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On the Stationary Distribution of Self-Sustained Oscillators around Bifurcation Points

Tamás Tél[1]

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A double expansion in powers of the damping coefficient and noise intensity is shown to be a powerful method for obtaining the stationary distribution of systems that after rescaling become weakly damped conservative ones. Systems undergoing Hopf bifurcations belong to this class. As an illustrative example, the generalized van der Pol oscillator is considered around its bifurcation point. A calculation is carried out up to third order in both the noise intensity and the bifurcation parameter (damping coefficient).

KEY WORDS: Stationary distribution; weak noise expansion; Hopf bifurcation; van der Pol oscillator.

1. INTRODUCTION

Time-periodic asymptotic behavior governed by a limit cycle attractor in the phase space is characteristic for nonequilibrium dissipative systems. Such self-sustained oscillations can be found in different fields of physics, electronics, chemistry, biology, and other disciplines.[1-6] The appearance of this temporal order via bifurcation is a phenomenon of special interest. Around instability points the external noise may play an essential role, which makes a stochastic description[7-12] necessary. The asymptotic behavior of the system is then characterized by a stationary distribution. To determine the stationary distribution of nonlinear oscillators around their bifurcation points, we use a method first applied, in leading order, to the problem of codimension-two bifurcations.[13] It will be illustrated here that the method can be developed to be a systematic one.

The procedure is based, on one hand, on a weak noise expansion,

[1] Institute for Theoretical Physics, Eötvös University, H-1088 Budapest, Hungary.

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which has been applied in several cases for calculating asymptotic results.[14–36,13] On the other hand, it is based on the property that the system of interest, after rescaling, can be considered as a weakly damped conservative system with a damping constant proportional to the bifurcation parameter. At the bifurcation point, therefore, the rescaled deterministic system is purely conservative and possesses a conserved quantity. These seem to be typical features of instabilities associated with Hopf bifurcations (and have been known for codimension-two cases[6]). The limit cycle just after appearance is therefore a contour along which the conserved quantity is constant. As a further consequence, we show that the stationary distribution of the rescaled noisy system is constant along the contours of constant conserved quantity. Away from the bifurcation point the stationary distribution is demonstrated to appear as a power series in the bifurcation parameter.

The van der Pol oscillator is one of the most extensively studied systems in nonlinear dynamics (see, e.g., Refs. 37–40). The unforced system undergoes a Hopf bifurcation. In the vicinity of this point the deterministic dynamics on a sufficiently long time scale and in suitably chosen coordinates can be approximately described by a normal form.[6] This form possesses a potential which is identical to the laser potential.[41] The stationary distribution of the noisy van der Pol oscillator, just like that of other nonlinear oscillators,[42] is, however, dominated by this potential in a certain region of the parameter space only. We are interested here in how to determine the deviation of the distribution from the rotationally invariant one given by the potential of the normal form; therefore, we do not want to restrict ourselves to the very vicinity of the bifurcation point nor to asymptotically small noise intensities. It will be convenient to study a family of oscillators undergoing Hopf bifurcations, in which the van der Pol oscillator appears as a special case. We perform an expansion up to third nontrivial orders in both the bifurcation parameter and noise intensity.

The paper is organized as follows. In Section 2 a brief summary of the weak noise expansion method is given for Fokker–Planck models. A generalized stochastic van der Pol oscillator is introduced in Section 3, and the rescaled problem, which at the bifurcation point corresponds to a harmonic oscillator dynamics, is defined in Section 4. The expansion of the stationary distribution in powers of the bifurcation parameter is performed for this system in Sections 5–7 for subsequently increasing order contributions in the noise intensity. Some technical details are relegated to the Appendix. The explicit expression for the stationary distribution of the generalized van der Pol oscillator is given in Section 8. The paper closes with a few concluding remarks.

2. THE WEAK NOISE EXPANSION

We summarize here the weak noise expansion method for the stationary distribution of continuous stochastic processes. Let us consider a system of n variables, $-\infty < q^\nu < \infty$, $\nu = 1,..., n$, the dynamics of which is governed by the Langevin equation

$$\dot{q}^\nu = K^\nu(q) + \eta^{1/2} g_i^\nu \xi^i(t) \tag{2.1}$$

where K^ν stands for a deterministic drift, generally nonlinear in q, and g_i^ν are coupling constants. The noise ξ^i is assumed to be a Gaussian white one with $\langle \xi^i(t) \rangle = 0$, $\langle \xi^i(t)\, \xi^j(0) \rangle = \delta^{ij}\delta(t)$, and η denotes the noise intensity. (Summation over repeated lower and upper indices is implied.) For the sake of definiteness, we interpret Eq. (2.1) in the sense of Ito. The stationary distribution of the process can always be written as

$$P(q; \eta) = \exp[-\Phi(q; \eta)/\eta] \tag{2.2}$$

It has been shown[16c,27c] by means of the path integral solution of the corresponding Fokker–Planck equation that for extremely weak noise, i.e., for $\eta \to 0$, the limit $\Phi(q; 0) \equiv \phi^{(0)}(q)$ exists and that the first correction to $\phi^{(0)}$ is proportional to the noise intensity. It is expected, in general, that Φ is analytic in η and can be represented by the series

$$\Phi(q; \eta) = \phi^{(0)}(q) + \eta\phi^{(1)}(q) + \eta^2\phi^{(2)}(q) + \cdots \tag{2.3}$$

