11 CONTROLLING CHAOS ON FRACTAL BASIN BOUNDARIES

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The OGY method of controlling chaos (Ott, Grebogi and Yorke 1990) has been applied to stabilize periodic orbits on fractal basin boundaries. As illustrative examples the fractal boundaries in an invertible two-dimensional map and in the driven damped pendulum are considered. The scaling behaviour in the number of trajectories controlled is discussed.

11.1 INTRODUCTION

Fractal basin boundaries (Moon and Li 1985; Thompson *et al.* 1987) are typical attributes of nonlinear systems. If two or more attractors coexist in a system, their basins of attraction are separated by basin boundaries. These boundaries might be *fractals* with the consequence that trajectories started in their vicinity exhibit very complicated and unpredictable *transient chaotic* motion (Tél 1990) before settling down onto one of the possible simple attractors. All kinds of basin boundaries are *stable manifolds* along which one or more *saddles*, situated between the attractors, can be approached (Grebogi *et al.* 1988). In the case of fractal boundaries at least one of the saddles is *chaotic*. In fact, many examples of transient chaos, including experiments (see e.g. Iansiti *et al.* 1985; Arecchi *et al.* 1982; Bergé and Dubois 1983; Arecchi and Lisi 1983; Kowalik *et al.* 1988; Papoff *et al.* 1988; Dangoisse *et al.* 1986), are due to trajectories starting close to a fractal boundary separating multiple attractors.

It might occur that one of the attractors is connected with an undesirable outcome, such as the capsizing of a ship (Thompson *et al.* 1990). Because of intertwined basin boundaries, it is then difficult to direct the trajectory towards a desirable attractor. Thus, it can be of practical importance to stop the trajectory *on the basin boundary*, or more precisely, on a periodic orbit of the chaotic saddle. This means that an orbit hesitating to go to any of the attractors (like Balaam's ass) is converted into an attracting orbit. In this chapter we show how this can be achieved by applying results from the theory of chaos control.

Controlling chaos by means of the OGY method (Ott *et al.* 1990; Romeiras *et al.* 1992; Auerbach *et al.* 1992; Lai *et al.* 1992) has attracted interest of late among both theoreticians and experimentalists. The approach can be summarized as follows. One selects

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a target region around a predetermined hyperbolic periodic orbit on the chaotic set. Then an ensemble of trajectories is started in some region of the phase space and one waits until trajectories enter the target region where the actual controlling algorithm is applied. The controlling perturbation is adjusted so that the predetermined periodic orbit is stabilized. Only small local perturbations are allowed, smaller in size than some value δ which is proportional to the linear extension of the target region.

The method was originally worked out for systems the dynamics of which is governed by an underlying chaotic attractor in phase space. All trajectories are then controlled since any region on the attractor is visited sooner or later. The time needed to wait at a given maximum perturbation δ, i.e. the expected time $\tau(\delta)$ to achieve control, however, diverges: $\tau(\delta) \sim \delta^{-\gamma}$ as the maximum perturbation approaches zero.

The method has been extended to controlling transient chaos (Tél 1991, 1993; Lai *et al.* 1993) asociated with non-attracting chaotic sets: chaotic repellers or saddles. These are unstable objects in the sense that trajectories starting from their neighbourhoods escape with an average rate κ, called the *escape rate*. One can select again any of the infinitely many periodic orbits on the chaotic set and apply the same procedure as for permanent chaos. The qualitatively new feature compared with the control of permanent chaos is that an orbit is stabilized now which *has nothing to do with the asymptotic behaviour* (attractor) of the system. The control of chaos on fractal basin boundaries is thus a special case of controlling transient chaos.

11.2 CONTROL OF A SIMPLE MAP

First, we investigate a simple two-dimensional invertible mapping introduced in McDonald (1985a, b) to illustrate the existence of a fractal boundary between two fixed point attractors. The map is defined in polar coordinates x and θ as

$$\theta_{n+1} = \theta_n + a\sin(2\theta_n) - b\sin(4\theta_n) - x_n \sin\theta_n, \qquad (1)$$
$$x_{n+1} = -J_0 \cos\theta_n.$$

Because of an invariance under the transformation $\theta \to 2\pi - \theta$, it is sufficient to consider the range $0 \leqslant \theta \leqslant \pi$ only.

