Physica **112A** (1982) 146–166 *North-Holland Publishing Co.*

FLUCTUATIONS IN THE LIMIT CYCLE STATE AND THE PROBLEM OF PHASE CHAOS

P. SZÉPFALUSY

Central Research Institute for Physics, H-1525 Budapest, P.O.B. 49, Hungary

and

Institute for Theoretical Physics, Eötvös University, H-1445 Budapest, P.O.B. 327, Hungary

and

T. TÉL

Institute for Theoretical Physics, Eötvös University, H-1445 Budapest, P.O.B. 327, Hungary

Received 4 November 1981

Gaussian fluctuations and first order fluctuation corrections to the deterministic solution are investigated in the framework of the generalized Ginzgurg–Landau type equation of motion exhibiting a hard mode transition leading to a homogeneous limit cycle state. It is shown that the stationary distribution of the fluctuations around the limit cycle is not of the form of a Ginzburg–Landau functional. The nature of the further instability in the post bifurcational region, resulting in the phase chaos in the deterministic problem, is found to be qualitatively changed by the presence of noise.

1. Introduction and summary

Hard mode instabilities in systems far from thermal equilibrium leading to the formation of homogeneous limit cycles[1,2]) in the ordered phase are good examples demonstrating, besides the similarities, also the differences between equilibrium and non-equilibrium phase transitions. It is of particular interest, for example, whether in the asymptotic state, reached after a sufficiently long time, a stationary distribution exists in a frame of reference moving together with the limit cycle, and if so, how this probability distribution looks like. Another question of interest is how further instabilities occuring in the ordered phase are influenced by fluctuations. In order to answer such general questions the investigation of model systems is highly useful.

As is well known, the time-dependent Ginzburg–Landau (TDGL) model played an essential role in understanding the dynamics near continuous phase transitions in systems close to thermal equilibrium[3,4]). Investigations of non-equilibrium phase transitions have shown that after an adiabatic elimination procedure the equation of motion obtained near the instability point for the slowly relaxing modes (i.e. for the components of the order parameter field) are similar to a TDGL equation[2]). In particular, in the case of hard mode instabilities that represent a normal Hopf bifurcation (i.e. the order parameter grows out continuously as entering the post-bifurcational region) the equation of motion for the complex order parameter field is of the same form as that near equilibrium phase transitions. There is, however, an essential difference, namely now the parameters in the equation of motion turn out to be complex numbers[5–7]). This equation of motion including a Langevin type additive Gaussian white noise can naturally be called the generalized TDGL model. The model possesses typical properties of systems far from thermal equilibrium: the fluctuation–dissipation theorem does not hold and the equation of motion admits no potential.

The investigations in this paper will be based on the generalized TDGL model and our aim is to study the nature and the consequences of the inhomogeneous fluctuations around the deterministic solution with emphasis on the behaviour in the ordered phase. In order to illustrate the most important points it will be sufficient to use a Gaussian approximation which makes also possible to evaluate first order fluctuation corrections to various quantities. Since the form of the distribution of the fluctuations does not follow from general arguments in this model, it had to be constructed directly from the equation of motion. One of the main results of the paper is the finding that the stationary distribution of the fluctuations around the limit cycle in the ordered phase is not of Ginzburg–Landau form and it exhibits a singular dependence on the wave-number and the amplitude of the limit cycle, which is a striking qualitative difference as compared to the equilibrium distribution near critical points at phase transitions. Assuming that this behaviour is not a pathology of the Gaussian approximation one can conclude that the existence of a Ginzburg–Landau type equation of motion in the general case does not involve a Ginzburg–Landau type functional for the distribution in the asymptotic state.

By means of the stationary distribution obtained, we calculate corrections to the deterministic results for both the amplitude and the frequency of the limit cycle, which turn out to diverge below four dimensions as the instability point is approached. As contrasted to the case of the usual TDGL model, we find two Ginzburg-like criteria specifying the region where the corrections are small. It is pointed out that a Ginzburg–Landau type functional cannot lead to any correction for the frequency of the limit cycle, therefore the existence of a deviation of the limit cycle frequency from the result of the deterministic approach can be taken generally as a signal of the unusual form of the stationary distribution. It is to be noted, on the other hand, that the absence of such a correction would not mean that the frequency of the limit cycle at the bifurcation point is not modified, since the fluctuations shift the bifurcation point itself in any case.

Finally we turn to another question of interest, namely to the effect of fluctuations on the transition where the limit cycle state loses its stability. This instability was discovered within the framework of the deterministic TDGL model[7]) and it was observed that the new phase emerging shows a type of chaotic behaviour[8-10]), termed phase chaos. In the Gaussian approximation we find that due to the enhanced phase fluctuations the local fluctuations of the order parameter field are diverging at this instability point indicating that the zeroth order approximation loses its meaning there and that the nature of this instability may change qualitatively due to the presence of non-linear fluctuations. Indeed, taking into account first order fluctuation corrections a tendency is exhibited that the amplitude of the limit cycle decreases when approaching the instability point, which suggests that this instability in a noisy system might be correctly specified by the vanishing of the limit cycle amplitude. It is then found that fluctuations shrink the region of the stable limit cycle state and the shift of the instability point is scaled in d dimensions with the power $2/(4-d)$ of the strength of the noise correlations.

All the calculations are carried out in the m-component version of the generalized TDGL model, i.e. when the order parameter field is an m-component complex vector. Though the most relevant case concerning applications corresponds to $m = 1$, such an extension of the model is a useful theoretical device similarly as it has proved to be at equilibrium phase transitions[11,3]).

The paper is organized as follows: After defining the generalized TDGL model (section 2) we consider a special case characterized by a certain relationship among the parameters in the equation of motion (section 3). Turning to the general case we derive in the Gaussian approximation the correlation functions in both the pre- and post-bifurcational regions (section 4), as well as the stationary distribution the anomalous properties of which are discussed (section 5). Then, the corrections to the amplitude and the frequency of the limit cycle are determined (section 6) followed by the investigation of the effect of fluctuations near the border-line where the limit cycle state loses its stability (section 7). A generalization of the model describing a behaviour analogous with that near tricritical and higher order critical points of equilibrium phase transitions, as well as, some expressions of correlation and response functions, not treated in the main text, are given in the appendices.