Equations for $\phi^{(j)}$ can then be easily derived from the time-independent Fokker–Planck equation by substituting (2.3) and collecting terms of the same order of magnitude in η. Thus, one obtains for $\phi^{(0)}$ a Hamilton–Jacobi-type equation:

$$\frac{1}{2} Q^{\nu\mu} \frac{\partial\phi^{(0)}}{\partial q^\nu} \frac{\partial\phi^{(0)}}{\partial q^\mu} + K^\nu(q) \frac{\partial\phi^{(0)}}{\partial q^\nu} = 0 \tag{2.4}$$

where the diffusion matrix is related to the coupling constants via $Q^{\nu\mu} = \sum_i g_i^\nu g_i^\mu$. For the sake of simplicity $Q^{\nu\mu}$ will be assumed here to be a constant. The quantity $\phi^{(0)}$, called the nonequilibrium potential, is of considerable interest in its own right. It is, e.g., the Liapunov function of the deterministic dynamics; consequently, it must be minimal on the attractors of the deterministic motion.[16,27] On the other hand, it generalizes the concept of free energy for nonequilibrium systems. As has been shown in Ref. 27, the coexistence of attractors may lead to the appearance of merely piecewise differentiable potentials. This, however, will not be the case in systems to be studied in this paper. From the point of view of the weak noise expansion, $\phi^{(0)}$ is the starting point of the procedure, and is to be determined by solving (2.4) with the boundary condition that $\phi^{(0)}$ is minimal on the attractors.

The equation for $\phi^{(1)}$ is obtained from (2.4) in the form

$$\left(K^{\nu}+Q^{\nu\mu}\frac{\partial\phi^{(0)}}{\partial q^{\mu}}\right)\frac{\partial\phi^{(1)}}{\partial q^{\nu}}=\frac{\partial K^{\nu}}{\partial q^{\nu}}+\frac{1}{2}Q^{\nu\mu}\frac{\partial^{2}\phi^{(0)}}{\partial q^{\nu}\,\partial q^{\mu}} \tag{2.5}$$

This is a linear problem for $\phi^{(1)}$, or for the so-called prefactor $z \equiv \exp(-\phi^{(1)})$, after $\phi^{(0)}$ has been calculated. Since the potential is constant on the attractors, the dominating contribution to the stationary distribution on attractors is given by the prefactor z.[25]

Similarly, if both $\phi^{(0)}$ and $\phi^{(1)}$ are known, the third term can be obtained by solving

$$\left(K^{\nu}+Q^{\nu\mu}\frac{\partial\phi^{(0)}}{\partial q^{\mu}}\right)\frac{\partial\phi^{(2)}}{\partial q^{\nu}}=\frac{1}{2}Q^{\nu\mu}\left(\frac{\partial^{2}\phi^{(1)}}{\partial q^{\nu}\,\partial q^{\mu}}-\frac{\partial\phi^{(1)}}{\partial q^{\nu}}\frac{\partial\phi^{(1)}}{\partial q^{\mu}}\right) \tag{2.6}$$

We shall see that the single-valuedness of Φ prescribed as a solvability condition makes the solution of (2.4)–(2.6) unique.

The main drawback of the scheme sketched above is the nonlinearity of Eq. (2.4). Therefore, as demonstrated in Refs. 21, 26, 27, and 13, in order to find analytic results one has, in general, to seek for an appropriate parameter and expand $\phi^{(0)}$ in powers of it. In the case of self-sustained oscillators this parameter can be chosen to be the bifurcation parameter.

3. THE MODEL

We consider the generalized van der Pol oscillator, the deterministic dynamics of which is defined by the following equations[40]:

$$\begin{aligned}\dot{x}&=v\\ \dot{v}&=-x+v(\alpha-\gamma x^{2}-\delta v^{2})\end{aligned} \tag{3.1}$$

where x and v denote the position and the velocity, respectively, of a particle of unit mass. Here α, γ, and δ are real parameters. The van der Pol oscillator is recovered for $\delta=0$. The case $\gamma=0$ corresponds to the Rayleigh equation,[43] which can be mapped (by the transformation $v\to x$) into the van der Pol form with $\gamma=3\delta$. The oscillator (3.1) exhibits a supercritical Hopf bifurcation at $\alpha=0$ for $\gamma+3\delta>0$, which can easily be seen by applying van der Pol's method of adiabatic elimination.[1]

When studying the influence of noise on this oscillator it is natural to allow for a Langevin force (of intensity η) in the momentum equation only.[40] Thus, the diffusion matrix has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that the equivalence of the cases $\gamma = 0$ and $\delta = 0$ is no longer valid, due to the asymmetric role the noise terms play. No exact solution is known for the stationary distribution of the stochastic system except the special case $\gamma = \delta$,[44] when "detailed balance" holds[45] and

$$\Phi = -\alpha(x^2 + v^2) + \frac{\gamma}{2}(x^2 + v^2)^2 \tag{3.2}$$

In this case Φ is independent of the noise intensity. It is thus identical with the nonequilibrium potential and happens to be the same as the potential of the normal form mentioned in Section 1. For $\gamma \neq \delta$ the quantities Φ, $\phi^{(0)}$, and the potential of the normal form are different.

