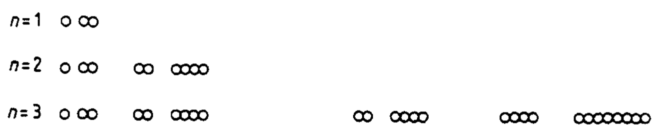

Figure 2. The first few steps in the construction of the growing asymmetric Cantor set.

After n steps the linear size is 4^n. It is therefore convenient to use $l=2^n$ as the box size. It can be easily observed that the number of non-empty boxes is then F_n, a Fibonacci number defined by $F_0=1$, $F_1=2$, $F_n=F_{n-1}+F_{n-2}$. Let $M_i^{(n)}$, $i=1,\ldots,F_n$ denote the mass of the ith non-empty box at the nth step of construction ($i=1$ corresponds to the leftmost box). The distribution $M_i^{(n)}$ is related to that of two previous generations for $n>2$ by

$$M_i^{(n)}=\begin{cases}3M_i^{(n-2)} & i=1,\ldots,F_{n-2}\\ 2M_{F_{n-2}+i}^{(n-1)} & i=1,\ldots,F_{n-1}\end{cases} \tag{8}$$

($M=3^n$) as can be checked directly. The 'initial values' for this recursion are $M_i^{(1)}=1,2$, $M_i^{(2)}=3,2,4$. At this point it is already possible to calculate a few characteristic exponents for the system. Since the number of non-empty boxes is F_n and $F_{n+1}/F_n\approx w$ for large n, where $w=(\sqrt{5}+1)/2$ is the golden mean, we find from (4) $D_0=\ln(w)/\ln(2)$ for the fractal dimension of the complete set. The densest box is the rightmost one with mass 2^n. Therefore, $\alpha_{\min}$ is obtained from (2) as $\alpha_{\min}=D_\infty=\ln(2/3)/\ln(1/2)=0.585\ldots.$ The most rarified non-empty interval is the first or the second one depending on the parity of n. For its mass we find $3^{(n-1)/2}$ for an odd n and $2\times 3^{n/2-1}$ for n even. Consequently, $\alpha_{\max}=D_{-\infty}=\ln(1/3)/\ln(1/4)=0.792\ldots.$

The picture simplifies further by observing that there are only $n+1$ different mass values in the set $M_i^{(n)}$. Denoting these quantities by $\tilde{M}_j^{(n)}$, $j=1,\ldots,n+1$ one obtains for $n>1$

$$\tilde{M}_j^{(n)}=\begin{cases}3^{[n/2]} & j=1\\ 2\tilde{M}_{j-1}^{(n-1)} & j=2,\ldots,n+1\end{cases} \tag{9}$$

with $\tilde{M}_j^{(n)}=1,2$ where [ ] denotes the integer part. Let the numbers $N_j^{(n)}$ denote how many times the value $\tilde{M}_j^{(n)}$ appears in $M_j^{(n)}$. They are found to follow a two-step recursion

$$N_j^{(n)}=N_{j-1}^{(n-1)}+N_j^{(n-2)} \qquad 1<j<n \tag{10}$$

and $N_j^{(n)}=N_n^{(n)}=N_{n+1}^{(n)}=1$. Now it is possible to derive an equation for the $f(\alpha)$ spectrum using (2), (3), (9) and (10). The solution is found to be

$$f(\alpha)=\left[(\alpha-\alpha_{\min})\ln\left(\frac{\alpha-\alpha_{\min}}{1-\alpha}\right)+2(\alpha_{\max}-\alpha)\right.$$
$$\left.\times\ln\left(\frac{2(\alpha_{\max}-\alpha)}{1-\alpha}\right)\right]\frac{1}{2\ln(1/2)(\alpha_{\max}-\alpha_{\min})}. \tag{11}$$

Finally, one can obtain from (5) and (11) an expression for the D_q spectrum.

