J. Phys. A: Math. Gen. 22 (1989) L691-L697. Printed in the UK

LETTER TO THE EDITOR

On the organisation of transient chaos—application to irregular scattering

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Received 25 May 1989

Abstract. It is shown how the local structure of chaotic repellers, being responsible for transient chaotic behaviour, is deduced from the properties of hyperbolic periodic orbits. Relations between static and dynamical multifractal spectra, with respect to the natural invariant measure on the repeller, are derived for invertible maps of the plane. The results obtained for maps with unit Jacobian apply to Hamiltonian systems with two degrees of freedom which exhibit the phenomenon of irregular scattering and are characterised by an exponential decay of trapping probabilty.

Chaotic behaviour can often be observed on finite timescales both in experiments [1] and in numerical simulations [2-14]. This transient chaos is associated with a so-called chaotic repeller (more precisely, chaotic saddle) in the phase space [4-7]. In contrast to chaotic attractors, these invariant objects are 'double' fractals [15]; they also possess structures on all length scales along their unstable directions [7]. Such strange sets play a role in different fields of physics ranging from hydrodynamics [1] through certain problems of disordered systems [5] to irregular scattering [16-23], a phenomenon of recently revived interest.

Irregular scattering occurs in scattering problems characterised by an extended and non-trivial interaction region. There exists then, in the phase space, a fractal set of initial conditions for which trajectories stay within the interaction region for arbitrarily long time and exhibit chaotic properties. For a review, see [21]. The set of the bounded trajectories can be considered as a repeller, whereas trajectories coming close to it are transiently chaotic. Although such systems are Hamiltonian, we point out that their chaotic and multifractal properties can be understood by applying the same laws as in dissipative cases, and finally letting the dissipation disappear. The only condition found for this is an exponential decay (in time) for the number of particles staying inside the interaction region.

Permanent chaos is organised around (unstable) periodic orbits [24-30]. Based on the fact that strange attractors are densely covered by such orbits, it has been shown [28, 29] that the location and stability of periodic orbits determine the structures of strange attractors in their neighbourhoods. Our aim here is to extend this approach to the problem of transient chaos. We shall show that the local structure of chaotic repellers also can be derived from the properties of hyperbolic orbits, and that useful relations follow among multifractal spectra of entropies and partial dimensions. Certain

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0305-4470/89/140691+07$02.50

aspects of this problem have been treated in § VII of [30]. We apply here a different method which enables us to deduce the fractal structure not only along the stable direction but also along the unstable one.

We investigate invertible maps of the plane $x' = M(x)$ and assume that the repeller is hyperbolic, i.e. to each point of it there exist distinct stable and unstable manifolds. Hyperbolicity is much more common for repellers than for attractors since often the folds of the unstable manifold (where homoclinic tangencies may occur) do not belong to the repeller [7].

The main problem in describing the metric properties of transient chaos is how to find an invariant distribution. In fact, sooner or later, all trajectories, except for a set of measure zero, leave any neighbourhood of the repeller. The probability that a randomly chosen point has not yet escaped a given neighbourhood after n steps decays, typically, as $\exp(-\kappa n)$, where κ is the escape rate [5]. Nevertheless, an invariant distribution can be defined by compensating for this escape. Let us start by distributing a great number of initial points on a surrounding S of the repeller. In other words, we start with an initial measure on this region. By subsequent applications of the map, the boundaries of the image region approach the repeller along the stable direction but escape is taking place along the unstable one. The final measure would be zero on the original neighbourhood S of the repeller. If, however, after each step the measure is multiplied by $\exp(\kappa)$, i.e. if exactly that number of particles is pumped onto S which escaped it, a well defined limit exists [3, 7, 8]. The procedure is the analogue of the Bowen–Ruelle–Sinai construction of the natural measure on chaotic attractors [31].

The limiting measure, known as the *conditionally invariant measure* [3] (or c-measure for short) is in hyperbolic cases *smooth* along the unstable manifold but has fractal structure along the stable one [3, 8]. The crowding index [32] along the stable direction will be denoted by α_2. The *true* invariant distribution sitting on the repeller can then be approached by restricting the c-measure to a refining coverage of the repeller and normalising so that the total measure stays constant, where the refining partition is obtained by taking the *cross sections* of the nth images and pre-images of the surrounding S, for $n \gg 1$. This natural invariant distribution can be obtained in experiments or numerical simulations from asymptotically long chaotic transients [7]. It will be crucial for what follows that the fractal properties of the true invariant distribution differ from those of the c-measure only along the unstable direction where the former has non-trivial crowding indices, α_1.