4. THE RESCALED PROBLEM

Scaling the parameters and the variables of the oscillator defined in the previous section by

$$\begin{aligned} &\alpha = \beta s, \qquad s \equiv \operatorname{sgn}(\alpha), \qquad \eta = \beta \bar{\eta} \\ &x = \beta^{1/2} \bar{x}, \qquad t = \bar{t}, \qquad v = \beta^{1/2} \bar{v} \end{aligned} \tag{4.1}$$

and omitting bars henceforth, we obtain the equations of motion in the form

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x + \beta v(s - \gamma x^2 - \delta v^2) + \eta^{1/2} \xi^1 \end{aligned} \tag{4.2}$$

The deterministic dynamics is thus that of a harmonic oscillator with a nonlinear damping. It is to be noted that the amplitude of the limit cycle now remains finite when approaching the bifurcation point from above ($s = 1$). Around the bifurcation point, i.e., $\beta \ll 1$, the damping is weak. When evaluating the stationary distribution $\exp[-\Phi(x, v; \eta)/\eta]$, it is therefore convenient to use the energy

$$E = (x^2 + v^2)/2 \tag{4.3}$$

as a new state variable. We eliminate v by

$$|v| \equiv v(x, E) \equiv (2E - x^2)^{1/2} \tag{4.4}$$

and define

$$\Phi^{\pm}(x, E; \eta) \equiv \Phi(x, \pm v(x, E); \eta) \tag{4.5}$$

where $\pm$ corresponds to the sign of v. In what follows our aim is to calculate $\Phi^{\pm}(x, E; \eta)$ for the rescaled problem (4.2) and to turn back to the original variables at the end.

5. THE NONEQUILIBRIUM POTENTIAL

The equation for the nonequilibrium potential $\phi^{(0)\pm}(x, E) = \Phi^{\pm}(x, E; \eta=0)$ follows from (4.2) using the x, E representation of the Hamilton–Jacobi equation (2.4), which has the following form:

$$\frac{|v|}{2}\left(\frac{\partial \phi^{(0)\pm}}{\partial E}\right)^2 + \beta\, |v|\, (s - \gamma x^2 - \delta\, |v|^2) \frac{\partial \phi^{(0)\pm}}{\partial E} \pm \frac{\partial \phi^{(0)\pm}}{\partial x} = 0 \tag{5.1}$$

In this and the next two sections we use $|v|$ as a short-hand notation for $v(x, E)$.

The single-valuedness of the potential provides a solvability condition for (5.1). Since the contour integral of $d\phi^{(0)\pm}$ along a constant E curve must vanish, we obtain

$$\int_{-R(E)}^{R(E)} \left.\frac{\partial \phi^{(0)+}}{\partial x}\right|_E dx + \int_{R(E)}^{-R(E)} \left.\frac{\partial \phi^{(0)-}}{\partial x}\right|_E dx = 0 \tag{5.2}$$

for the physically acceptable solution of (5.1), where $R(E) = (2E)^{1/2}$ is the radius of the $E = \text{const}$ circle in the phase space of the harmonic oscillator. It will be seen that the solution of (5.1) is unique (up to an additive constant) with condition (5.2).

Assuming analyticity, the potential appears as a power series in β. Since in a conservative system the potential is constant,[13,32] $\phi^{(0)\pm}(x, E)$ must be independent of (x, E) for $\beta \to 0$. Therefore, we look for a solution in the form

$$\phi^{(0)\pm}(x, E) = \beta \phi_1^{(0)\pm}(x, E) + \beta^2 \phi_2^{(0)\pm}(x, E) + \beta^3 \phi_3^{(0)\pm}(x, E) + \cdots \tag{5.3}$$

where $\phi_i^{(0)\pm}$ is independent of β.

After substituting (5.3) into (5.1), one finds in leading order, i.e., in order β,

$$\partial \phi_1^{(0)\pm} / \partial x = 0 \tag{5.4}$$

It is solved by

$$\phi_1^{(0)\pm}(x, E) = F_1(E) \tag{5.5}$$

where F_1 is a still arbitrary function of E.

The order β^2 contribution to (5.1) reads

$$\frac{|v|}{2}\left(\frac{\partial\phi_1^{(0)\pm}}{\partial E}\right)^2 + |v|\,(s-\gamma x^2-\delta\,|v|^2)\frac{\partial\phi_1^{(0)\pm}}{\partial E} \pm \frac{\partial\phi_2^{(0)\pm}}{\partial x} = 0 \qquad (5.6)$$

The solvability condition (5.2) applied to $\phi_2^{(0)\pm}$ immediately determines $F_1(E)$. From (5.2), (5.5), and (5.6) we find

$$F_1'(E) = -2\left(s - \gamma\frac{\overline{|v|\,x^2}}{\overline{|v|}} - \delta\frac{\overline{|v|^3}}{\overline{|v|}}\right) \qquad (5.7)$$

where prime denotes differentiation, and

$$\overline{A(E)} \equiv \frac{1}{R(E)}\int_0^{R(E)} A(x, E)\,dx \qquad (5.8)$$

stands for the phase space average of the quantity A along a constant E contour. The application of the formula

$$\overline{|v|^n\,x^m} = 2^{n/2-1}B\left(\frac{n+2}{2}, \frac{m+1}{2}\right)E^{m/2} \qquad (5.9)$$

(m even, B is the beta function) leads to the result

$$F_1(E) = -2Es + \frac{\gamma+3\delta}{2}E^2 \qquad (5.10)$$

Since $F_1(E)$ must be minimal on the attractor, one recovers for $\gamma+3\delta>0$, $s=1$ the result $E_c = 2/(\gamma+3\delta)$, which is the radius of the limit cycle after its appearance (in the rescaled variables).