2. The model

A wide class of hard mode instabilities has turned out to be describable by a generalized time-dependent Ginzburg–Landau (TDGL) model for a complex order parameter field, the parameters of which are also complex numbers[5-7]). In order to simulate fluctuations and to study their effects a complex noise term is included into the equation of motion. We shall consider the m-component version of this model which corresponds to a situation where m modes become simultaneously unstable at the bifurcation point and assume isotropy in the component space. Let $\phi_j(x, t)$, $j = 1, 2, \dots m$ denote the slowly relaxing complex order parameter field (the critical mode) with momentum cut-off Λ in a d-dimensional system of volume unity. The equation of motion of ϕ_j in the vicinity of the bifurcation point is the following

$$\dot{\phi}_j(x, t) = -\left(-a\nabla^2 + u_2 + u_4 \sum_{l=1}^{m} |\phi_l|^2\right)\phi_j(x, t) + \zeta_j(x, t), \tag{2.1}$$

where the coefficients a, u_2 and u_4 are complex numbers. We shall use the notation for complex numbers z: $\mathrm{Re}\, z \equiv z^{(1)}$ and $\mathrm{Im}\, z \equiv z^{(2)}$. The real part of u_2 is supposed to depend linearly on the control parameter, λ. In suitable units

$$u_2^{(1)} = \lambda_0 - \lambda, \tag{2.2}$$

where λ_0 is the value of the control parameter at the instability point of the deterministic problem. All the other parameters of eq. (2.1) are taken to be independent of λ. To ensure the existence of a stable homogeneous stationary state $a^{(1)}$ and $u_4^{(1)}$ should be positive. By using suitable units we set $a^{(1)} = 1$.

As the simplest possibility, we assume the complex noise ζ to be a Gaussian white noise (similarly as in references[12-16]) with zero mean value and correlation functions as

$$\langle \zeta_j^{(1)}(x, t)\zeta_{j'}^{(1)}(x', t')\rangle = 2\Gamma\delta(x - x')\delta(t - t')\delta_{j,j'}, \tag{2.3}$$

$$\langle \zeta_j^{(2)}(x, t)\zeta_{j'}^{(2)}(x', t')\rangle = 2\Gamma\delta(x - x')\delta(t - t')\delta_{j,j'}. \tag{2.4}$$

$\zeta_j^{(1)}$ and $\zeta_j^{(2)}$ are independent random variables, Γ is a real constant.

Note the symmetry of eqs. (2.1), (2.3) and (2.4): if we perform the gauge transformation $\phi_j(x, t) \to \phi_j(x, t)\exp(-\mathrm{i}\omega t)$ for $j = 1, 2, \dots, m$, where ω is a (real) constant, we get a similar equation as (2.1) but now u_2 is replaced by $u_2 - \mathrm{i}\omega$.

The model defined above describes a noisy system exhibiting a normal Hopf bifurcation: for control parameter values $\lambda < \lambda_c$ (λ_c is the value of the control parameter at the bifurcation point) the system has a homogeneous steady state while for $\lambda > \lambda_c$ a homogeneous limit cycle is approached asymptotically. The amplitude of the limit cycle, Ψ, is considered to be the order parameter which sets in continuously when λ goes through its critical value, λ_c. The order parameter, Ψ, is in general a complex m-component vector. However, by making use of the isotropy of the system in the component space we can choose it to point in the direction of the $j = 1$ axis; furthermore, it can be always chosen to be real. In the asymptotic state, reached for $t \to \infty$, the average of $\phi_j(x, t)$ is obtained for $\lambda \gtrsim \lambda_c$ then as

$$\langle \phi_j(x, t) \rangle = \Psi \exp(-i\omega_{lc}(\lambda)t)\delta_{j,1}, \tag{2.5}$$

where $\omega_{lc}(\lambda)$ denotes the frequency of the limit cycle at control parameter value λ, and Ψ is real.

A possible generalization of the model, the critical behaviour of which is analogous to that near higher order critical points, is given in the appendix A.

3. A special case

Before turning to the approximate evaluation of the effects of fluctuations it is worth discussing a special case in which considerable simplification occurs and some exact results can be obtained. This case is specified by the condition

$$a^{(2)} = \frac{u_4^{(2)}}{u_4^{(1)}}. \tag{3.1}$$

Writing down the Fokker–Planck equation associated with the process (2.1)–(2.4) and using the condition (3.1) it can easily be shown that in the pre-bifurcational region ($\lambda < \lambda_c$) a stationary distribution $P_{st}\{\phi_j\}$ is reached for $t \to \infty$ which is of the Ginzburg–Landau form defined by

$$P_{st}^{GL}\{\sigma_j\} \propto \exp(-\mathscr{F}/\Gamma), \tag{3.2}$$

$$\mathscr{F} = \tfrac{1}{2} \int d^d x \left[\sum_{j=1}^{m} |\nabla \sigma_j|^2 + u_2^{(1)} \sum_{j=1}^{m} |\sigma_j|^2 + \tfrac{1}{2} u_4^{(1)} \left(\sum_{j=1}^{m} |\sigma_j|^2 \right)^2 \right]. \tag{3.3}$$

Note that $\mathscr{F}$ contains only the real parts of the parameters of the equation of motion. Furthermore, one can convince oneself that after performing the gauge transformation $\phi_j \to \phi_j \exp(-i\omega t)$ one obtains a stationary distribution of the same form as (3.2)–(3.3) for the transformed field for $\lambda < \lambda_c$.

It is important for the following that, when (3.1) holds, there is a particular gauge transformation

$$\phi_j(x, t) = \psi_j(x, t) \exp(-i\omega_0(\lambda)t), \tag{3.4}$$

$$\omega_0(\lambda) = u_2^{(2)} - (u_4^{(2)}/u_4^{(1)})u_2^{(1)}, \tag{3.5}$$

which leads for ψ_j to the equation of motion

$$\dot{\psi}_j(x,t) = -L\left(-\hat{a}^{(1)}\nabla^2 + \hat{u}_2^{(1)} + \hat{u}_4^{(1)}\sum_{l=1}^{m}|\psi_l|^2\right)\psi_j(x,t) + \zeta_j(x,t), \tag{3.6}$$

where the coefficients with a hat denote the original ones divided by Γ and

$$L = \Gamma(1 + \mathrm{i}u_4^{(2)}/u_4^{(1)}). \tag{3.7}$$

This equation of motion corresponds to a well-known model in critical dynamics of equilibrium phase transitions[17,4]), namely to the TDGL model with a complex kinetic coefficient (but otherwise with real parameters). Note that the noise is proportional to $L^{(1)} = \Gamma$. This connection of our problem with the equilibrium phase transition is especially useful in the post-bifurcational region, $\lambda > \lambda_c$. Namely, one can immediately conclude that there is a non-zero stationary value of ψ_j for $\lambda > \lambda_c$ which by the usual convention can be chosen to be real and to point in the direction of the first axis:

$$\langle\psi_j(x,t)\rangle = \langle\bar{\psi}_j(x,t)\rangle = \Psi\delta_{j,1}. \tag{3.8}$$

Here and in the following bar denotes complex conjugation and bracket stands for the average taken in the asymptotic state of the system.

