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Modeling Two-Dimensional Fluid Flows with Chaos Theory

John C. Sommerer, Edward Ott, and Tamás Tél

Experiments at the Applied Physics Laboratory have shown that chaos theory can be used to quantitatively model two-dimensional fluid flows. These results support the utility, under some circumstances, of low-degree-of-freedom approaches to the theoretically infinite-degree-of-freedom phenomena described by the Navier–Stokes equations. However, the types of predictions made by chaos theory are fundamentally different from the predictions that result from usual approaches such as computational fluid dynamics. Chaotic models of fluids thus require further investigation before their overall utility can be evaluated.
(Keywords: Chaos, Fluid dynamics, Fractals.)

INTRODUCTION: FLUID DYNAMICS AND CHAOS THEORY

At a meeting of the British Association for the Advancement of Science in 1932, shortly before his death, Sir Horace Lamb[1] declared that he hoped God would provide enlightenment on two scientific issues: quantum electrodynamics and turbulence. Lamb reportedly commented that he was optimistic about getting a satisfactory explanation of the first mystery. Less whimsically, Steven Strogatz[2] of Cornell has attempted to provide a two-parameter classification for all dynamical systems. The two parameters are nonlinearity and the number of degrees of freedom. At the extreme end of both scales is a cluster of seriously intractable problems: ecosystems, turbulent fluids, and finally, life itself.

These anecdotes reflect the ability of fluid systems to produce extremely complicated behavior, the treatment of which has largely resisted general analytical approaches. Consequently, fluid mechanics is a field with a bewildering number of specialty areas; progress in many special cases and niche problems, using many different methods, has been necessary to address the field as a whole.

When any new approach to such a difficult field arises, there is a temptation to hope for a breakthrough, followed by disappointment when none is forthcoming, followed in turn by a more careful examination of the methods, including an assessment of their strengths and weaknesses as they are added to the overall tool kit. Chaos theory is in the process of moving from the second to the third stage of this process.

Chaos may be incompletely but accurately described as the ability of simple nonlinear systems to produce complicated behavior. The temporal behavior of chaotic systems is, in fact, so suggestive of turbulent fluid behavior that words like "turbulent" and "laminar" were quickly misappropriated by dynamicists. However, although fluid systems are nonlinear, they are also spatially distributed, and the underlying dynamical equations are infinite-dimensional. Chaos theory, on the other hand, concerns finite- (especially low-) dimensional nonlinear systems. This explains the temptation stage: if the behavior of solutions to the Navier–Stokes equations could be adequately captured in a low-dimensional system, the practical implications would be tremendous. In fact, much of the modern interest in chaos theory was precipitated by Edward Lorenz's modeling of Rayleigh–Bénard convection,[3] and several early experiments testing the applicability of chaos theory involved fluid systems.[4,5]

Unfortunately, chaos theory is no panacea for resolving the many problems of fluid dynamics. Its applicability is limited primarily to relatively low excitations, and its predictive capability is also limited. As a result, some veteran fluid dynamicists have indulged in a "we told you so" attitude toward perhaps too-ingenuous nonlinear dynamicists. However, chaos theory has made some important conceptual contributions to fluid dynamics. For example, the incorrect Landau picture of the onset of turbulence[6] was displaced by the (chaotic) Ruelle–Takens–Newhouse route[7,8] (although the latter route also is now recognized as an incomplete description[9,10]). Further, a nonlinear dynamics approach has proven fruitful in elucidating the nature of intermittency in the turbulent boundary layer.[11]

We have our own particular niche in the application of chaos theory to fluid dynamics. For several years we have been investigating, with a number of different collaborators, the applicability of chaos theory to relatively low-excitation flows in two dimensions. Such flows have long been studied analytically, because of various mathematical simplifications that are possible and because of greater tractability in computational fluid dynamics. Further, two-dimensional flows are relevant to a number of practical problem areas, including geostrophic flows in the atmosphere and oceans (Fig. 1).

Our work has shown that low-dimensional chaotic models can provide excellent quantitative descriptions of some of the complicated phenomena observed in such flows. However, the nature of the descriptions is generally very different from those provided by more conventional approaches. Although the utility of some of these descriptions is still not settled, chaos theory clearly provides additional insight into the fluid systems.

(a)

(b)