At parameter values $a = 1.32$, $b = 0.9$, $J_0 = 0.3$ the only attractors are the points $A_1 = (0, -J_0)$ and $A_2 = (\pi, J_0)$. Three further fixed points H_0 and $H_{1,2}$ lie along the line $x = -J_0 \cos\theta$ and are hyperbolic. Their θ values are $\theta_0 = \pi/2$, θ_1, and $\theta_2 = \pi - \theta_1$, respectively, with $\theta_1 = \arccos[(2a + J_0)/4b]/2$. All of these fixed points are on the chaotic saddle and belong, therefore, to the basin boundary (see Figure 11.1). Note that the position of the fixed point H_0 does not depend on the system parameters, and therefore it is more difficult to control than $H_{1,2}$. Only the control of the latter will be discussed here.

We shall demonstrate the stabilization of H_1 below by adding a weak time-dependent perturbation p_n to the parameter a, i.e. by replacing a in equation (1) by $a_n = a + p_n$. In order to achieve this, one has to use a large *ensemble* of points starting from some region of phase space including the chaotic saddle and concentrate on long-lived chaotic transients. Next, define a target region, a disc of radius r, around H_1, and denote by ξ_n the distance of the nth point of the trajectory from the desired fixed point.

Some of the trajectories will stay around the chaotic saddle over many time steps

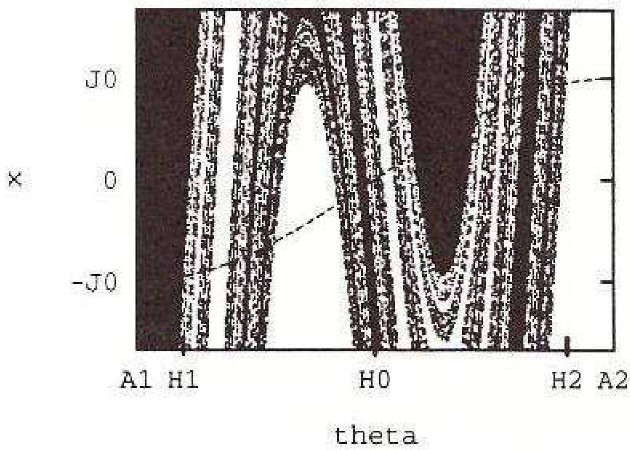

Figure 11.1 Basin boundary for map (1) with parameters $a = 1.32$, $b = 0.9$, $J_0 = 0.3$ plotted in the range $|x| \leqslant 0.5$, $0 \leqslant \theta \leqslant \pi/2$. The fixed points lie on the curve $x = -J_0 \cos\theta$ (dashed line). The θ values of the attractors $A_{1,2}$ as well as of the hyperbolic points H_0 and $H_{1,2}$ are indicated on the horizontal axis.

and might fall near H_1. Therefore, wait until ξ_n of any trajectory enters the target region around H_1 and then change the actual value of p_n to be different from zero. Pick p_n so that the next iterate ξ_{n+1} falls on the stable manifold of H_1 of the *uncontrolled* map. This mechanism is completely the same as for chaotic attractors; therefore, the result for the appropriate choice of p_n can be taken over from OGY. The computation based on the *linearized* dynamics around a fixed point of a two-dimensional map says (Ott *et al.* 1990) that

$$p_n = \frac{\lambda_u}{\lambda_u - 1} \frac{\xi_n \boldsymbol{f}_u}{\boldsymbol{g}\boldsymbol{f}_u}. \tag{2}$$

Here λ_u and $\boldsymbol{f}_u$ are the unstable eigenvalue of the fixed point in the uncontrolled map ($p = 0$) and the corresponding left eigenvector, respectively. The quantity $\boldsymbol{g}p$ yields the shift of the fixed point's position when changing the perturbation parameter by a small amount of p. It is assumed that the parameter p can be varied in a small range $|p| < \delta$ only. Thus, if $|p_n|$ happens to be greater than the maximum perturbation δ, then we set $p_n = 0$. This last condition also specifies the size of the target region where control is activated.