Substituting (3.8) into (3.4) we see that the average value of ϕ oscillates in the asymptotic state, and comparing it with equation (2.5) we obtain for the frequency of the limit cycle

$$\omega_{lc}(\lambda) = u_2^{(2)} - (u_4^{(2)}/u_4^{(1)})u_2^{(1)}, \quad \lambda \geq \lambda_c. \tag{3.9}$$

Note that (3.9) is of the same form as the frequency of the limit cycle specified by the deterministic version of equation (2.1) obtained by disregarding the noise term. Since, however, due to the fluctuations there is a shift in the critical value of λ from λ_0 to some λ_c, the region of validity of (3.9) is different from that of the deterministic case and there is also a shift in the frequency of the limit cycle at the bifurcation point. To exhibit it explicitly we can rewrite (3.9) as follows

$$\omega_{lc}(\lambda) = \omega_{lc}(\lambda_c) + (u_4^{(2)}/u_4^{(1)})(\lambda - \lambda_c), \tag{3.10}$$

where

$$\omega_{lc}(\lambda_c) = u_2^{(2)} + (u_4^{(2)}/u_4^{(1)})(\lambda_c - \lambda_0). \tag{3.11}$$

Concerning the stationary distribution in the post-bifurcational region ($\lambda > \lambda_c$) it is again of the Ginzburg–Landau form (3.2)–(3.3) for ψ_j but, of course, there exists no stationary distribution for ϕ_j itself. It is more appropriate to consider the distribution not for ψ_j but for its deviation from the order parameter, i.e. for ψ_j' defined as

$$\psi_j(x,t) = \psi_j'(x,t) + \Psi\delta_{j,1}. \tag{3.12}$$

It can be obtained by substituting (3.12) into $P_{\mathrm{st}}^{\mathrm{GL}}\{\psi_j\}$, which we do not write down. For future reference, however, we give the stationary distribution in the Gaussian approximation keeping only the terms quadratic in ψ'

$$P_{\mathrm{st}}^{(0)\mathrm{GL}}\{\psi_j'\} \propto \exp(-\Phi^{\mathrm{GL}}/\Gamma), \tag{3.13}$$

$$\Phi^{\mathrm{GL}} = \Phi_1^{\mathrm{GL}} + \Phi_2^{\mathrm{GL}}, \tag{3.14}$$

$$\Phi_1^{\mathrm{GL}} = \tfrac{1}{2}\int \mathrm{d}^d x(|\nabla\psi_1'|^2 + u_4^{(1)}\Psi^2|\psi_1'|^2 + \tfrac{1}{2}u_4^{(1)}\Psi^2\psi_1'^2 + \tfrac{1}{2}u_4^{(1)}\Psi^2\bar{\psi}_1'^2), \tag{3.15}$$

$$\Phi_2^{\mathrm{GL}} = (1/2)\int \mathrm{d}^d x \sum_{j=2}^{m} |\nabla\psi_j'|^2. \tag{3.16}$$

Our main interest in this paper will be in the general case when condition (3.1) does not hold. Then the situation is much more complicated. Firstly, it remains true only in the Gaussian approximation that the frequency of the limit cycle agrees with the deterministic result. Secondly, while the stationary distribution in the pre-bifurcational region in the Gaussian approximation is the quadratic part of the Ginzburg–Landau form (3.2)–(3.3), in the post-birfurcational region, already in the Gaussian approximation, we find a significant deviation from the expression given by (3.13).

4. Correlation functions in the asymptotic state

In order to describe the dynamical properties of the system we shall use the response field formalism[18–22]). Accordingly, the path probability distribution associated with a stochastic process can be written as

$$W\{\tilde{\phi},\phi\} \propto \exp\mathcal{A}\{\tilde{\phi},\phi\} = \exp\int \mathrm{d}t A\{\tilde{\phi},\phi\}, \tag{4.1}$$

where $\tilde{\phi}_j(x,t)$, $j = 1,2,\ldots,m$ denotes the m-component complex response field and $\mathcal{A}$ is the action of the process. The correlation functions of the random variables ϕ and $\tilde{\phi}$ in the asymptotic state are to be calculated by means of W taking the time integration in it between minus infinity and plus infinity[21]).

If we are interested in the Gaussian fluctuations it is sufficient to keep terms of first order in the equation of motion. Consequently, $\mathcal{A}\{\tilde{\phi},\phi\}$ will be a quadratic expression.

In particular, from eq. (2.1) we obtain

$$\mathscr{A}\{\tilde{\phi}, \phi\} = \int \mathrm{d}t \int \mathrm{d}^d x \left[\sum_{l=1}^{m} \{-\Gamma|\tilde{\phi}_l|^2 + (\mathrm{i}/2)[\tilde{\phi}_l(\dot{\phi}_l + (-a\nabla^2 + u_2)\phi_l) + \mathrm{c.c.}]\} \right], \tag{4.2}$$

in the pre-bifurcational region disregarding the non-linear coupling. In this approximation $\lambda_c = \lambda_0$. Evaluating the correlation function of ϕ in the Fourier space we find

$$\langle \phi_{j,k,\omega} \bar{\phi}_{j,k,\omega} \rangle = 4\Gamma|-\mathrm{i}\omega + ak^2 + u_2|^{-2}, \quad j = 1, 2 \dots m, \tag{4.3}$$

where $\phi_{j,k,\omega}$ denotes the Fourier transform of ϕ_j in space and time.

In the post-bifurcational region we introduce new variables, ψ_j, through

$$\psi_j(x,t) = \phi_j(x,t) \exp(\mathrm{i}\omega_{\mathrm{lc}}(\lambda) t), \quad j = 1, 2, \dots, m, \tag{4.4}$$

where $\omega_{\mathrm{lc}}(\lambda)$ is the frequency of the limit cycle, and use the separation (3.12) where the real quantity, Ψ, stands for the order parameter of the system, defined by (2.5). From the equation of motion obtained for ψ'_1 one finds in the linear approximation the following condition for a stationary solution

$$\Psi(u_2 - \mathrm{i}\omega_{\mathrm{lc}} + u_4\Psi^2) = 0. \tag{4.5}$$

This can be considered as a complex "equation of state". Its real part determines the order parameter

$$\Psi = \begin{cases} \pm[(\lambda - \lambda_0)/u_4^{(1)}]^{1/2}, & \lambda \geqslant \lambda_0, \\ 0, & \lambda \leqslant \lambda_0, \end{cases} \tag{4.6}$$

while the imaginary part yields the frequency of the limit cycle

$$\omega_{\mathrm{lc}}^{(0)}(\lambda) = u_2^{(2)} + u_4^{(2)}\Psi^2 = u_2^{(2)} + (u_4^{(2)}/u_4^{(1)})(\lambda - \lambda_0), \quad \lambda \geqslant \lambda_0, \tag{4.7}$$

agreeing with that of the deterministic problem.