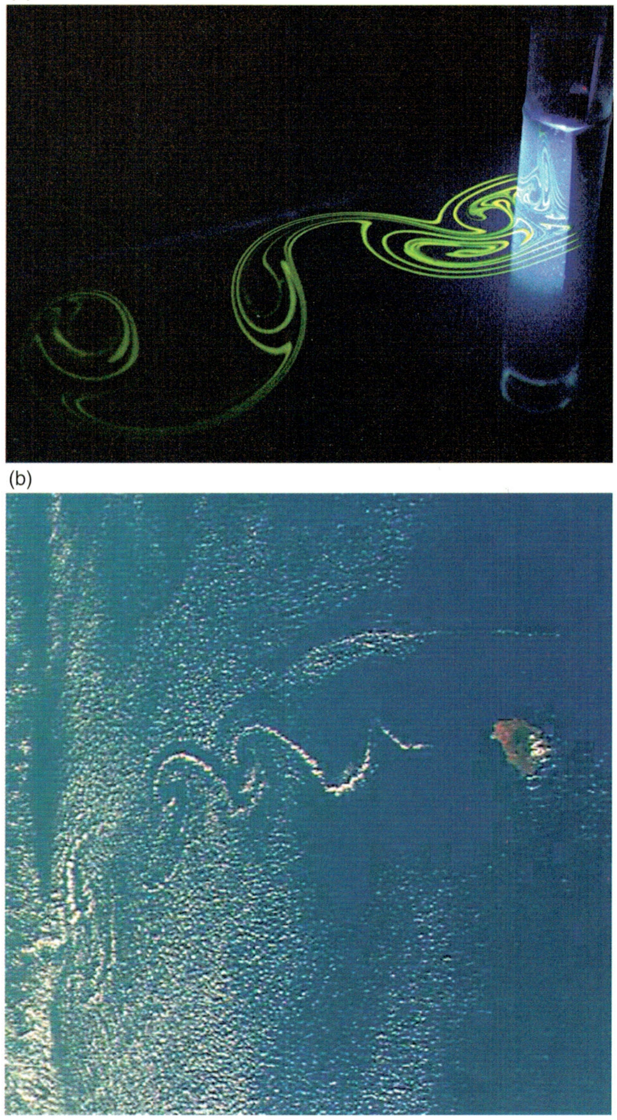

Figure 1. Examples of the von Karman vortex street in two-dimensional fluid flows. (a) The wake behind a cylinder moving through the APL stratified towing tank. (b) Cloud patterns downwind of Sorocco Island, Mexico, imaged from the space shuttle (mission STS044). The problem scales, working fluids, reasons for two-dimensionality of the flows, and means of visualizing the wakes are completely different in the two cases, yet the phenomena are the same.

This article summarizes two very different types of two-dimensional fluid flows. For these two types of flows, experiments performed at APL have confirmed the validity of a common, chaos-based approach that connects fundamental dynamical quantities to the static geometry of a flow visualization at an instant in time. We first discuss closed surface flows. These can exhibit a type of time-varying strange attractor, whose geometric properties can be predicted in terms of dynamical quantities. We then discuss an open flow, where the chaotic behavior is unstable and transient, but which nevertheless leaves characteristic traces that can be quantitatively assessed. Finally, we conclude by mentioning several fluid models that appear to be amenable to chaotic techniques but for which experiments have not yet been done.

SURFACE FLOWS

Background

The strange attractor of a dissipative dynamical system (one in which phase-space volumes shrink with time) is a fractal set approached asymptotically at large times by typical system trajectories. Being fractal, the set has zero volume in the embedding space. (See the earlier *Technical Digest* article by Sommerer[12] for a more complete discussion of fractals.) This behavior seems very unlike the behavior of fluid elements in an incompressible fluid; the very word incompressible belies the possibility of fluid elements collapsing to a set of zero volume. However, this counterintuitive behavior is possible if we consider a set of passive tracers *confined to the surface of the fluid.* The incompressibility condition $\nabla \cdot \mathbf{v} = 0$ does indeed prevent an arbitrary initial volume of tracers, advected with the fluid velocity $\mathbf{v}$, from concentrating on a zero-volume set. However, if the tracers are constrained to the fluid surface (for which $z = z_0$, $v_z = 0$), then their advection can be compressible, even though $\nabla \cdot \mathbf{v} = 0$. In particular, $(\partial v_x/\partial x + \partial v_y/\partial y)|_{z=z_0} = -\partial v_z/\partial z|_{z=z_0} \neq 0$. This situation, which Ott observed in a pedagogical demonstration where patterns of foam formed on the surface of a moving fluid, led him to consider a simple dynamical model that caricatured the surface flow. Several theoretical results concerning the onset of chaos in such models were forthcoming.[13,14]

The dynamical model was a random mapping of the form

$$\mathbf{r}_{n+1} = \mathbf{F}_n(\mathbf{r}_n), \tag{1}$$

where at each time step n, the mapping $\mathbf{F}_n$ is chosen at random from a prescribed family of mappings, thus transforming the system state $\mathbf{r}_n$ into the new system state $\mathbf{r}_{n+1}$. Here, the system state corresponds to a position on the surface of the fluid. Such systems belong to a class included in a powerful theorem of Ledrappier and Young.[15] This theorem connects the information dimension (see the boxed insert) of the mapping's strange attractor to the Lyapunov exponents of the system. Lyapunov exponents are fundamental quantities of a dynamical system that describe how nearby system trajectories track one another. Geometrically, the Lyapunov exponents λ_1 and λ_2 can be interpreted as follows. Given an initial infinitesimal circle of radius dr, for very large n the image of the circle after n applications of the mapping in Eq. (1) will be an ellipse of semimajor and semiminor axes on the order of $dr\ \exp(n\lambda_1)$ and $dr\ \exp(n\lambda_2)$, respectively. (Note that for a dissipative situation, the area of the initially circular region shrinks, implying $\lambda_1 + \lambda_2 < 0$, while for chaos one of the λ values, say λ_1, is positive.) Having at least one positive Lyapunov exponent is associated

GENERALIZED DIMENSIONS

Most readers are familiar with the notion of Euclidean dimensions: a point is zero-dimensional, a line is one-dimensional, etc. The usual description is that the dimensionality of a geometric set is associated with the minimum number of coordinates needed to specify a point in the set. In fact, the notion of dimension can be generalized to accommodate geometrically complex objects that are difficult to categorize with simple Euclidean geometry. Such objects, known as fractals, can have dimensions that are not integers. This statement demands a rather precise definition in terms not depending on coordinates, since one clearly can have only an integer number of coordinates.