The knowledge of recursions (9) and (10) makes it possible to calculate the quantity χ_q (4) numerically for any finite stage, n, of the construction. Comparing two subsequent stages of the growth process we define an n-dependent $D_q(n)$ by

$$\frac{\chi_{q,n+1}/M_{n+1}^q}{\chi_{q,n}/M_n^q}=\left(\frac{l_{n+1}/L_{n+1}}{l_n/L_n}\right)^{(q-1)D_q(n)} \tag{12}$$

where the subscript denotes quantities at the nth stage of the growth. For large n, $D_q(n)\to D_q$ as can be seen from (4). The results plotted in figure 3 show a rapid convergence as a function of n towards D_q, but note that the linear size in this model grows as 4^n.

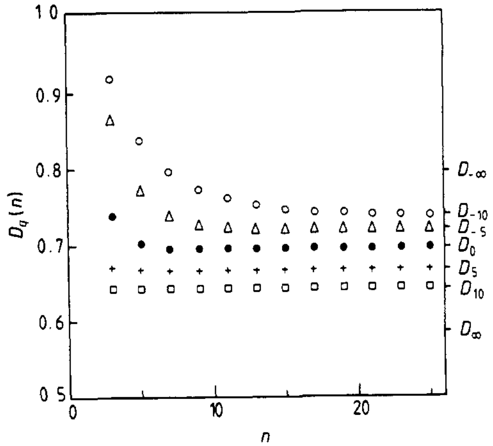

Figure 3. Convergence towards the generalised dimensions as a function of the actual size of the growing one-dimensional multifractal. The $D_q(n)$ have been calculated using (9), (10) and (12). The actual approach is alternating. Therefore, to make the figure more transparent we have plotted here the $D_q(n)$ values for odd n only. ○, $D_{-10}(n)$; △, $D_{-5}(n)$; ●, $D_0(n)$; +, $D_5(n)$; □, $D_{10}(n)$.

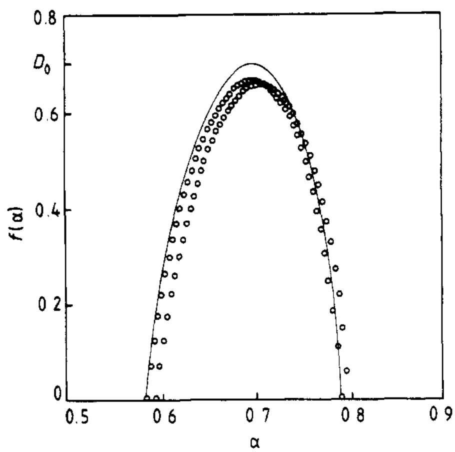

Figure 4. $f(\alpha)$ obtained from (11) is shown by the full curve. The open circles represent the function $f_j^{(n)}$ against $\alpha_j^{(n)}$ (see the text) for $n=100$. The special doubled internal structure of the latter plot is a consequence of the regular geometry of the model (it would be destroyed for a random configuration). A similar plot for $n=300$ (not shown) has circles practically all lying on the exact $f(\alpha)$ curve.

We have also calculated numerically the function $f_j^{(n)} = -\ln(N_j^{(n)})/\ln(l_n/L_n)$ against $\alpha^{(n)}j = \ln(\tilde{M}_j^{(n)}/M_n)/\ln(l_n/L_n)$ for increasing values of n. These quantities converge to $f(\alpha)$ and α as was shown above. The convergence is, however, slow since the constant factors not written out explicitly in (2) and (3) lead to contributions proportional to n^{-1}. The result for $n=100$ together with the exact solution is displayed in figure 4. Finally, we point out an important consequence of our results. In the previous examples the relationship

$$M \sim L^{-D} \tag{13}$$

between the mass and the linear size is fulfilled with $D = D_{-\infty}$ (i.e. with D different from the fractal dimension of the object, D_0!) just after the nth step of the construction ($L = L_n$, $M = M_n$). We conclude, therefore, that in a continuously growing cluster large fluctuations of D as the linear size L of the cluster increases might be, in general, a sign of geometrical multifractality. This behaviour is expected to be most pronounced for inhomogeneous growth occurring under non-stationary external conditions.

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