Making use of the "equation of state" (4.5) we obtain for the transverse components the contribution to the action as

$$\mathscr{A}\{\tilde{\psi}'_T, \psi'_T\} = \int \mathrm{d}t \int \mathrm{d}^d x \left[\sum_{l=2}^{m} \{-\Gamma|\tilde{\psi}'_l|^2 + (\mathrm{i}/2)[\tilde{\psi}'_l(\dot{\psi}'_l - a\nabla^2\psi'_l) + \mathrm{c.c.}]\} \right], \ \lambda \geqslant \lambda_0, \tag{4.8}$$

where the subscript T stands for any $j \geqslant 2$. The correlation function $\langle \psi'_{T,k,\omega} \bar{\psi}'_{T,k,\omega} \rangle$ will be a similar expression as (4.3) but u_2 will be replaced by zero in it.

Turning to the longitudinal ($j = 1$) component, its equation of motion in the linear approximation reads

$$\dot{\psi}'_1(x,t) = -s^+\psi'_1(x,t) - s^-\bar{\psi}'_1(x,t) + \zeta_1(x,t), \tag{4.9}$$

with

$$s^+ \equiv -a\nabla^2 + u_4\Psi^2 = -a\nabla^2 + (1 + \mathrm{i}u_4^{(2)}/u_4^{(1)})(\lambda - \lambda_0) \tag{4.10}$$

and

$$s^- \equiv u_4\Psi^2 = (1 + \mathrm{i}u_4^{(2)}/u_4^{(1)})(\lambda - \lambda_0). \tag{4.11}$$

The corresponding action is given by

$$\mathscr{A}\{\tilde{\psi}'_1, \psi'_1\} = \int \mathrm{d}t \int \mathrm{d}^d x\{-\Gamma|\tilde{\psi}'_1|^2 + (\mathrm{i}/2)[\tilde{\psi}'_1(\dot{\psi}'_1 + s^+\psi'_1 + s^-\bar{\psi}'_1) + \mathrm{c.c.}]\}. \tag{4.12}$$

The total action is obviously the sum of (4.12) and (4.8). The non-vanishing two-point correlation functions of ψ'_1, evaluated in the Fourier space, are obtained from (4.12) as

$$\langle \psi'_{1,k,\omega}\overline{\psi'_{1,k,\omega}}\rangle = 4\Gamma(\omega^2 + |s_k^+|^2 + |s_k^-|^2 + 2\omega s_k^{+(2)})/N_k, \tag{4.13}$$

$$\langle \psi'_{1,k,\omega}\psi'_{1,-k,-\omega}\rangle = -8\Gamma s_k^- \overline{s_k^+}/N_k, \tag{4.14}$$

where the common denominator is

$$N_k = (\omega^2 - |s_k^+|^2 + |s_k^-|^2)^2 + 4\omega^2 s_k^{+(1)2} \tag{4.15}$$

and s_k^+, s_k^- denote the spatial Fourier transform of s^+ and s^-, given by (4.10) and (4.11), respectively. The fact that in the post-bifurcational region $\psi'_{1,k,\omega}$ and $\psi'_{1,-k,-\omega}$ become correlated is a manifestation of the symmetry breaking.

At the bifurcation point, $\lambda = \lambda_0$, the correlation function $\langle \psi'_{1,k,\omega}\psi'_{1,-k,-\omega}\rangle$ must vanish and $\langle \psi'_{1,k,\omega}\overline{\psi'_{1,k,\omega}}\rangle$ and $\langle \psi'_{T,k,\omega}\overline{\psi'_{T,k,\omega}}\rangle$ become identical with (4.3) with $u_2 = 0$.

It is worth noting that the correlation functions determined in the Gaussian approximation can also be considered as the zeroth order propagators in a systematic perturbation expansion of the path probability W. In order to make the result more complete we give the expressions of some additional correlation functions in the appendix B where also the temporal behaviour of the correlation functions will be discussed.

From (4.13) and (4.14) we obtain the equal-time correlation functions of the asymptotic state

$$\langle \psi'_{1,k}\overline{\psi'_{1,k}}\rangle \equiv C_{\mathrm{L}}^{(0)}(k) = \frac{2\Gamma}{k^2}\frac{(k\xi)^4 + (k\xi)^2\sigma/|a|^2 + (1+\kappa^2)/(4|a|^2)}{((k\xi)^2 + 1/2)((k\xi)^2 + \sigma/|a|^2)}, \tag{4.16}$$

$$\langle \psi'_{1,k}\psi'_{1,-k}\rangle \equiv D_{\mathrm{L}}^{(0)}(k) = -\frac{\Gamma}{k^2}\frac{(k\xi)^2[\sigma + \mathrm{i}(\kappa - a^{(2)})] + (1+\kappa^2)/2}{((k\xi)^2 + 1/2)((k\xi)^2 + \sigma/|a|^2)|a|^2}, \tag{4.17}$$

where

$$\kappa \equiv u_4^{(2)}/u_4^{(1)}, \tag{4.18}$$

$$\sigma \equiv 1 + a^{(2)}\kappa, \tag{4.19}$$

and a correlation length, ξ, has been introduced

$$\xi^{-2} = 2u_4^{(1)}\Psi^2 = 2(\lambda - \lambda_0), \quad \lambda > \lambda_0. \tag{4.20}$$

It is to be noted that in the special case specified by (3.1) the right-hand side of eqs. (4.14) and (4.17) are real. The property that this type of correlation function is complex reflects an essential feature of the model as will be seen in section 5.

Finally, we should emphasize that (4.16) is meaningful only if its denominator is positive which is fulfilled for any k only if $\sigma > 0$. The singular behaviour of (4.16), (4.17) at $\sigma = 0$ indicates that an instability develops at this point the onset of which will be discussed in section 7. In sections 5 and 6 we shall assume σ to be of order unity.