We will provide the definition in terms of a *probability* distribution living within some Euclidean space. The usual notion of a geometric object is less general than a distribution, but dovetails as follows: the geometric object is the set having nonzero probability. One can calculate an entire spectrum of generalized dimensions for the probability distribution, using the conceptual aid of a grid of boxes of side length ϵ overlaid on the distribution. The total probability within the ith box is denoted p_i. Then, the generalized dimension of the distribution is defined to be

$$d_q = \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\log \sum_{i=1}^{N(\epsilon)} p_i^q}{\log 1/\epsilon},$$

where $N(\epsilon)$ is the number of boxes of side length ϵ having nonzero probability.

The definition of the generalized dimension is valid for all q (including $q = 1$, where L'Hôpital's rule must be used to make sense of the definition), but a few particular values of q are most widely used. For $q = 0$ all parts of the geometric set are treated on equal footing; d_0 corresponds to our usual ideas of geometric dimension for "ordinary" objects, though it, too, can attain fractional values for geometrically complicated objects.[12] For $q = 1$, the definition produces the *information* dimension, which emphasizes the contributions of the most probable parts of the distribution. This is a valuable distinction in an experimental context, because finite data on the distribution make one liable to undercount low-probability boxes. Thus, the information dimension d_1 is less sensitive than d_0 to whether all of the low-probability boxes have been found. For $q = 2$, even more emphasis is put on high-probability boxes; this special case is termed the *correlation* dimension.

The experimental estimation of generalized dimensions is easy for images, where image intensity can be associated with probability values, and pixels (or groups of pixels) can be associated with the ϵ grid of boxes. Of course, the image pixel represents the smallest experimentally attainable value of ϵ, so in practice the limit in the definition is not taken. Instead, the generalized dimension is estimated as the slope of a plot of the numerator versus the denominator in the formula defining d_q.

with sensitive dependence on initial conditions, which is synonymous with chaos. The connection of fundamental dynamical quantities (the Lyapunov exponents) with a flow visualization result (the strange attractor) would allow a stringent test of whether the dynamical model was more than a caricature of the fluid flow, i.e., whether chaos theory could actually describe the flow; also, not incidentally, it would suggest one possible reason why fractals are ubiquitous in nature.

Experiments

Our apparatus for investigating this possibility experimentally is shown schematically in Fig. 2. A bulk flow (if one can apply that term to an appartus with a flow basin only 10 cm in diameter) was produced by pumping the working fluid (viscous sucrose solution) over an annular sill and recovering it from the center of the enclosed basin. With perfect symmetry, steady pumping producing a steady flow would lead to convergence of any passive tracer confined to the surface at a point above the fluid recovery port. However, fluid instabilities produce recirculation cells on the surface of the fluid. Steady pumping producing a steady flow in reality would drive passive floaters onto one or more closed curves or fixed points. (The Poincaré–Bendixon theorem precludes a steady two-dimensional vector field from producing a strange attractor.[16]) Instead, we pumped the fluid in a sequence of equally energetic pulses (that is, for the same interval and at the same speed), allowing the fluid to come to rest between pulses.

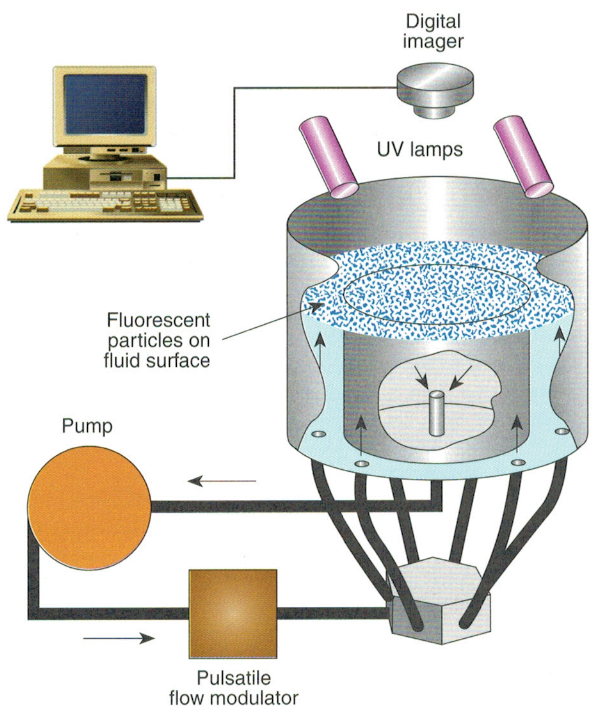

Figure 2. Schematic of the apparatus for surface flow experiments.

This iterative process takes each point in a two-dimensional region (the stationary fluid surface) and maps it onto another point on the same surface. Thus, the pumping action produces a physical analog to the mathematical system of Eq. 1, where $\mathbf{r} = (x,y)$. The fluid instabilities and other perturbing factors make each pulse different in detail from all of the others, although they are statistically identical. In particular, the axis of the nascent recirculation structure changes randomly from iteration to iteration. Thus, we considered these random effects to be the random choice from a family of mappings. Parametrically, we could control the pumping rate and interval, the viscosity of the working fluid (by controlling its temperature), and the height of the fluid surface above the sill. As tracers, we used small fluorescent particles that were constrained to the fluid surface by a combination of buoyancy and surface tension. These tracers were initially distributed approximately uniformly over the fluid surface. A sequence of pumping pulses produced a complicated aggregate pattern of particles, visualized by exciting the fluorescent dye in the particles with ultraviolet light and imaging the distribution of particles with a charge-coupled device (CCD) camera. The aggregate of particles was a candidate strange attractor of the system in Eq. 1. Note that no explicit identification of the mapping is necessary to test this hypothesis, because the Ledrappier–Young theorem relates the *consequences* of the map $\mathbf{F}$, whatever its explicit form.