Chaotic repellers seem to be the closures of the set of all hyperbolic periodic orbits [4, 5]. Let us consider a small box of size l_1 (l_2) along the unstable (stable) manifold around an element x_0 of a hyperbolic m-cycle, $m \gg 1$. The c-measure inside this box $P_c(l_1, l_2)$ can be expressed, according to the definition of the crowding index as

$$P_c(l_1, l_2) \sim l_1 l_2^{\alpha_2}. \tag{1}$$

Here no anomalous scaling appears in l_1 since the c-measure is smooth along direction 1. The mth pre-image of the box is of size ($l_1 \exp(-\lambda_1^{(m)}), l_2 \exp(-\lambda_2^{(m)})$) where $\exp(\lambda_i^{(m)})$ denote the eigenvalues of the m-fold iterated map at x_0. Due to the quasi-invariance of the c-measure

$$\begin{aligned} P_c(l_1, l_2) &= P_c(l_1 \exp(-\lambda_1^{(m)}), l_2 \exp(-\lambda_2^{(m)})) \exp(\kappa m) \\ &\sim l_1 l_2^{\alpha_2} \exp(-\lambda_1^{(m)} + \kappa m + \lambda_2^{(m)} \alpha_2). \end{aligned} \tag{2}$$

In the last line, we used the fact that the mth pre-image of the box is again around x_0. Thus, the relation

$$\lambda_1 - \kappa + \lambda_2\alpha_2 = 0 \tag{3}$$

follows where the local Lyapunov exponents $\lambda_i = \lambda_i^{(m)}/m$, $i = 1, 2$, have been introduced. By repeating the argument now for the true invariant distribution (no compensation, but non-trivial α_1), one obtains a second equation (which holds also for attractors [28, 29]),

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0. \tag{4}$$

Consequently,

$$\alpha_1 = 1 - \kappa/\lambda_1 \tag{5}$$

i.e. along the unstable manifold the same relation holds as for repellers of one-dimensional maps [33, 34], as might be expected. Equations (3)–(5) imply that fractal properties of the repeller can be deduced by determining the local Lyapunov exponents of a nearby hyperbolic periodic orbit. The number of cycle points $N_i(\alpha_i)$ with a given partial crowding index $\alpha_i (i = 1, 2)$ defines a function $f_i(\alpha_i)$ via $N_i(\alpha_i) \sim \varepsilon_i^{-f_i(\alpha_i)}$, where ε_i is the length scale of the corresponding box in the above-mentioned refining partition. By using cycles of length m, these local scales can be approximated [28, 29] as $\varepsilon_1 \sim \exp(-\lambda_1^{(m)})$ and $\varepsilon_2 \sim \exp(\lambda_2^{(m)})$. The partial generalised dimensions $D_q^{(i)}$ [35, 36] can then be obtained as the Legendre transform of $f_i(\alpha_i)$ $(i = 1, 2)$:

$$(q-1)D_q^{(i)} = q\alpha_i(q) - f_i(\alpha_i(q)) \tag{6}$$

where $f_i'(\alpha_i(q)) = q$, and the total generalised dimension D_q is their sum $D_q = D_q^{(1)} + D_q^{(2)}$ [36].

As for the multifractal properties of the local Lyapunov exponents, the latter are connected with the generalised entropies, or the so-called dynamical multifractal spectrum, $f_0(\alpha_0)$ [37–39] ($g(\Lambda)$ in the notation of [37]). In the case of strange attractors, the path probability $W_j^{(m)}$ for trajectories of length m having the same symbolic fate as the m-cycle j, is proportional to the reciprocal value of the expansion rate $\exp(\lambda_1^{(m)})$ [37]. For transient chaos, the latter is to be multiplied with the probability that the trajectory has not yet escaped, which leads to [7, 39]

$$W_j^{(m)} \sim \exp(\kappa m)\exp(-\lambda_1^{(m)}). \tag{7}$$

The dynamical scaling index α_0, defined in [37] via $W_j \sim \exp(-\alpha_0 m)$ is then obtained as

$$\alpha_0 = \lambda_1 - \kappa. \tag{8}$$

The number of trajectories with a given α_0 grows like $\exp(mf_0(\alpha_0))$ [37, 38]. This can be used to calculate the generalised entropies K_q [40]. One finds

$$(q-1)K_q = q\alpha_0(q) - f_0(\alpha_0(q)) \tag{9}$$

with $f_0'(\alpha_0(q)) = q$.