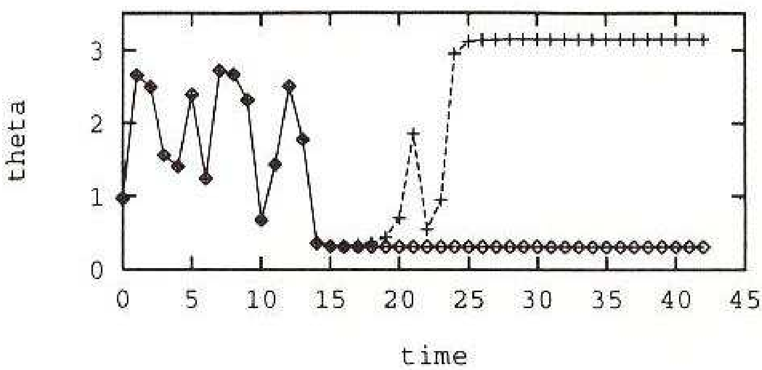

Figure 11.2 The θ component of a trajectory of map (1) starting at the initial point $x_0 = 0.154,839$, $\theta_0 = 0.962\,746$. Crosses and diamonds represents the uncontrolled and controlled trajectories, respectively. Control sets in at step 16 where the trajectory enters the control region, a disc of radius $r = 0.01$, around the hyperbolic point H_1.

Figure 11.2 shows one coordinate of a trajectory that moves around on the basin boundary until, at the 16th step, it enters the target region where control is applied, and then the trajectory becomes captured at H_1. Without control, this trajectory goes away from this fixed point and eventually runs into the attractor A_2.

11.3 THE DRIVEN DAMPED PENDULUM

Our second example, the forced driven pendulum, is a system with a continuous time dynamics. The equation of motion for the angle Φ of the pendulum in appropriate dimensionless variables is (Gwinn and Westervelt 1985, 1986a, b):

$$\ddot{\Phi} + \beta\dot{\Phi} + \sin\Phi = \rho\cos\omega t, \quad (3)$$

where β and ρ measure the damping and driving strengths, respectively, and dots denote time derivatives. Here we assume a pure sinusoidal torque drive of frequency ω without an additional constant torque. In the following, we fix $\beta = 0.2$ and $\omega = 1$, and consider ρ as our control parameter.

It is known (Gwinn and Westervelt 1985, 1986a, b) that for $\rho = 2$ (and other ρ values in a certain region around this point) the system possesses two periodic attractors with fractal basin boundaries. The attractors correspond to a motion in which the pendulum keeps winding around but each rotation is interrupted by a complete swinging cycle. If we introduce a two-dimensional Poincaré section in the system by measuring Φ and $\dot{\Phi}$ when the driving

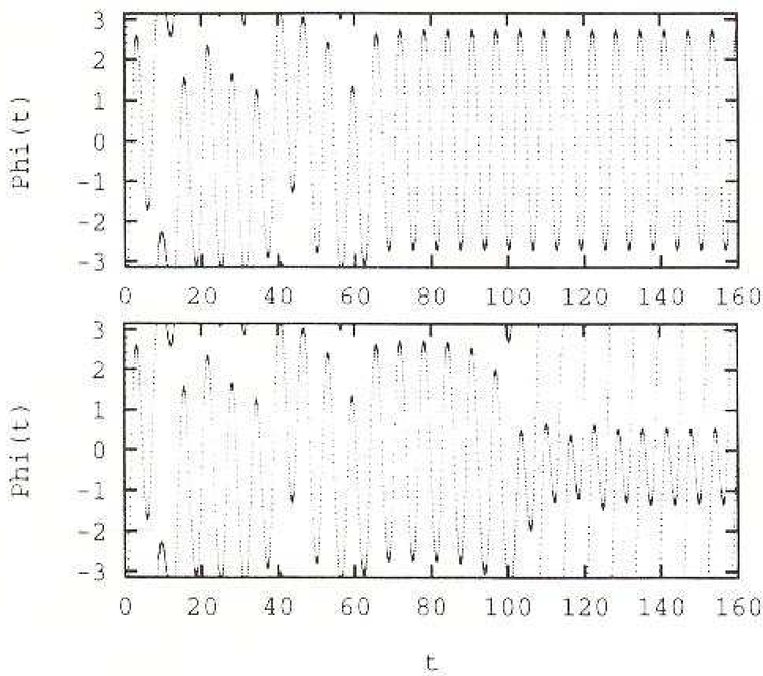

Figure 11.3 Time dependence of the angle Φ (modulo π) started from $\Phi_0 = 0.580\,685\,8$ and $\dot{\Phi}_0 = 2.752\,419\,0$ at $t = 0$. The trajectory enters the target region, a disc of radius $r = 0.05$ around the unstable fixed point $(-2.631\,308\,5,\ 0.651\,685\,6)$ on the Poincaré surface, after 10 driving cycles where control sets in causing the pendulum to keep swinging without rotation (upper box). Without control (lower box), the same trajectory is attracted to the stable winding motion.