Results

For a mapping such as Eq. 1, with Lyapunov exponents $\lambda_1 > 0 > \lambda_2$, $\lambda_1 + \lambda_2 < 0$, the Ledrappier–Young theorem predicts that the information dimension of the strange attractor is given by

$$d_1 = 1 + \frac{\lambda_1}{|\lambda_2|}, \qquad (2)$$

implying that the experimental tracer pattern on the surface of the fluid should be fractal for large values of the time step n. A typical large-time particle distribution for a particular set of flow parameters is shown in Fig. 3. Measured values of the information dimension of such fractal patterns agreed well with predictions made on the basis of the Lyapunov exponents for the surface flow (Fig. 4); the Lyapunov exponents were estimated by noting the deformation of small blobs of tracers under the influence of the flow.[17,18] This agreement is strong evidence that the simple dynamical model captures essential features of the fluid behavior, in spite of the model's much lower dimensionality.

More recently, new theoretical[19,20] and experimental[21] results connecting the *correlation* dimensions of the

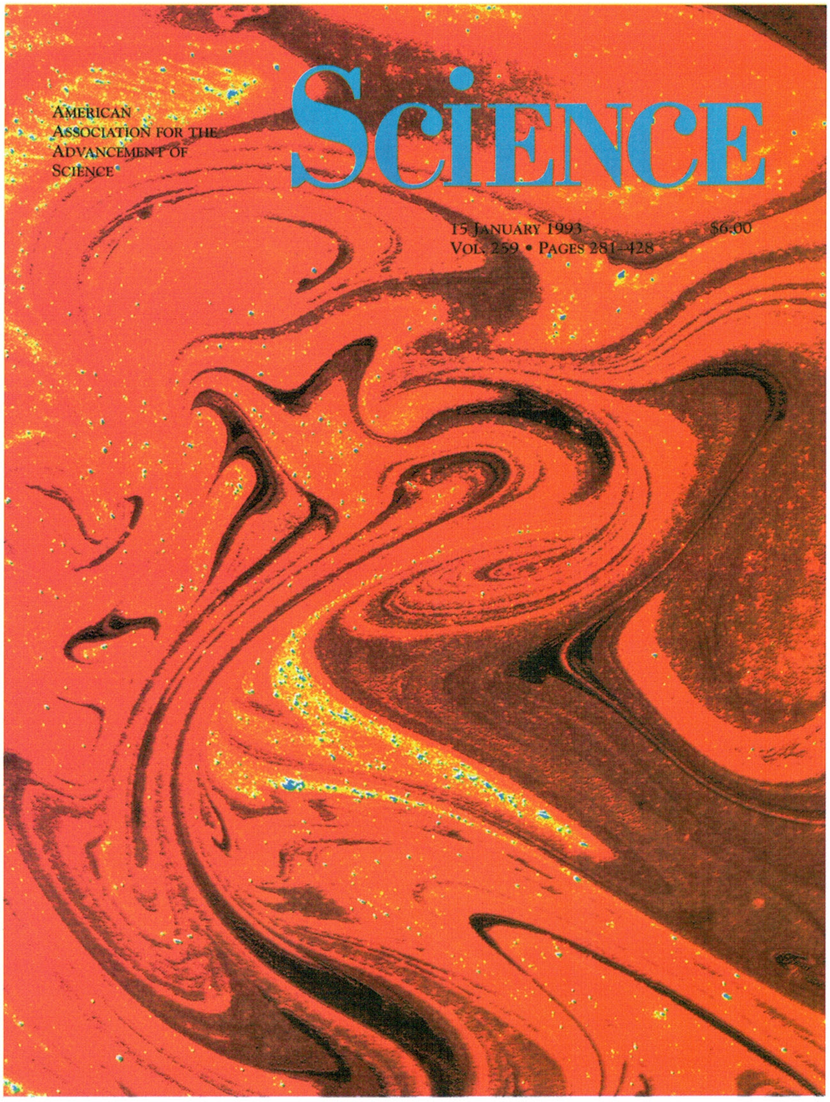

Figure 3. Fractal pattern formed on the surface of a moving fluid. This false-color image shows differences in tracer concentration. High particle density (yellow-blue) indicates regions of past compressive surface flow or downwelling. Low tracer density (black-red) indicates past upwelling. (Reprinted with permission from Ref. 17. Copyright 1993 American Association for the Advancement of Science.)

tracer patterns with power spectra of the patterns provide additional evidence that chaotic models adequately describe the dynamics of these surface flows. (Readers of David Mermin's *Boojums All the Way Through*[22] will appreciate that a measure of the significance of this work is the willingness of the journal *Physical Review Letters* to use the word "scum" in an article title.)