It is worth noting that $\phi_1^{(0)\pm}$ is the potential belonging to the normal form of (3.1), expressed in the rescaled variables. Furthermore, $\phi_1^{(0)\pm}$ has a more general statistical meaning, too. Arguments along the line of Refs. 46 and 12 show that the stationary distribution of (4.2) in the limit $\beta\to 0$ is $\exp[-\phi_1^{(0)\pm}(E)/\eta]$ not only for $\eta\to 0$, but at *arbitrary* noise intensity. This, however, does not hold for finite damping coefficients.

The rotational invariance of the nonequilibrium potential is lost by $\phi_2^{(0)\pm}$ already. This is in accord with the fact that the shape of the limit cycle is destorted by increasing β. By integrating (5.6) from zero to an arbitrary $x<R(E)$ and observing that $\arcsin[x/R(E)]$ terms exactly cancel, one obtains

$$\phi_2^{(0)\pm} = \pm\frac{\gamma-\delta}{2}x\,|v|^3\left(s - \frac{\gamma+3\delta}{2}E\right) + F_2(E) \qquad (5.11)$$

The function $F_2(E)$ is fixed by the order β^3 contribution,

$$|v| \left(s - \gamma x^2 - \delta \, |v|^2 + \frac{\partial \phi_1^{(0)\pm}}{\partial E} \right) \frac{\partial \phi_2^{(0)\pm}}{\partial E} \pm \frac{\partial \phi_3^{(0)\pm}}{\partial x} = 0 \tag{5.12}$$

to (5.1). Applying the solvability condition (5.2) for $\phi_3^{(0)\pm}$, one finds

$$F_2'(E) = 0 \tag{5.13}$$

since the x-dependent part of $\phi_2^{(0)\pm}$ changes sign with v. An integration of (5.12) over x then yields

$$\begin{aligned}\phi_3^{(0)\pm} = \frac{\gamma - \delta}{4} \{ & x^2 E[6 - (11\gamma + 21\delta) \, Es + 5(\gamma + 3\delta)(\gamma + \delta) \, E^2] \\ & - x^4[6 - 5(5\gamma + 3\delta) \, Es + (\gamma + 3\delta)(17\gamma - 3\delta) \, E^2]/4 \\ & + x^6[-(7\gamma - 3\delta)s + 2(\gamma + 3\delta)(4\gamma - 3\delta)E]/6 \\ & - x^8(\gamma + 3\delta)(\gamma - \delta)/8 \} + F_3(E) \end{aligned} \tag{5.14}$$

The evaluation of F_3 is given in the Appendix and leads to a fourth-order polynomial in E.

6. THE PREFACTOR

The equation for the η-independent term $\phi^{(1)\pm}(x, E)$ of the negative logarithm of the stationary distribution is obtained from (4.2) and (2.5), after changing in the latter to the variables x, E, as

$$\begin{aligned} & |v| \left[\beta(s - \gamma x^2 - \delta \, |v|^2) + \frac{\partial \phi^{(0)\pm}}{\partial E} \right] \frac{\partial \phi^{(1)\pm}}{\partial E} - \beta \frac{s - \gamma x^2 - 3\delta \, |v|^2}{|v|} \\ & \quad - \frac{|v|}{2} \frac{\partial^2 \phi^{(0)\pm}}{\partial E^2} - \frac{1}{2 \, |v|} \frac{\partial \phi^{(0)\pm}}{\partial E} \pm \frac{\partial \phi^{(1)\pm}}{\partial x} = 0 \end{aligned} \tag{6.1}$$

The prefactor $z^{\pm}(x, E)$ is then given by $\exp(-\phi^{(1)\pm})$. Similarly to the case of the nonequilibrium potential, the single-valuedness of the prefactor provides a solvability condition for (6.1). Its form is that of (5.2) with $\phi^{(1)\pm}$ replacing $\phi^{(0)\pm}$.

Since the prefactor can be nonzero even in conservative systems,[13,32] we set

$$\phi^{(1)\pm}(x, E) = \phi_0^{(1)\pm}(x, E) + \beta \phi_1^{(1)\pm}(x, E) + \beta^2 \phi_2^{(1)\pm}(x, E) + \cdots \tag{6.2}$$

After substitution one finds that $\phi_0^{(1)\pm}$ does not depend on x; consequently,

$$\phi_0^{(1)\pm}(x, E) = G_0(E) \tag{6.3}$$

where G_0 is an arbitrary function.