torque is maximal (and positive), the attractors are represented as fixed points of the two-dimensional map induced in the section. The perpetual swinging ('pendulum-like') behaviour never making a full rotation appears as an unstable fixed point of the map lying on the basin boundary.

This swinging motion can be stabilized by applying the control method described earlier. The parameter ρ is used for this purpose: we take $\rho_n = 2 + p_n$ as the driving torque amplitude. p_n is kept fixed between two returns to the Poincaré surface when its value is refreshed according to the control rules. We started a large number of orbits from a region containing a part of the saddle, and looked for those that stay long enough on the boundary to be able to come close to the unstable fixed point. Figure 12.3 presents the continuous time dynamics of an orbit that wanders on the basin boundary until it gets close to the swinging motion. Then control is turned on, pulling the orbit into the desired fixed point. This behaviour can be compared to the motion, also shown in Figure 12.3, started from the same initial condition but never subject to control: it ends up following the attractor characterized by winding in a counterclockwise sense.

11.4 SCALING PROPERTIES

Quantitatively, another novel feature distinct from the permanent case is that many trajectories escape before falling into the target region. Consequently, the number $N(\delta)$ of trajectories controlled at a given maximum perturbation δ behaves as

$$N(\delta) \sim \delta^{\gamma(\kappa)} \tag{4}$$

where $\gamma(\kappa)$ is an exponent depending on the escape rate κ. Its explicit form has been found to be (Tél 1991):

$$\gamma(\kappa) = 1 + \frac{\ln|\lambda_u| - \kappa}{\ln 1/|\lambda_s|} \tag{5}$$

where λ_s denotes the stable eigenvalue of the controlled fixed point. κ is the escape rate from the chaotic saddle the stable manifold of which forms the basin boundary. In other words, $1/\kappa$ is the average chaotic lifetime on the basin boundary.

Since the time to achieve control cannot be longer than the chaotic life time, we obtain that $\tau(\delta)$ is independent of the maximum allowed perturbation δ and is limited from above by $1/\kappa$:

$$\tau(\delta) = \text{const.} \leqslant 1/\kappa. \tag{6}$$

We mention in passing that the escape rate κ which appeared in the formulae above can be estimated in the knowledge of the *uncertainty exponent* α (Grebogi *et al.* 1983a). This exponent plays a central role in characterizing fractal boundaries where uncertainty in initial conditions leads to enhanced uncertainty in the final state. It is easy to see (Tél 1990) that

$$\alpha = \frac{\kappa}{\lambda} + D_1^{(1)} - D_0^{(1)} \leqslant \frac{\kappa}{\lambda} \tag{7}$$

where λ, $D_0^{(1)}$ and $D_1^{(1)}$ stand for the average Lyapunov exponent, and for the partial fractal and information dimensions of the chaotic saddle along its unstable direction, respectively. The inequality is expected to be close to saturation since the difference between $D_0^{(\mathrm{I})}$ and $D_1^{(\mathrm{I})}$ is typically less than a few percentage points. By assuming that the local Lyapunov exponent of the controlled trajectory is close to the average one, the approximate relation

$$\kappa \approx \alpha \ln \lambda_u \tag{8}$$

can also be used.

11.5 CLOSING REMARKS

We have demonstrated that chaotic trajectories moving in the vicinity of a fractal basin boudary can be controlled, i.e. directed towards a periodic orbit on the chaotic saddle organizing the boundary. Such trajectories never reach the attractors of the system and represent a qualitatively new asymptotic behaviour. We saw that controlling chaos on fractal boundaries is more difficult than the control of permanent chaos as one has to use ensembles of trajectories, but it is also simpler since the time needed for control does not grow with decreasing perturbation. By applying recent techniques like targeting (Shinbrot *et al.* 1990) or control via synchronization (Lai and Grebogi 1993) one can hope that even the average number of trajectories needed to achieve control can be drastically reduced.

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