CHAOTIC SCATTERING MODEL OF A FLUID WAKE

Background

At a 1993 NATO Advanced Studies Workshop, where Sommerer presented the initial experimental results on surface flows, Tél presented theoretical and numerical work (done with Jung and Ziemniak) on a very different two-dimensional fluid problem[23–26] (see also Ref. 27). This work considered the modeling of a fluid wake via *chaotic scattering*. Chaotic scattering[28] refers to the possibility that a system can manifest symptoms of chaos (such as sensitive dependence on initial conditions) for a finite time, even though the system's phase-space trajectory is asymptotically free during earlier and later epochs (i.e., the system is *open*). This can occur when the system's phase-space trajectory shadows a genuinely chaotic but unstable set, which, as such, is not directly observable in a typical experiment. Chaotic scattering has been identified as a possible phenomenon in a wide variety of contexts, including celestial mechanics, microwave scattering, solar physics, geophysics, optics, and atomic and nuclear physics, as well as fluid dynamics. However, there have been almost no laboratory demonstrations of chaotic scattering in any kind of system. Tél and Sommerer discussed the possibility of experimentally testing the chaotic wake model with some trepidation.

First, the experiment would have to be much larger physically than the surface flow experiment to explore the relevant fluid parameters.

Second, the flow would be open, leading to time constraints on experimental runs. In a two-dimensional incompressible flow, there exists a time-dependent stream function $\psi(x,y,t)$ such that the fluid velocity field $\mathbf{v}(x,y,t)$ is given by

$$v_x(x,y,t) = \frac{\partial}{\partial y}\psi(x,y,t), \qquad (3)$$

$$v_y(x,y,t) = -\frac{\partial}{\partial x}\psi(x,y,t). \qquad (4)$$

A fluid element (and an advected passive tracer) will have as its equations of motion simply

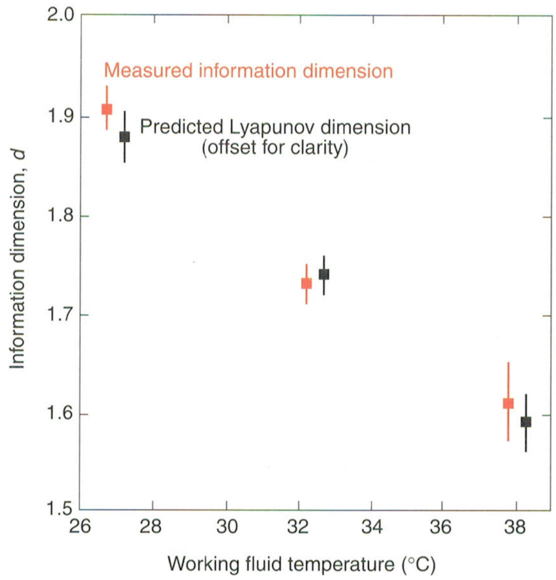

Figure 4. Summary of experimental results for surface flows. Measurements of fractal dimension (red symbols) agree well with dynamical predictions (black symbols) for all flow conditions and trials. Shift in absolute values with different flow conditions indicates that measured quantity is diagnostic for flow condition.

$$\frac{d}{dt}x(t) = v_x(x(t), y(t), t), \quad (5)$$

$$\frac{d}{dt}y(t) = v_y(x(t), y(t), t), \quad (6)$$

so a passive tracer's equations of motion have the exact form of Hamilton's equations, with $\psi(x,y,t)$ playing the role of a time-dependent Hamiltonian.[29] Note that in this case the phase space of the Hamiltonian system is just the physical space of the two-dimensional fluid flow. Thus, phase-space behavior can be observed directly. In a Hamiltonian system, phase-space volumes are conserved. Thus, there are no attractors (strange or otherwise) in a Hamiltonian system; consequently, diagnosis of the system properties in an experiment would have to occur "on the fly" rather than after the experiment settled down to long-term behavior. The chaotic behavior would be transient and unstable and, therefore, not so easily amenable to diagnosis using typical techniques of nonlinear dynamics.

Finally, our quantitative demonstration of chaotic scattering had to be indirect. The most spectacular manifestation of chaotic scattering is that a generic scattering function exhibits an uncountable number of singularities, located on a fractal support in the space of impact parameters. In the case of a fluid wake, we take the *time delay* of fluid elements passing the perturbing body to be the scattering variable; almost all fluid elements eventually leave the wake and join free-streaming fluid elements, but some can be delayed for a long time. Unfortunately, exhibiting an actual scattering function is impractical for a real fluid wake, where accurate determination of the impact parameter of a fluid element exiting the wake region is currently impossible. The test of the theory in this case would depend on applying a result of Kantz and Grassberger[30] to connect *three* aspects of the experiment: the geometry of the wake (fractal dimensions again), the local dynamics of the wake (Lyapunov exponents again), and the characteristic time delay of fluid elements interacting with the wake.

This connection can be outlined qualitatively as follows. The unstable, fractal chaotic set in the wake is a *saddle*, in that it has both attracting and repelling directions (just like a saddle point on a potential surface). Fluid elements having impact parameters corresponding to the attracting directions get very close to the chaotic set, and they spend a long time in the disturbed part of the wake. They also tend to leave the wake region along the unstable directions. Having been close to the unstable set, the fluid elements leaving the neighborhood of the unstable chaotic set along the unstable directions retain some of the geometric properties of the set. On the other hand, fluid elements having impact parameters far from the stable directions won't get very close to the unstable chaotic set, won't spend a long time in the wake region, and won't have geometry closely related to the chaotic set.