The order β contribution to (6.1) can be written as

$$|v| \left(s - \gamma x^2 - \delta |v|^2 + \frac{\partial \phi_1^{(0)\pm}}{\partial E} \right) \frac{\partial \phi_0^{(1)\pm}}{\partial E} - \frac{s - \gamma x^2 - 3\delta |v|^2}{|v|}$$
$$- \frac{|v|}{2} \frac{\partial^2 \phi_1^{(0)\pm}}{\partial E^2} - \frac{1}{2|v|} \frac{\partial \phi_1^{(0)\pm}}{\partial E} \pm \frac{\partial \phi_1^{(1)\pm}}{\partial x} = 0 \tag{6.4}$$

Applying the solvability condition and using (5.5), (5.10), and (6.2) we obtain

$$\overline{|v| (s - \gamma x^2 - \delta |v|^2)}\, G_0' = -\frac{\gamma - 3\delta}{2} \overline{|v|} + \gamma \overline{\left(\frac{x^2}{|v|} \right)} - \frac{\gamma + 3\delta}{2} \overline{\left(\frac{1}{|v|} \right)} E \tag{6.5}$$

The right-hand side turns out, according to (5.9), to be zero. Thus,

$$G_0'(E) = 0 \tag{6.6}$$

i.e., there is no rotationally invariant contribution to the prefactor. The integration of (6.4) over x then yields

$$\phi_1^{(1)\pm} = \pm\tfrac{3}{4}(\gamma - \delta) x |v| + G_1(E) \tag{6.7}$$

after arcsin terms cancel again. As in the case of $\phi_2^{(0)\pm}$, the solvability condition applied in order β^2 leads to

$$G_1'(E) = 0 \tag{6.8}$$

and one obtains, after integration,

$$\phi_2^{(1)\pm} = \frac{\gamma - \delta}{32} [36x^2 s - 4(13\gamma + 33\delta) x^2 E + (13\gamma + 15\delta) x^4] + G_2(E) \tag{6.9}$$

From the next order equation in β we find $G_2(E)$ to be proportional to E^2, as shown in the Appendix.

7. THE NEXT CORRECTION IN THE NOISE INTENSITY

In the variables x, E the equation for $\phi^{(2)\pm}$ takes the form

$$|v| \left[\beta(s - \gamma x^2 - \delta |v|^2) + \frac{\partial \phi^{(0)\pm}}{\partial E} \right] \frac{\partial \phi^{(2)\pm}}{\partial E} - \frac{|v|}{2} \frac{\partial^2 \phi^{(1)\pm}}{\partial E^2}$$

$$- \frac{1}{2|v|} \frac{\partial \phi^{(1)\pm}}{\partial E} + \frac{|v|}{2} \left(\frac{\partial \phi^{(1)\pm}}{\partial E} \right)^2 \pm \frac{\partial \phi^{(2)\pm}}{\partial x} = 0 \tag{7.1}$$

with a similar solvability condition as for $\phi^{(0)\pm}$ and $\phi^{(1)\pm}$.

By setting

$$\phi^{(2)\pm}(x, E) = \phi_0^{(2)\pm}(x, E) + \beta \phi_1^{(2)\pm}(x, E) + \beta^2 \phi_2^{(2)\pm}(x, E) + \cdots \tag{7.2}$$

we find that $\phi_0^{(2)\pm}$ may depend on E only. The order β contribution to (7.1) then fixes this dependence to be a constant, and yields

$$\phi_1^{(2)\pm}(x, E) = I_1(E) \tag{7.3}$$

with an arbitrary I_1.

The use of (6.6)–(6.9), (A.6), and of the solvability condition applied on the order β^2 part of (7.1) leads to the result

$$I_1'(E) = 0 \tag{7.4}$$

according to which the first nontrivial term in $\phi^{(2)\pm}$ can be $\phi_2^{(2)\pm}$. By integration one finds then (arcsin terms cancel again)

$$\phi_2^{(2)\pm} = \pm \frac{\gamma - \delta}{128} 3(35\gamma + 57\delta) x |v| + I_2(E) \tag{7.5}$$

The function I_2 is to be determined by the order β^3 contribution to (7.1). In this equation $\phi_3^{(1)\pm}$ shows up, which we have not calculated. It is, however, sufficient to know that $\phi_3^{(1)\pm}$ and $\phi_4^{(0)\pm}$ are odd functions of v (as follows from the results of the previous sections and of the Appendix) to see that

$$I_2'(E) = 0 \tag{7.6}$$

is the restriction provided by the solvability condition.

8. THE STATIONARY DISTRIBUTION

The formulas of Sections 5–7 and of the Appendix together with Eq. (2.3) provide an expression for the stationary distribution of (4.2) valid up to third nontrivial orders in β and η. Since the degree of the polynomial contributions $\phi_i^{(j)\pm}(x, E)$ increases linearly with i, we obtain in the rescaled variables

$$E \ll \beta^{-1}, \qquad x \ll \beta^{-1/2} \tag{8.1}$$

as a condition for the validity of the β-expansion. This is, however, not a very strong restriction, since the dominating part of the distribution, that around the deterministic attractor characterized by an E of order unity, lies well inside this region. Therefore, in cases when the distribution is normalizable, the results obtained give also a global approximation for the stationary distribution.