5. Stationary distribution in the Gaussian approximation

In this section it is convenient to use the Fourier components of the order parameter field, $\phi_{j,k}(t)$ and $\psi'_{j,k}(t)$, for $\lambda < \lambda_0$ and $\lambda > \lambda_0$, respectively. In order to calculate the stationary distribution, reached for $t \to \infty$, we consider the path probability $P\{\phi\} \propto \exp(-\int L\, d\tau)$[23-25] which can be obtained by integrating $W\{\tilde{\phi}, \phi\}$, given by (4.1), over $\tilde{\phi}$. In the Gaussian approximation we can treat each component separately. Let us take first $\lambda > \lambda_0$ and start with the longitudinal component. The Lagrangian associated with the coupled stochastic equation (4.9) is given by

$$L(\dot{\psi}'_{1,k}(\tau), \psi'_{1,k}(\tau)) = (4\Gamma)^{-1} \sum_k |\dot{\psi}'_{1,k} + s_k^+ \psi'_{1,k} + s_k^- \overline{\psi'_{1,-k}}|^2, \tag{5.1}$$

where s_k^+ and s_k^- denote the Fourier transforms of s^+ and s^- ((4.10) and (4.11)), respectively, as in the previous section. Graham pointed out[26] that for a linear process of a random variable q, the stationary probability distribution $P_{st}(q)$ can be expressed as $\exp(-\int_{-\infty}^{t} L\, d\tau)$ where the Lagrangian $L(\dot{q}, q)$ is to be integrated along the most probable path with boundary condition $q(\tau = t) = q$. Applying this procedure in our case, where $\psi'_{1,k}$ and $\overline{\psi'_{1,k}}$ are considered as the basic variables we obtain for the Euler–Lagrange equation determining the most probable path

$$\ddot{\psi}'_{1,k}(\tau) + (s_k^+ - \overline{s_k^+})\dot{\psi}'_{1,k}(\tau) - (|s_k^+|^2 + |s_k^-|^2)\psi'_{1,k}(\tau) - 2s_k^- \overline{s_k^+ \psi'_{1,-k}}(\tau) = 0. \tag{5.2}$$

with the boundary condition $\psi'_{1,k}(\tau = t) = \psi'_{1,k}$. Since the stationary distribution must not depend on the initial values at $\tau = -\infty$, it is sufficient to consider instead of the solution of the second order equation (5.2) those of the equations

$$\dot{\psi}'_{1,k} = -s_k^+\psi'_{1,k} - s_k^-\overline{\psi'_{1,-k}}, \tag{5.3}$$

$$\dot{\psi}'_{1,k} = \overline{s_k^+}\psi'_{1,k} + (s_k^-\overline{s_k^+}/s_k^+)\overline{\psi'_{1,-k}}. \tag{5.4}$$

It can be easily checked that the solutions of (5.3) and (5.4) are also solutions of (5.2). The solution of (5.3), however, gives a vanishing Lagrangian which cannot characterize the most probable path. Furthermore, the Lagrantian (5.1) along the most probable path specified by (5.4) turns out to be a "time" derivative:

$$L(\dot{\psi}'_{1,k}(\tau), \psi'_{1,k}(\tau)) = \frac{1}{\Gamma}\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau}, \tag{5.5}$$

where

$$\Phi_1 = \frac{1}{2}\sum_k \left\{ s_k^{+(1)}|\psi'_{1,k}|^2 + \left(\frac{s_k^{+(1)}\overline{s_k^-}}{2\overline{s_k^+}}\psi'_{1,k}\psi'_{1,-k} + \text{c.c.}\right)\right\} \tag{5.6}$$

and Φ_1 is considered as a function of the variables, $\psi'_{1,k}$ and $\overline{\psi'_{1,k}}$. Consequently, the stationary distribution associated with eq. (4.9) is obtained as

$$P_{\text{st}}\{\psi'_{1,k}\} \propto \exp(-\Phi_1/\Gamma). \tag{5.7}$$

It is to be noted that in the special case (3.1) we obtain $\Phi_1 = \Phi_1^{\text{GL}}$ in accordance with the results of section 3 where Φ_1^{GL} is given by (3.15). In the general case, however, additional terms appear and one can rearrange (5.6) as

$$\Phi_1 = \Phi_1^{\text{GL}} + \sum_k \left\{\frac{\mathrm{i}(a^{(2)}u_4^{(1)} - u_4^{(2)})k^2\Psi^2}{4(\bar{a}k^2 + \bar{u}_4\Psi^2)}\psi'_{1,k}\psi'_{1,-k} + \text{c.c.}\right\}. \tag{5.8}$$

Using the same method we obtain for the stationary distribution of the transverse components for $\lambda > \lambda_0$

$$P_{\text{st}}\{\psi'_{T,k}\} \propto \exp\left\{-(2\Gamma)^{-1}\sum_k\sum_{j=2}^m k^2|\psi'_{j,k}|^2\right\} \tag{5.9}$$

and for the stationary distribution in the pre-bifurcational region, $\lambda < \lambda_0$, for the field $\phi_{j,k}$

$$P_{\text{st}}\{\phi_{j,k}\} \propto \exp\{-(2\Gamma)^{-1}\sum_{k,j}(k^2 + u_2^{(1)})|\phi_{j,k}|^2\}. \tag{5.10}$$

Both are expressions of the usual Ginzburg–Landau form in the Gaussian approximation. The absence of the constant term in the coefficient of $|\psi'_{j,k}|^2$ in (5.9) is a manifestation of Goldstone's theorem similarly as at equilibrium phase transitions.

Let us turn to a discussion of the stationary distribution for the longitudinal component. The most striking property of (5.8) is its deviation from a Ginzburg–Landau functional due to the fact that the coefficient of $\psi'_{1,k}\psi'_{1,-k}$ cannot be considered as a constant. The strange behaviour of the stationary distribution can be traced back to the fact that the equation of motion in the generalized TDGL model does not have a potential i.e. it is not derivable from the stationary distribution itself. The deviation from the Ginzburg–Landau form makes the behaviour of the system in the asymptotic state much reacher than otherwise. First, it will be seen (section 6) that there is a correction to the frequency of the limit cycle coming from the imaginary part of $D_L^{(0)}$ (see (6.6)). No Ginzburg–Landau form would lead to such a correction since $D_L^{(0)}$ is necessarily real in this case. Furthermore, the instability mentioned at the end of section 4 (and discussed in detail in section 7) can be considered as a consequence of the deviation from the Ginzburg–Landau form, too. Finally, we mention that in the special case (3.1) the anomalous term of (5.8) vanishes in accordance with the general results of section 3.

6. Corrections to the amplitude and frequency of the limit cycle

By means of the correlation functions obtained in the previous sections one can calculate corrections to the leading order terms. Here we shall be interested in the correction to the complex "equation of state" derived in section 4.