Experiments

Our experiments were conducted in APL's $1 \times 2 \times 8$ m stratified flow facility, which allows experiments with arbitrary, stable-density profiles (Fig. 5). The facility was designed by Harold Gilreath and Joseph Hopkins of APL's Milton S. Eisenhower Research and Technology Development Center. We produced a thin (1–5 cm), strong density gradient at mid-depth, between layers of concentrated brine on the bottom and freshwater on the top. We made all measurements within the thin mixing layer, where the Brunt–Väisälä frequency $N = [g\rho^{-1}|d\rho/dz|]^{1/2}$ was in the range $6\ s^{-1} \leq N \leq 14\ s^{-1}$. This very strong stratification effectively suppressed motion in the vertical direction, especially on timescales characteristic of fluid motion in the horizontal directions.

The fluid wake was created by a cylinder (with vertical symmetry axis, radius R_{cyl} = 5 cm) moving horizontally along the center of a channel (width 20 cm) between false walls. We towed the cylinder from an overhead track at low velocities U_{cyl} of a few millimeters per second, yielding Reynolds numbers $Re = 2R_{cyl}U_{cyl}/v$ in the range $100 \leq Re \leq 250$ (v is the kinematic viscosity). In this range of Reynolds number, after an initial

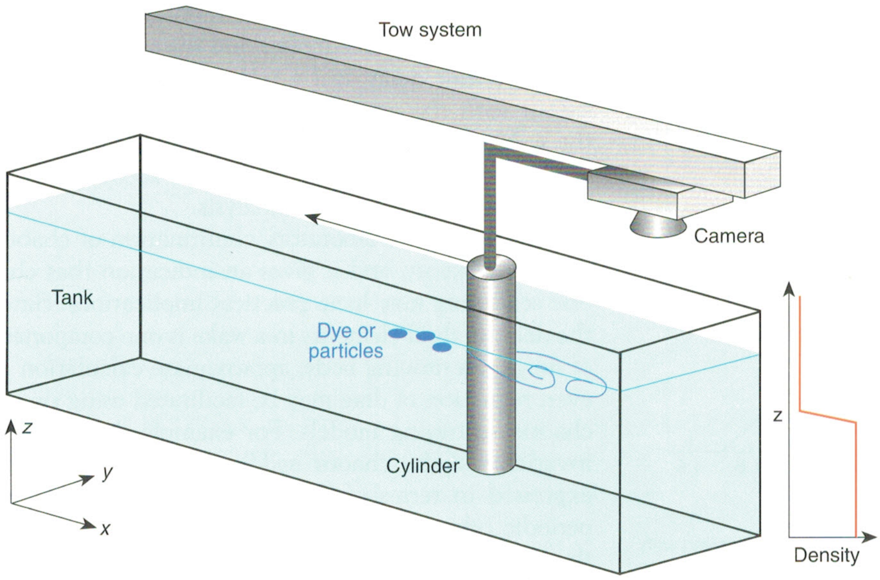

Figure 5. Schematic of the apparatus for cylinder wake experiments.

transient, the velocity field in the wake region is (ideally) time-periodic. This time dependence provides sufficient phase-space degrees of freedom (x, y, and t mod T_c, where T_c is the period of the velocity field) to support chaos. The period T_c also provides a characteristic timescale against which to measure the dynamics (at Re = 100, T_c = 595 s; at Re = 250, T_c = 205 s). It should be emphasized that although the velocity field is simply time-periodic, the *trajectories* of fluid elements can be chaotic. The dynamical system can be considered a three-dimensional continuous-time system, or, by considering snapshots taken at intervals of T_c, it can be viewed as a two-dimensional area-preserving mapping.

The flow field in the wake is dominated by vortices that form behind the cylinder, alternating sides in the cross-stream direction. The nearby walls and viscosity quickly suppress this vorticity, so the entire von Karman street is reduced to only two vortices at any time.

The wake region was visualized from above using a CCD camera fixed with respect to the cylinder. Ultraviolet lamps inside the cylinder excited tiny fluorescent tracer particles or fluorescent dye, depending on the experiment.

To check that the experiment was indeed approximately two-dimensional, with a time-periodic velocity field, APL's Hwar Ku performed a strictly two-dimensional direct numerical simulation of the flow field at the limiting Reynolds numbers. The simulation used a multigrid domain decomposition approach incorporating the pseudospectral element method.[31] We then compared computed pathlines with observed dye lines produced by a comb of outlets upstream from the cylinder. The correspondence was excellent.

Results

Fractal Nature of Wake

In an experiment where the cylinder traverses a cross-stream stripe of dye, one expects that dye leaving the wake after being delayed for a while should have picked up the geometric properties of the unstable chaotic set, since the delayed fluid elements have been close to the set. Thus, dye delayed for a long time should make an approximately fractal pattern in the fluid. (From a practical standpoint, because the duration of the experiment is finite and because the dye also undergoes diffusion, smearing over the fine structure in any patterns, we needed to look at dye that hadn't been delayed for *too* long.)

At a Reynolds number Re = 250, the information dimension of the dye pattern in the wake was measured to be $d = 1.3 \pm 0.1$. Due to the camera's limited resolution, the precision of this estimate is low, but the result is statistically bounded away from an integer.

Time Delay Statistics

We also considered the time delay of fluid elements leaving the wake. By moving the cylinder though a cross-stream stripe of dye, we marked an ensemble of impact parameters. The first (freestreaming) dye to reach a strip $10R_{cyl}$ behind the cylinder (which was also behind the alternating vortices) had not interacted with the wake, and it was used to define the zero of time delay t_{fs}. We recorded the radiometric intensity of the dye that subsequently passed through the strip. The decay of the remaining dye (shown in Fig. 6) was initially exponential, indicating the predicted interaction with a chaotic saddle. Longer delays showed a more complicated time dependence due to interactions with the cylinder wall, another marginally stable set from a dynamical point of view.