The stationary distribution of the generalized van der Pol oscillator can be obtained by turning back to the original variables. By inverting the transformation (4.1), different β factors appear in different polynomials, leading to a rearragement of the series. The highest order polynomial of $\phi_i^{(j)\pm}$ always gives β-independent terms, i.e., terms that do not vanish at the bifurcation point. It is to be noted that no singular contributions (with a negative power of β) show up, due to the fact that $\phi_0^{(1)\pm}$, $\phi_0^{(2)\pm}$, $\phi_1^{(2)\pm}$ are irrelevant. The contributions to the distribution are thus obtained as

$$\begin{aligned}\phi^{(0)}(x, v) = {} & \frac{\gamma+3\delta}{2} E^2 - \frac{\gamma-\delta}{4}\left[(\gamma+3\delta)\left(xv^3E - 5(\gamma+\delta)\, x^2E^3\right.\right.\\ & \left. + \frac{17\gamma-3\delta}{4} x^4E^2 - \frac{4\gamma-3\delta}{3} x^6E + \frac{\gamma-\delta}{8} x^8\right)\\ & \left. + \frac{1}{128}\left(\frac{359}{3}\gamma^2 + 626\gamma\delta + 753\delta^2\right) E^4\right]\\ & - \alpha 2E + \alpha\frac{\gamma-\delta}{4}\left[2xv^3 + \frac{23\gamma+49\delta}{8} E^3 - (11\gamma+21\delta)\, x^2E^2\right.\\ & \left. + \frac{5(5\gamma+3\delta)}{4} x^4E - \frac{7\gamma-3\delta}{6} x^6\right] - \alpha^2\frac{\gamma-\delta}{8} 3(E^2 - 4x^2E + x^4)\end{aligned} \tag{8.2}$$

$$\begin{aligned}\phi^{(1)}(x, v) = {} & \frac{\gamma-\delta}{8}\left[6xv + \frac{35\gamma+57\delta}{8} E^2 - (13\gamma+33\delta)\, x^2E + \frac{13\gamma+15\delta}{4} x^4\right]\\ & + \alpha\frac{\gamma-\delta}{8} 9x^2\end{aligned} \tag{8.3}$$

$$\phi^{(2)}(x, v) = \frac{\gamma-\delta}{128} 3(35\gamma+57\delta)\, xv \tag{8.4}$$

The energy $E \equiv (x^2 + v^2)/2$ has een used here as a short-hand notation. The restriction (8.1) is now replaced by

$$x, v \ll 1 \tag{8.5}$$

The dominating part of the distribution is situated in a region where x, v are of order $\alpha^{1/2}$.

An essential feature of this solution is to be emphasized. Although the weak noise expansion applied in the rescaled variables would imply [see (4.1)] $\bar{\eta} \equiv \eta/|\alpha| \ll 1$, the applicability of expressions (8.2)–(8.4) with Eq. (2.4) is much broader; they are valid for *arbitrary* small values of α and η. This property has been suggested by the fact that $\phi^{(j)}(x, v)$ is found to be analytic in α, but it can be shown by more rigorous arguments, too. The Hamilton–Jacobi equation (2.4) of the original problem formulated in Section 3 cannot be solved by an α-expansion, since no analytic results are known for the nonequilibrium potential at $\alpha = 0$. It is possible, however, to make an expansion in powers of $\varepsilon \equiv (\gamma - \delta)/(\gamma + \delta)$, since for $\gamma = \delta$ an exact expression holds. It has been shown[36] that this ε-expansion is valid for arbitrary α, and for $\alpha \ll 1$ our results are recovered. On the other hand, a direct substitution of (8.2)–(8.4) into the weak noise expansion equations (2.4)–(2.6) of the original problem proves that (8.2)–(8.4) are indeed polynomial approximations for $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$ of the generalized van der Pol oscillator and are valid also at $\alpha = 0$.

It is worth comparing the results with those obtained by other methods. A direct polynomial approximation for the stationary distribution has been applied in Refs. 39 and 40. In the general case γ, δ arbitrary, only the $(\gamma - \delta)$-independent part of $\phi^{(0)}$ and the first term of $\phi^{(1)}$ have been given.[40] For the special case $\gamma = 0$, $\delta = 1$, $\phi^{(0)}$ and $\phi^{(1)}$ have been calculated[39] up to fourth- and second-degree polynomials in x, v, respectively. These results are in agreement with ours. But, as mentioned above, we do not see any reason to restrict the expressions to certain regions of the parameter plane α, η, in contrast to Refs. 39 and 40. The comparison also suggests that the method applied here is more powerful than a direct polynomial approximation, especially close to the bifurcation point, where high-degree polynomials give the main contribution.

Finally, we note that the results also illustrate how different the stationary distribution of a certain problem and of its rescaled version can be due to the fact not only the variables, but the noise intensity should be rescaled as well.

9. CLOSING REMARKS

It is to be mentioned that *no* polynomial ansatz has been incorporated in our method. We have thus proved that $\phi^{(j)}(x, v)$ with $j \leqslant 2$ is a polynomial. A similar rule is conjectured for j arbitrary. This simple structure seems to be a consequence of the harmonic oscillator dynamics obtained for $\beta = 0$. In more complex cases, e.g., for a nonlinear oscillator governing the dynamics at a codimension-two bifurcation, the logarithm of the stationary density is not a polynomial.[13] In general, a polynomial form that can be an appropriate local approximation is not expected to be a correct global form, especially if coexisting attractors characterize the deterministic system.