We start by giving the equation of motion for the longitudinal field $\psi'_{1,k}$ including the non-linear terms

$$\begin{aligned}\dot{\psi}'_{1,k} = &-(u_2 - \mathrm{i}\omega_{\mathrm{lc}} + u_4\Psi^2)\Psi\delta_{k,0} - (u_2 - \mathrm{i}\omega_{\mathrm{lc}} + 2u_4\Psi^2 + ak^2)\psi'_{1,k} \\ &- u_4\Psi^2\overline{\psi'_{1,-k}} - u_4\Psi\Big(\sum_{j=1}^{m}\sum_{k}\psi'_{j,k'}\overline{\psi'_{j,k'-k}} + \sum_{k'}\psi'_{1,k}\overline{\psi'_{1,k'-k}} \\ &+ \sum_{k'}\psi'_{1,k'}\psi'_{1,k-k'}\Big) - u_4\sum_{j,k',k''}\psi'_{1,k'}\psi'_{j,k''}\overline{\psi'_{j,k'+k''-k}} + \zeta_{1,k}. \qquad (6.1)\end{aligned}$$

Then we take the average of (6.1) in the asymptotic state. According to the definitions (3.8), (3.12) $\langle\psi'_{1,k}\rangle = 0$ which in the $k = 0$ case provides the complex "equation of state". In calculating the right-hand side of eq. (6.1) two and three point equal-time correlation functions appear. We evaluate them by using the Gaussian approximation for the stationary distribution given in the previous section; the three point term vanishes and we obtain

$$0 = \Psi\left[u_2 - \mathrm{i}\omega_{\mathrm{lc}} + u_4\Psi^2 + u_4 \int_k \{(m-1)C_{\mathrm{T}}^{(0)}(k) + 2C_{\mathrm{L}}^{(0)}(k) + D_{\mathrm{L}}^{(0)}(k)\}\mathrm{d}^d k/(2\pi)^d\right], \tag{6.2}$$

where the correlation functions $C_{\mathrm{L}}^{(0)}$ and $D_{\mathrm{L}}^{(0)}$ are determined by (4.16) and (4.17), respectively, and the transverse one by

$$C_{\mathrm{T}}^{(0)}(k) = \Gamma 2/k^2 \tag{6.3}$$

as it follows from the distribution (5.9). It is convenient to rearrange (6.2) by adding and substracting the quantity

$$\Gamma 2 u_4 \Psi(m+1) \int_k k^{-2}\, \mathrm{d}^d k/(2\pi)^d \equiv u_4 B\Psi, \tag{6.4}$$

where

$$B \equiv \Gamma(m+1)K_d 2\Lambda^{d-2}/(d-2) \tag{6.5}$$

and $K_d(2\pi)^d$ denotes the area of the d-dimensional unit sphere, resulting finally in the "equation of state"

$$0 = \Psi\left[u_2 - \mathrm{i}\omega_{\mathrm{lc}} + u_4 B + u_4\Psi^2 + u_4 \int_k \{2(C_{\mathrm{L}}^{(0)}(k) - \Gamma 2k^{-2}) + D_{\mathrm{L}}^{(0)}(k)\}\, \mathrm{d}^d k/(2\pi)^d\right]. \tag{6.6}$$

Since the correction to the deterministic result is assumed to be small one can use approximately $\lambda_c = \lambda_0$ in the term under the integral in (6.6). Consequently, this term vanishes at $\lambda = \lambda_c$, so does Ψ^2, and one immediately obtains for the critical value of the control parameter and for the frequency of the limit cycle at the bifurcation point

$$\lambda_c = \lambda_0 + u_4^{(1)} B \tag{6.7}$$

and

$$\omega_{\mathrm{lc}}(\lambda_c) = u_2^{(2)} + u_4^{(2)} B, \tag{6.8}$$

respectively.

Next, we turn to the evaluation of the corrections in the post-bifurcational region, $\lambda > \lambda_c$. Using the expressions (4.16) and (4.17), introducing ξ through (4.20), and making use of the fact that around the bifurcation point $\Lambda\xi \gg 1$, we can replace the integral over k by a dimensionless one running to infinity (and being convergent for $2 < d < 4$). So we arrive at the equation for $\lambda > \lambda_c$

$$0 = u_4 - i\omega_{lc}(\lambda) + u_4 B + u_4 \Psi^2 - \Gamma u_4 \xi^{2-d} K_d \int_0^\infty \frac{A_1/2 + x^2 A_2}{(x^2 + \frac{1}{2})(x^2 + \sigma/|a|^2)} x^{d-3}\,dx, \tag{6.9}$$

with

$$A_1 \equiv [-|a|^2 + \sigma(4|a|^2 - 2 - \sigma)]/(a^{(2)}|a|)^2, \tag{6.10}$$

$$A_2 \equiv (2|a|^2 + \sigma)/|a|^2 + i(\sigma - |a|^2)/(a^{(2)}|a|)^2, \tag{6.11}$$

where σ is the combination defined by (4.18) and (4.19).

The real part of (6.9) determines the order parameter Ψ. We obtain

$$u_4^{(1)}\Psi^2 = \lambda - \lambda_c + u_4^{(1)}\xi^{2-d} K_d \Gamma \frac{2^{(2-d)/2} B((d-2)/2, (4-d)/2)}{1 - 2\sigma/|a|^2} \times \left[2 - \frac{2\sigma}{|a|^2} - \left(\frac{2\sigma}{|a|^2}\right)^{(d-4)/2} \frac{1}{a^{(2)2}}\left(1 - \frac{\sigma}{|a|^2}\left(4 - 5\sigma + \frac{2\sigma^2}{|a|^2}\right)\right)\right], \tag{6.12}$$

where λ_c has been given by (6.7) and B stands for the beta function. Similarly as in the case of equilibrium phase transitions one finds a correction which is proportional to ξ^{2-d}. This reflects that the expansion goes in powers of $u_4^{(1)}\xi^{4-d}$. Consequently, one obtains a small correction to the deterministic result if

$$u_4^{(1)}\xi^{4-d} \ll 1. \tag{6.13}$$

In the immediate vicinity of the bifurcation point this criterion, of course, cannot be fulfilled. Note, however, that at non-equilibrium phase transitions the critical region is extremely narrow and one can hope at best to reach the region where the first corrections, though small, become detectable.

The frequency of the limit cycle is specified by the imaginary part of (6.9). Comparing this equation with (6.12) we obtain

$$\omega_{lc}(\lambda) = \omega_{lc}(\lambda_c) + \kappa(\lambda - \lambda_c) - u_4^{(1)}(\kappa - a^{(2)})\xi^{2-d}\Gamma \frac{1+\kappa^2}{1 + a^{(2)2}} K_d \times \frac{2^{(2-d)/2} B((d-2)/2, (4-d)/2)}{1 - 2\sigma/|a|^2}\left(1 - \left(\frac{2\sigma}{|a|^2}\right)^{(d-2)/2}\right), \tag{6.14}$$

where κ has been defined by (4.18). This shows that the correction to the deterministic result $\omega_{lc}^{(0)}(\lambda) = (u_2^{(2)} + \kappa(\lambda - \lambda_0)) = \omega_{lc}(\lambda_c) + \kappa(\lambda - \lambda_c)$ is proportional to ξ^{2-d} and suggests that the expansion of $\omega_{lc}(\lambda)$ would go in powers of $u_4^{(1)}(\kappa - a^{(2)})\xi^{4-d}$. A small correction will be found if

$$u_4^{(1)}(\kappa - a^{(2)})\xi^{4-d} \ll 1. \tag{6.15}$$

It is worth mentioning here that in the special case of $\kappa = a^{(2)}$ the correction in (6.12) goes over to that appearing in this context at equilibrium phase transitions. At the same time no correction is obtained to the deterministic expression of the frequency of the limit cycle. All these are in full agreement with the general results obtained for the special case (section 3). If $\kappa \neq a^{(2)}$ but $|\kappa - a^{(2)}| \ll 1$ the correction to the frequency of the limit cycle is much less important than the correction to its amplitude. In principle the opposite case $|\kappa - a^{(2)}| \gg 1$ might also happen which implies that the quantity in which a deviation can be detected first is the frequency of the limit cycle.