Prediction of Wake Geometry

Kantz and Grassberger[30] derived relationships among several quantities characteristic of the chaotic saddle of a two-dimensional mapping: the information dimensions of geometric sets associated with its stable and unstable directions, the Lyapunov exponents describing how fast neighboring fluid elements diverge

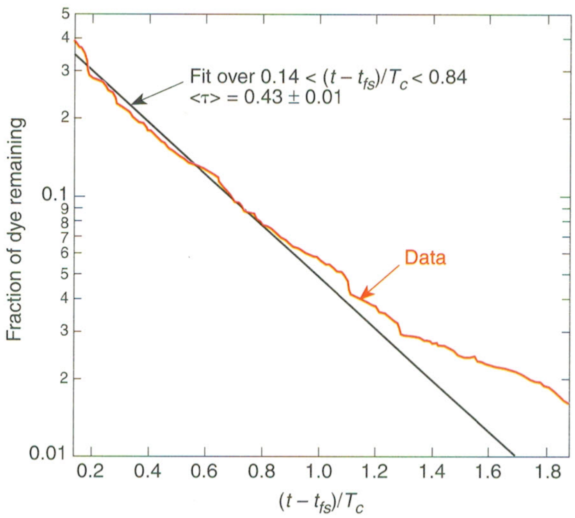

Figure 6. Average time delay statistics for the dye interacting with the wake. Since the space of impact parameters includes the time at which the fluid element encounters the cylinder, relative to the phase of the periodic velocity field, the average is taken over eight runs with different upstream placement of the dye stripe.

from one another near the saddle, and the characteristic delay time $\langle\tau\rangle$ of randomly initialized trajectories shadowing the saddle. For this experiment, the relevant prediction is that the information dimension of the saddle's unstable direction in the wake is given by

$$d_1 = \frac{\lambda_1 - 1/\langle\tau\rangle}{|\lambda_2|} \,. \qquad (7)$$

In the preceding paragraphs, we described the measurement of the geometric properties and the characteristic time delays. Measuring Lyapunov exponents required a different experimental technique: particle tracking. Unlike the experiments described earlier, which involved moving the cylinder through blobs of dye, determination of Lyapunov exponents requires that particular fluid elements be marked in a way that allows them to be identified at a later time. This marking can be done by adding passively advected fluorescent particles to the flow and tracking them as they move. In an earlier *Technical Digest* article, Steven Diamond discussed particle tracking in more detail.[32] In principle, the theory of Kantz and Grassberger[30] requires the determination of *both* the Lyapunov exponents, including that corresponding to atypical trajectories that approach one another along the stable direction (i.e., $\lambda_2 < 0$). However, in the case of this two-dimensional Hamiltonian system, the conservation of phase-space volumes implies that $\lambda_2 = -\lambda_1$. Thus, we had all the information necessary to provide a quantitative test of the chaotic scattering model of the wake.

The predicted fractal dimension, based on the Lyapunov exponent estimates and the measured characteristic time delay, is $d = 1.26 \pm 0.03$, which is consistent with the experimental direct measurement of the wake information dimension. Sommerer, Ku, and Gilreath[33] provide a more complete account of the experiment and associated analysis.

This is the first laboratory confirmation of chaotic scattering theory, and it gives an indication that chaotic scattering may have practical implications. Since the delay of fluid elements in a wake is one component of drag on a moving body, approximate calculation or even reduction of drag may be facilitated using simple chaotic scattering models. For example, because the invariants of the chaotic saddle in the wake may be expressed in terms of the properties of the unstable periodic orbits embedded in the saddle, disruption of the periodic orbits should affect the drag properties directly. Analysis of periodic orbits in simple models is much easier than direct numerical simulation of the Navier–Stokes equations.

OTHER CHAOTIC FLUID MODELS

Several other fluid models also appear to be amenable to chaotic techniques, although no experiments have yet been done to confirm their potential utility.

Leapfrogging of Vortex Rings

It is a striking phenomenon that two coaxial vortex rings with the same sense of rotation can leapfrog one another: the rear vortex ring attempts to pass through the front one. The leading ring then widens because of the mutual interaction and travels more slowly. Simultaneously, the other ring shrinks, travels faster, and penetrates the first one. In an ideal case this process is repeated continuously. The two-dimensional version of this motion, the leapfrogging of two pairs of ideal point vortices moving along a joint symmetry axis, is a model where tracer trajectories can be studied.[34] Unstable periodic orbits and the entire chaotic saddle have been found in numerical simulations. Simulated tracers introduced into the flow in front of the leapfrogging system were shown to mark, at sufficiently long times, a chaotic saddle's unstable direction. This pattern, just like that in a three-dimensional simulation,[35] is in very good qualitative agreement with the photographs of a 1978 experiment[36] of smoke rings where the smoke particles play the role of the tracer, marking a fractal foliation.