A crucial feature of the method we applied is the property that the deterministic system of interest, after rescaling, becomes at the bifurcation point a conservative one. Close to this point, i.e., at weak but nonvanishing dissipation, the equipotential lines of the nonequilibrium potential turn out to be approximated by the trajectories of the conservative system. A necessary condition for the applicability of the method is the existence of *closed* trajectories; otherwise the density associated with the nonequilibrium potential is not normalizable. If this condition holds, the double expansion in powers of the noise intensity and the bifurcation parameter can be a powerful method for determining stationary distributions.

APPENDIX

A1. Calculation of F_3

The solvability condition (5.2) applied for the β^4 contribution to Eq. (5.1) yields

$$\overline{\tfrac{1}{2}|v|\,(\partial\phi_2^{(0)\pm}/\partial E)^2} + \overline{|v|\,(s - \gamma x^2 - \delta\,|v|^2 + \partial\phi_1^{(0)\pm}/\partial E)\,\partial\phi_3^{(0)\pm}/\partial E} = 0 \qquad \text{(A.1)}$$

Substituting (5.10), (5.11), and (5.14) into (A.1), one finds an equation for F_3'. A straightforward but tedious calculation with subsequent applications of (5.9) leads to the equation

$$[-2s + (\gamma + 3\delta)E]\, F_3'(E) = \mathscr{P}(E) \qquad \text{(A.2)}$$

where $\mathscr{P}$ is a fourth-order polynomial. Since $\mathscr{P}(E)$ turns out to be divisible by $[2s - (\gamma + 3\delta)E]$, one obtains a polynomial expression for F_3' in the form

$$F_3'(E) = (\gamma - \delta)\left[-\frac{3}{8}E^2 + \frac{23\gamma + 49\delta}{32}E^3 s - \frac{1}{512}\left(\frac{359}{3}\gamma^2 + 626\gamma\delta + 753\delta^2\right)E^4 \right] \qquad \text{(A.3)}$$

A2. Calculation of G_2

The order β^3 contribution to Eq. (6.1) reads

$$|v| \left(\frac{\partial \phi_1^{(0)\pm}}{\partial E} \frac{\partial \phi_2^{(1)\pm}}{\partial E} + \frac{\partial \phi_2^{(0)\pm}}{\partial E} \frac{\partial \phi_1^{(1)\pm}}{\partial E} \right)$$
$$+ |v| \, (s - \gamma x^2 - \delta \, |v|^2) \frac{\partial \phi_2^{(1)\pm}}{\partial E}$$
$$- \frac{|v|}{2} \frac{\partial^2 \phi_3^{(0)\pm}}{\partial E^2} - \frac{1}{2\,|v|} \frac{\partial \phi_3^{(0)\pm}}{\partial E} \pm \frac{\partial \phi_3^{(1)\pm}}{\partial x} = 0 \qquad \text{(A.4)}$$

The application of the solvability condition and Eq. (5.9), after some algebra, leads to

$$[-2s + (\gamma + 3\delta)E] \, G_2'(E) = \mathscr{R}(E) \qquad \text{(A.5)}$$

Since $\mathscr{R}(E)$ happens to be proportional to $E[2s - (\gamma + 3\delta)E]$ we obtain

$$G_2(E) = (\gamma - \delta) \frac{35\gamma + 57\delta}{64} E^2 \qquad \text{(A.6)}$$

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REFERENCES

1. A. A. Andronow and C. E. Chaikin, *Theory of Oscillations* (Princeton University Press, Princeton, 1949).
2. N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, New York, 1962).
3. W. Ebeling, *Strukturbildung bei irreversiblen Prozessen* (Teubner Verlag, Leipzig, 1976).
4. G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).
5. H. Haken, *Synergetics, An Introduction* (Springer-Verlag, Berlin, 1977); *Advanced Synergetics* (Springer-Verlag, Berlin, 1983).
6. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1983).
7. R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
8. Z. Schuss, *Theory and Applications of Stochastic Differential Equation* (Wiley, New York, 1980).