Concluding this section it is to be noted that for finding a small departure in the asymptotic behaviour of the system from the predictions obtained in a Gaussian approximation not only the condition (6.13) (a straightforward generalization of the Ginzburg criterion of critical statics[3,27]) is required but also (6.15) which is a novel feature of the generalized TDGL model.

7. The onset of an instability induced by phase fluctuations

In this section we are going to investigate how the instability at $\sigma = 0$, mentioned at the end of section 5, develops, i.e. we are interested in the regime where σ is a small positive number. It follows from (4.16) and (4.17) that the equal time correlation functions $C_{\mathrm{L}}^{(0)}(k)$, $D_{\mathrm{L}}^{(0)}(k)$ at $\sigma = 0$ diverge as k^{-4} for $k \to 0$ which would mean an infinite local fluctuation in $\psi_1'(x)$ for any dimensions below four. Considering the stationary distribution (5.6), (5.7) one observes that the determinant of the quadratic form associated with a given value of k becomes proportional to k^4 at $\sigma = 0$.

The stationary distribution, (5.9), of the transverse components does not depend on σ, therefore it is sufficient to concentrate on the $j = 1$ component when treating the problem of the onset of this instability. Since the amplitude of the limit cycle is chosen to be real it follows from (3.12) that the imaginary part of ψ_1' can be written as $\psi_1'^{(2)}(x, t) = \Psi\Theta(x, t)$ supposing $\Theta(x, t)$, the phase of the complex field $\psi_1 = \psi_1' + \Psi$, to be small. Consequently, they are just the phase fluctuations which are described by $\psi_1'^{(2)}$. The Fourier transform of $\psi_1'^{(2)}$ reads

$$(\psi_1'^{(2)})_{k,\omega} = (\psi_{1,k,\omega}' - \overline{\psi_{1,-k,-\omega}'})/(2\mathrm{i}). \tag{7.1}$$

First, we investigate its equal time correlation function. We are interested in the long wavelength fluctuations of $\psi_1'^{(2)}$ at a given value of $\lambda (\lambda > \lambda_0)$, therefore the region $k\xi \ll 1$ will be discussed (ξ has been defined by (4.20)). In this region we obtain from (4.16), (4.17)

$$\langle (\psi_1'^{(2)})_k (\psi_1'^{(2)})_{-k} \rangle = \frac{\Gamma(|u_4|/u_4^{(1)})^2}{k^4 \xi^2 (|a|^2 + (kl_c)^{-2})}, \tag{7.2}$$

where a new correlation length

$$l_c = \xi\sigma^{-1/2} \tag{7.3}$$

has been introduced which goes to infinity as $\sigma \to 0$. As for the other correlation functions, we find that $\langle(\psi_1'^{(1)})_k(\psi_1'^{(2)})_{-k}\rangle$ tends to a constant as $k \to 0$ for $\sigma > 0$ and becomes proportional to k^{-2} at $\sigma = 0$, while $\langle(\psi_1'^{(1)})_k(\psi_1'^{(1)})_{-k}\rangle$ is constant for both $\sigma > 0$ and $\sigma = 0$ when $k \to 0$, i.e. these correlation functions do not describe diverging local fluctuations.

Turning to the time dependent correlation function of $\psi_1'^{(2)}$, we obtain from (4.13)–(4.15) in the small frequency limit ($\omega\xi^2 \ll 1$) a Lorentzian form

$$\langle(\psi_1'^{(2)})_{k,\omega}(\psi_1'^{(2)})_{-k,-\omega}\rangle = \frac{2\Gamma(|u_4|/u_4^{(1)})^2}{\omega^2 + \gamma^2(k)}, \tag{7.4}$$

with

$$\gamma(k) = \sigma k^2 + |a|^2 k^4 \xi^2 = k^4\xi^2(|a|^2 + (kl_c)^{-2}). \tag{7.5}$$

The expression (7.4) implies that the relaxation rate of the phase is $\gamma(k)$.

At this point we can make contact with the work by Kuramoto and coworkers[7–10]) who treated the deterministic generalized TDGL model. It was first pointed out by them that the limit cycle state becomes unstable when $\sigma \to 0$ (the control parameter σ corresponds to ν in their notation). As a matter of fact the phase relaxation rate obtained here in the Gaussian approximation for the correlation function (7.4) agrees with that of Kuramoto and Yamada[8,10]). Treating the fluctuations in a more accurate way, however, the agreement with the deterministic result is expected to be lost. As for the region beyond this instability Kuramoto and Yamada found[8–10]) in the deterministic model that a chaotic behaviour develops which comes from the irregular motion of the phase of the local oscillations. They called this type of behaviour a phase chaos.

Having calculated the corrections to the deterministic expressions of the amplitude and the frequency of the limit cycle (section 6) we can gain some information on how the inclusion of the interaction of the fluctuations of ψ_1' may modify the picture above. We consider only the case $2 < d < 4$.

First we note that the correction to the frequency of the limit cycle remains finite even at $\sigma = 0$ (see (6.14)). This is not the case, however, concerning the amplitude of the limit cycle. As far as σ is close to $|a|^2$ (i.e. to the special case) the correction in (6.12) is positive but when σ is reducing we arrive at a point, $\sigma = \tilde{\sigma}$, where the correction vanishes. For $\sigma < \tilde{\sigma}$ the correction becomes negative which results in a decrease in the amplitude of the limit cycle. To be explicit we give $\tilde{\sigma}$ for $a^{(2)} \gg 1$ when its expression is simple

$$\tilde{\sigma} = 2^{(d-6)/(4-d)}(a^{(2)})^{(4-2d)/(4-d)}. \tag{7.6}$$

As the simplest possibility let us assume that this tendency does not break down till the amplitude of the limit cycle vanishes which then defines a new instability point. This point is specified by a critical value $\sigma = \sigma_c$. Although we cannot calculate the shift, σ_c, exactly since the validity of our procedure is restricted to small corrections, we can estimate it by equating the two terms on the right-hand side of (6.12) and keeping terms of leading order in $u_4^{(1)}$ only

$$\sigma_c = \frac{|a|^2}{\lambda - \lambda_0} \Gamma^{2/(4-d)} \left[K_d B\left(\frac{d-2}{2}, \frac{4-d}{2}\right) \frac{u_4^{(1)}}{2a^{(2)}} \right]^{2/(4-d)} . \tag{7.7}$$

This shows that external fluctuations make the region of the limit cycle state smaller, and the strength of the noise correlation, Γ, is a measure of the shift. Furthermore, the instability at σ_c differs qualitatively, in this picture, from that described by Kuramoto[10]) in the deterministic model since in the present case the amplitude of the limit cycle vanishes at σ_c and, consequently, the phase loses its meaning.