We note in passing that based on the definition

$$\langle (W_j^{(m)})^{(q-1)} \rangle \sim \exp(m(1-q)K_q) \tag{10}$$

another relation also follows. The bracket $\langle\ \rangle$ denotes here the average over the invariant distribution. From (7) and a cumulant expansion, we obtain

$$K_q = \bar{\lambda}_1 - \kappa + \sum_{l=2} \frac{1}{l!}(1-q)^{l-1}Q_l \tag{11}$$

where mQ_l is the order l cumulant of the fluctating quantity $\lambda_1^{(m)}$ [36] on the repeller. $Q_1 \equiv \bar{\lambda}_1$ denotes the averaged Lyapunov exponent. Equation (11) is an extension of the formula derived in [36] for chaotic attractors.

Equations (3)-(5) and (8) can easily be checked on the example of a generalised Baker's transformation [8]

$$x' = \begin{cases} ax & y < c \\ \frac{1}{2} + bx & y > c \end{cases}$$
$$y' = \begin{cases} sy & y < c \\ 1 - t(1-y) & y > c. \end{cases} \tag{12}$$

For $0 < a, b, c < \frac{1}{2}$ and $sc, t(1-c) > 1$ this map possesses a chaotic repeller which is a prototype for a hyperbolic horseshoe [41] and is expected to be present locally in every hyperbolic repeller.

In what follows, we derive further relations between static and dynamical quantities. Since there is a unique connection between λ_1 and α_1, the number of trajectories $\exp(mf_0(\alpha_0))$ with a given scaling index α_0 is the same as the number of boxes with a crowding index $\alpha_1 = \alpha_0/(\alpha_0 + \kappa)$ (see (5) and (8)). Since the latter is $\varepsilon_1^{-f_1(\alpha_1)}$, we obtain

$$\frac{f_1(\alpha_1)}{\alpha_1} = \frac{f_0(\alpha_0)}{\alpha_0}\bigg|_{\alpha_0 = \alpha_1\kappa/(1-\alpha_1)}. \tag{13}$$

By differentiating and taking into account (6) and (9) an implicit relation is found:

$$D_q^{(1)} = \frac{K_{q'}}{K_{q'} + \kappa}\bigg|_{q' = q-(q-1)D_q^{(1)}} \tag{14}$$

which is the analogue of what is valid for repellers of one-dimensional maps [10].

Since α_2 is expressed in terms of two variables λ_1, λ_2 (see (3)), one cannot derive an equation between the partial dimensions $D_q^{(2)}$ and the entropies. Nevertheless, numerically, $f_2(\alpha_2)$ can always be determined by calculating the escape rate and λ_1, λ_2 for all cycles of length m ($\gg 1$).

In maps with a constant Jacobian J, however,

$$\lambda_1 + \lambda_2 = \ln J \tag{15}$$

holds leading to a unique relation between α_2 and α_0. By repeating the argument above, we find

$$\frac{f_2(\alpha_2)}{\alpha_2} = \frac{f_0(\alpha_0)}{\alpha_0}\bigg|_{\alpha_0 = \alpha_2(\kappa - \ln J)/(1-\alpha_2)} \tag{16}$$

or

$$D_q^{(2)} = \frac{K_{q'}}{K_{q'} + \kappa - \ln J}\bigg|_{q' = q-(q-1)D_q^{(2)}}. \tag{17}$$

As a consequence

$$\frac{f_1(\alpha_1)}{\alpha_1} = \frac{f_2(\alpha_2)}{\alpha_2}\bigg|_{\alpha_2 = \kappa\alpha_1/(\kappa - \ln J(1-\alpha_1))} \tag{18}$$

and

$$D_q^{(1)} = \frac{D_{q'}^{(2)}(\kappa - \ln J)}{\kappa - \ln J D_{q'}^{(2)}}\bigg|_{q' = q+(q-1)D_q^{(1)}\ln J/(\kappa - \ln J)}. \tag{19}$$

Thus, repellers of two-dimensional maps with constant Jacobian are particularly simple multifractals since any of the spectra $f_i(\alpha_i)$, $i = 0, 1, 2$, completely determines the other two. For $q \to 1$, equations (14), (17) and (19) become explicit, we recover the expression [7] $\kappa = (1 - D_1^{(1)})\bar{\lambda}_1$ and (19) goes over into a Kaplan-Yorke formula [42]. Furthermore, in the limit $\kappa \to 0$, relations (14), (16) and (17) go over into those derived for hyperbolic attractors [43].