Systems of Unidirectional Vortices

Theoretical investigations and model simulations show that four or more vortices of equal sign in an infinite two-dimensional fluid (or three or more vortices in a circular container) exhibit chaotic motion in a finite domain around their center of vorticity.[37] Consequently, the advection of particles in the field of at least three (in a container, two) vortices is also chaotic. Dye droplets should trace out more and more convoluted shapes as time goes on. In this case no escape takes place, and compact regions of the fluid surface will be asymptotically spread by particles. Such chaotic vortex motion can be observed in magnetically confined plasma experiments.[38] Recent theoretical investigations show that this chaos can be controlled; i.e., vortices can be forced to follow periodic motion forever by suitably chosen weak time-dependent perturbations.[39]

Basins with Sinks and Sources

Consider a large bath tub with two sinks that are opened in an alternating manner. In the course of drainage, a rotational flow is formed around the sinks. Because particles are pushed toward one or the other sink alternately, transient chaotic mixing can take place before the tub is empty. A model of this phenomenon is the blinking vortex-sink system introduced by Aref et al.[37] The time-reversed version describes the periodic *injection* of fluid into the basin via two different sources accompanied by rotation. In both cases one observes complicated tracer trajectories before exiting via one of the sinks (or going out to large distances in the injection model). Again, a chaotic saddle is present, consisting of trajectories that stay bounded and never approach the sinks, both in the forward and in the backward dynamics. The corresponding fractal patterns might be observable experimentally, but an even simpler arrangement could be a set of three or more alternately working injection holes (without any induced rotation), where particle motion is again expected to be chaotic.

General Features

These examples and the wake experiment shed new light on the nature of tracer patterns. Convoluted shapes observed in classical experiments[36,40] did not attract special attention because the concept of fractals was not yet established at that time. The present observations show that dye droplets, or, more generally, streaklines, converge to fractal foliations whenever the advection dynamics is chaotic. The patterns can also be interpreted as fractal dye boundaries in open flows.[41] Any kind of fractality appears to be the fingerprint of an underlying chaotic set that contains an infinite number of bounded, nonescaping particle trajectories, including unstable periodic orbits. Measuring the fractality of the tracer patterns provides information on the underlying chaotic set.

CONCLUSIONS

We have shown that simple chaotic models provide good descriptions of relatively complicated two-dimensional fluid flows. Although chaos theory does not constitute a breakthrough capable of treating all fluid problems, it clearly takes its place as one of a spectrum of tools available. Further, the character of diagnostics appropriate to chaos theory is rather different from the detailed predictions of velocity fields that are produced by computational fluid dynamics. Our analysis and experiments have focused on the relation of local, dynamical quantities to global, static geometric measures. This nonstandard treatment has both disadvantages and potential advantages. If a detailed flow simulation is called for, then clearly chaos theory will not do. On the other hand, static measurements may be used to infer something about the time-averaged dynamics of the fluid system without the need for temporally resolved computations or measurements. Certainly, the theoretical and experimental exploration of such techniques continues apace.

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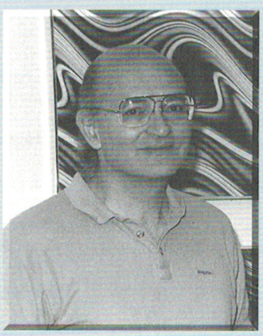
ACKNOWLEDGMENTS: We would like to thank our many collaborators in these endeavors. Our work was supported at various stages by the U.S. Office of Naval Research (support for E. O.); the U.S. Air Force Office of Scientific Research and the U.S. Navy Space and Naval Warfare Systems Command (support for J. C. S.); and the Hungarian Scientific Research Foundation (support for T. T.).

THE AUTHORS

JOHN C. SOMMERER is a Principal Professional Staff physicist in the Milton S. Eisenhower Research and Technology Development Center at APL. He has B.S. and M.S. degrees in systems science and mathematics from Washington University in St. Louis, an M.S. in applied physics from The Johns Hopkins University, and a Ph.D. in physics from the University of Maryland. Dr. Sommerer's work in nonlinear dynamics was recognized by APL with the 1992 R. W. Hart Prize for Excellence in Research and by the Maryland Academy of Sciences with its Distinguished Young Scientist Award in 1994. Dr. Sommerer teaches a course on chaos and its applications in the Applied Physics and Applied Mathematics Programs of the G.W.C. Whiting School of Engineering's Part Time Programs in Engineering and Applied Science. His e-mail address is John.Sommerer@jhuapl.edu.



EDWARD OTT is a Distinguished University Professor at the University of Maryland, College Park. He holds appointments in the Physics and Electrical Engineering Departments, as well as in the Institute for Systems Research. He has a B.S. in electrical engineering from the Cooper Union, and M.S. and Ph.D. degrees in electrophysics from the Polytechnic Institute of Brooklyn. He is a fellow of the American Physical Society, a fellow of the IEEE, and has been Divisional Associate Editor of *Physical Review Letters*. He has published over 250 papers on plasma physics, particle beams, and nonlinear dynamics. Dr. Ott is author of the book *Chaos in Dynamical Systems*, published by Cambridge University Press. He refuses to have an e-mail address.

TAMÁS TÉL is Associate Professor of Physics at the Institute for Theoretical Physics, Eötvös University in Budapest, Hungary. He has B.S, M.S., and Ph.D. degrees in physics from the same university. He has worked as a guest scientist in various research institutes in Italy, Germany, Israel, and the United States. Dr. Tél's work in nonlinear science was recognized by the Hungarian Academy of Sciences with its award in 1995. He teaches courses on chaos for students in physics and meteorology at the Eötvös University. His e-mail address is tel@poe.elte.hu.