9. N. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
10. P. Hänggi and H. Thomas, *Phys. Rep.* **88**:207 (1982).
11. C. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1983).
12. H. Risken, *The Fokker–Planck Equation* (Springer-Verlag, Berlin, 1984).
13. R. Graham and T. Tél, *Phys. Rev. A* **35**:1328 (1987).
14. J. K. Cohen and R. M. Lewis, *J. Inst. Math. Appl.* **3**:266 (1967).
15. A. D. Ventzel and M. I. Freidlin, *Russ. Math. Surveys* **25**:1 (1970); M. I. Freidlin and A. D. Ventzel, *Random Perturbations of Dynamical Systems* (Springer-Verlag, Berlin, 1984).
16. (a) R. Graham, in *Coherence and Quantum Optics*, L. Mandel and E. Wolf, eds. (Plenum Press, New York, 1973); (b) in *Fluctuations, Instabilities and Phase Transitions*, T. Riste, ed. (Plenum Press, New York, 1975); (c) in *Stochastic Processes in Nonequilibrium Systems*, L. Garrido, P. Seglar, and P. J. Shephard, eds. (Springer-Verlag, Berlin, 1978); (d) in *Stochastic Nonlinear Systems*, L. Arnold and R. Lefever, eds. (Springer-Verlag, Berlin, 1981).
17. R. Kubo, K. Matsuo, and K. Kitahara, *J. Stat. Phys.* **9**:51 (1973).
18. Yu. Kifer, *Math. SSSR Izv.* **8**:1083 (1974).
19. D. Ludwig, *SIAM Rev.* **17**:605 (1975).
20. K. Kitahara, *Adv. Chem. Phys.* **29**:85 (1975).
21. R. Graham and A. Schenzle, *Phys. Rev. A* **23**:1302 (1981); H. Schmidt, S. W. Koch, and H. Haug, *Z. Phys. B* **51**:85 (1983); P. Talkner and P. Hänggi, *Phys. Rev. A* **29**:768 (1984).
22. B. J. Matkowsky and Z. Schuss, *SIAM J. Appl. Math.* **42**:822 (1982); *Phys. Lett.* **95A**:213 (1983); E. Ben-Jacob, D. J. Bergmann, B. J. Matkowsky, and Z. Schuss, *Phys. Rev. A* **26**:2805 (1982).
23. P. Talkner and D. Ryter, *Phys. Lett.* **88A**:163 (1982); in *Noise in Physical Systems and 1/f Noise* (North-Holland, Amsterdam, 1983); D. Ryter, *Physica* **130A**:205 (1985); **142A**:103 (1987); P. Talkner, *Z. Phys. B* **68**:201 (1987).
24. W. G. Faris and G. Jona-Lasinio, *J. Phys. A* **15**:3025 (1982); G. Jona-Lasinio, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, N. Ghil, R. Benzi, and G. Parisi, eds. (North-Holland, Amsterdam, 1985).
25. M. Dörfle and R. Graham, *Phys. Rev. A* **27**:1096 (1983).
26. R. Graham and A. Schenzle, *Z. Phys.* **52**:61 (1983).
27. (a) R. Graham and T. Tél, *Phys. Rev. Lett.* **52**:9 (1984); (b) *J. Stat. Phys.* **35**:729 (1984); (c) *Phys. Rev. A* **31**:1109 (1985); (d) *Phys. Rev. A* **33**:1322 (1986).
28. H. Lemarchand and G. Nicolis, *J. Stat. Phys.* **37**:609 (1984).
29. R. Graham, D. Roekaerts, and T. Tél, *Phys. Rev. A* **31**:3364 (1985); D. Roekaerts and F. Schwarz, *J. Phys. A* **20**:L127 (1987).
30. A. Schenzle and T. Tél, *Phys. Rev. A* **32**:596 (1986).
31. H. R. Jauslin, *J. Stat. Phys.* **42**:573 (1986); *Physica* **144A**:179 (1987).
32. R. Reibold, *Z. Phys. B* **62**:397 (1986).
33. R. Graham, *Europhys. Lett.* **2**:901 (1986).
34. P. Hänggi, *J. Stat. Phys.* **42**:105 (1986).
35. V. Altares and G. Nicolis, *J. Stat. Phys.* **46**:191 (1987); E. Sulpice, A. Lemarchand, and H. Lemarchand, *Phys. Lett.* **121A**:67 (1987).
36. R. Graham, Macroscopic potentials, bifurcations and noise in dissipative system, preprint (1987).
37. B. van der Pol, *Phil. Mag.* **3**:65 (1927); M. L. Cartwright and J. E. Littlewood, *J. Lond. Math. Soc.* **20**:180 (1945); P. Holmes and D. Rand, *Q. Appl. Math.* **35**:495 (1978); U. Parlitz and W. Lauterborn, *Phys. Rev. A* **36**:1428 (1987).

38. F. Baras, M. Malek Mansour and C. Van den Broeck, *J. Stat. Phys.* **28**:577 (1982); D. Ryter, P. Talkner, and P. Hänggi, *Phys. Lett.* **93A**:447 (1983); P. Hänggi and P. Riseborough, *Am. J. Phys.* **51**:347 (1983).
39. L. Schimansky-Geier, A. V. Tolstopyatenko, and W. Ebeling, *Phys. Lett.* **108A**:329 (1985); W. Ebeling and L. Simansky-Geier, *Fluid Dyn. Trans.* **12**:7 (1985).
40. W. Ebeling, H. Herzel, W. Richert, and L. Schimansky-Geier, *Z. Angew. Math. Mech.* **66**:141 (1986).
41. H. Haken, *Phys. Rev. Lett.* **13**:329 (1964); H. Risken, *Z. Phys.* **186**:85 (1965); R. Graham, *Quantum Statistics in Optics and Solid-State Physics* (Springer-Verlag, 1973).
42. K. H. Hoffman, *Z. Phys. B* **49**:245 (1982).
43. Lord Rayleigh, *Theory of Sound*, Vol. I (Dover, New York, 1945).
44. M. O. Hongler and D. Ryter, *Z. Phys. B* **31**:333 (1978); W. Ebeling and H. Engel-Herbert, *Physica* **104A**:378 (1981).
45. R. Graham and H. Haken, *Z. Phys.* **243**:289 (1971); **245**:141 (1971).
46. H. Risken and K. Voigtlander, *J. Stat. Phys.* **41**:825 (1985).