Finally, let us turn to the Hamiltonian limit where a considerable simplification occurs. For $J = 1$, due to the reversibility of the motion on the repeller, $\lambda_1 = -\lambda_2$ and, consequently,

$$\alpha_1 = \alpha_2 \qquad f_1(\alpha_1) = f_2(\alpha_2) \tag{20a}$$

$$\frac{f_1(\alpha_1)}{\alpha'} = \frac{f_0(\alpha_0)}{\alpha_0}\bigg|_{\alpha_0 = \alpha_1\kappa/(1-\alpha_1)} \tag{20b}$$

$$D_q^{(1)} = D_q^{(2)} = \frac{K_{q'}}{K_{q'} + \kappa}\bigg|_{q' = q-(q-1)D_q^{(1)}} \tag{20c}$$

i.e. the multifractal spectra along stable and unstable directions are equivalent.

These formulae can be relevant for the phenomenon of irregular scattering. It has been pointed out [17, 20] that the set of unstable periodic orbits is responsible for the irregular scattering. In order to apply the formulae above, an extra condition is the exponential decay of the survival inside the interaction region. This is ensured if there are no KAM tori in the system, but might be fulfilled in more general cases too. The averaged lifetime of a particle inside the interaction region corresponds then to $1/\kappa$. If so, in systems with two degrees of freedom, where a Poincaré plane can be introduced (see Jung (1986) in [17]), relations (20) hold. They imply that the repeller with its natural invariant distribution is an isotropic multifractal.

As a special consequence of (20c), we obtain for the partial fractal dimensions

$$D_0^{(1)} = D_0^{(2)} = \frac{{}^{\kappa}D_0^{(1)}}{K_{D_0^{(1)}} + \kappa}. \tag{21}$$

The relevance of these quantities follows from the fact that $D_0^{(1)} = D_0^{(2)}$ can easily be measured in scattering problems. Let us fix, as done in [20], a straight line in the phase space sufficiently far away from the repeller. Start trajectories from this line and specify the intervals from which trajectories do not leave the interaction region until at least n collisions. In the limit $n \to \infty$ these intervals approach a Cantor set C. Since the stable manifolds of the repeller are assumed to extend smoothly to infinity, the fractal dimension of the set C is the same as that of the refining partition along the unstable direction, i.e. $D_0^{(1)}$. Furthermore, the dimension of C is independent of the orientation of the line on which C is sitting.

In the case of sufficiently rarified repellers, when $D_0^{(1)} \ll 1$, the quantity

$$d_f \equiv \frac{K_0}{K_0 + \kappa} \tag{22}$$

where K_0 denotes the topological entropy, might be a good approximation to the partial fractal dimensions. In fact, d_f has been introduced as an approximant to the dimension of the Cantor set C [20]. For exceptional monofractal repellers $K_q \equiv K_0$, $D_q^{(i)} \equiv D_0^{(i)}$ and one finds $D_0^{(i)} = d_f$, but otherwise the difference $d_f - D_0^{(i)}$ is proportional to the derivative $\mathrm{d}K_q/\mathrm{d}q$ taken at $q = 0$ which can be considered as a number characterising the multifractality of the repeller.

We have shown that the concept of transient chaos can successfully be applied to irregular scattering in unbounded systems. Measuring the escape rate and the distribution of either the local Lyapunov exponents ($f_0(\alpha_0)$) or the partial crowding indices ($f_1(\alpha_1)$) completely specifies all multifractal properties of such Hamiltonian repellers in systems with two degrees of freedom.

The author is indebted to I Procaccia for illuminating exchanges of ideas concerning the organisation of chaos. He benefited from discussions with B Eckhardt, G Eilenberger, P Grassberger, Z Kovács, and U Smilansky. The kind hospitality at the IFF of the KFA Jülich is acknowledged.

Note added in proof. It is worth giving also the global versions, containing sums over all hyperbolic periodic orbits of length $m \gg 1$, of a few relations treated above. The generalised entropies and partial dimensions of chaotic repellers, with respect to the natural measure, can be deduced from the rules

$$\sum_j e^{-\lambda_{1j}^{(m)} q} \sim e^{-(\kappa q + \tau_0(q))m} \tag{23a}$$

$$e^{\kappa q m} \sum_j e^{\lambda_{1j}^{(m)}(\tau_1(q) - q)} \sim 1 \tag{23b}$$

$$e^{\kappa q m} \sum_j e^{-\lambda_{2j}^{(m)} \tau_2(q) - \lambda_{1j}^{(m)} q} \sim 1 \tag{23c}$$

where $\tau_0(q) = (q-1)K_q$, $\tau_i(q) = (q-1)D_q^{(i)}$, $i = 1, 2$. Equation (14) and, in case of constant Jacobian, equations (17) and (19) immediately follow. For $\lambda_{1j}^{(m)} = -\lambda_{2j}^{(m)}$ relations (23) apply to irregular scattering.

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