It is an open question what type of behaviour will be found for $\sigma < \sigma_c$ in the noisy system. One cannot exclude the possibility that not in the immediate vicinity of σ_c a phase chaos could be found by introducing the concept of a kind of local value of the limit cycle amplitude.

Appendix A

We mention here a possible generalization of the model. Considering the equation of motion

$$\begin{aligned} \dot{\phi}_j(x, t) &= -\left(-a\nabla^2 + u_2 + u_{2\sigma}\left(\sum_{l=1}^{m} |\phi_l|^2\right)^{\sigma-1}\right)\phi_j(x, t) + \zeta_j(x, t), \\ u_2^{(1)} &= \lambda_0 - \lambda, \quad u_{2\sigma}^{(1)} > 0, \end{aligned} \tag{A.1}$$

instead of (2.1) means that the situation around $\lambda = \lambda_0$ is analogous to that of a critical point of order σ ($\sigma = 2$ ordinary critical point, $\sigma = 3$ tricritical, $\sigma = 4$ fourth order critical point). Repeating the calculation in the Gaussian approximation we find the only modification that the complex "equation of state" now reads

$$\Psi(u_2 - i\omega_{lc} + u_{2\sigma}\Psi^{2(\sigma-1)}) = 0. \tag{A.2}$$

Consequently, for $\lambda \geq \lambda_0$

$$\Psi = \pm[(\lambda - \lambda_0)/u_{2\sigma}^{(1)}]^{1/[2(\sigma-1)]}, \tag{A.3}$$

where $1/[2(\sigma - 1)]$ coincides with the exponent β of a critical point of order

σ^{27}), and

$$\omega_{lc}^{(0)}(\lambda) = u_2^{(2)} + u_{2\sigma}^{(2)}\Psi^{2(\sigma-1)} = u_2^{(2)} + (u_{2\sigma}^{(2)}/u_{2\sigma}^{(1)})(\lambda - \lambda_0), \tag{A.4}$$

i.e. $\omega_{lc}^{(0)}$ depends on λ linearly for any σ. Furthermore s^+ and s^- are now to be replaced by

$$s^+ = (-a\nabla^2 + (\sigma - 1)u_{2\sigma}\Psi^{2(\sigma-1)}), \tag{A.5}$$

$$s^- = (\sigma - 1)u_{2\sigma}\Psi^{2(\sigma-1)}. \tag{A.6}$$

The correlation function, and the stationary distribution in the pre-bifurcational region, just as the transverse correlation function and the corresponding stationary distribution in the post-bifurcational region are given by the same expressions (by (4.3), (5.10), (6.3) and (5.9), respectively) for any σ as in the case of the ordinary critical point. The longitudinal correlation and response functions, as well as the stationary distribution of the longitudinal component can be obtained by substituting s_k^+ and s_k^- in (4.13)–(4.15), (B.1)–(B.6) and in (5.6) by the Fourier transform of (A.5) and (A.6), respectively. All the other results and conclusions of the paper can be straightforwardly generalized for the case of an arbitrary σ.

Due to the fact that the parameters are complex, there are other possibilities, too, for finding instabilities analogous to higher order critical points. For the sake of simplicity we consider the case analogous to a tricritical point only. Let us suppose that only the real part of u_4 vanishes and $u_6^{(1)} > 0$. In experimental situations this can be reached by fitting one parameter only which is much easier than eliminating the complex u_4 corresponding to the case discussed above. The important consequence of having $u_4^{(1)} = 0$, $u_4^{(2)} \neq 0$ lies in a qualitative change in the frequency of the limit cycle. It follows from the "equation of state"

$$\Psi(u_2 - i\omega_{lc} + u_4^{(2)}\Psi^2 + u_6\Psi^4) = 0 \tag{A.7}$$

that near λ_0

$$\omega_{lc}(\lambda) = u_2^{(2)} + u_4^{(2)}/(u_6^{(1)})^{1/2}(\lambda - \lambda_0)^{1/2} \tag{A.8}$$

characterized by a stronger dependence on λ than the usual linear one (4.7). Therefore one can hope that instabilities of this type are experimentally observable in systems exhibiting limit cycle behaviour.

Appendix B

First, we give the two-point correlations of the real and imaginary parts of $\psi'_{1,k,\omega}$ for the post-bifurcational region in the Gaussian approximation. From

(4.12) it follows that

$$\langle \psi_{1,k,\omega}'^{(1)2} \rangle = \langle \psi_{1,k,\omega}'^{(2)2} \rangle = 2\Gamma(\omega^2 + |s_k^+|^2 + |s_k^-|^2 + 2\omega s_k^{+(2)})/N_k, \tag{B.1}$$

$$\langle \psi_{1,k,\omega}'^{(1)} \psi_{1,-k,-\omega}'^{(1)} \rangle = -\langle \psi_{1,k,\omega}'^{(2)} \psi_{1,-k,-\omega}'^{(2)} \rangle = -4\Gamma(s_k^- \overline{s_k^+})^{(1)}/N_k, \tag{B.2}$$

$$\langle \psi_{1,k,\omega}'^{(1)} \psi_{1,-k,-\omega}'^{(2)} \rangle = \langle \psi_{1,k,\omega}'^{(2)} \psi_{1,-k,-\omega}'^{(1)} \rangle = -4\Gamma(s_k^- \overline{s_k^+})^{(2)}/N_k, \tag{B.3}$$

$$\langle \psi_{1,k,\omega}'^{(1)} \psi_{1,k,\omega}'^{(2)} \rangle = \langle \psi_{1,-k,-\omega}'^{(1)} \psi_{1,-k,-\omega}'^{(2)} \rangle = 0, \tag{B.4}$$

where N_k has been determined by (4.15), furthermore, s_k^+ and s_k^- denote the Fourier transforms of (4.10) and (4.11), respectively.

Next, we turn to the two-point correlations between ψ_1' and the response field, $\tilde{\psi}_1'$, in the post-bifurcational region. From (4.12) we obtain

$$\langle \psi_{1,k,\omega}' \overline{\tilde{\psi}}_{1,k,\omega}' \rangle = \frac{-2\mathrm{i}s_k^-}{|s_k^+|^2 - |s_k^-|^2 - \omega^2 - 2\mathrm{i}\omega s_k^{+(1)}}, \tag{B.5}$$

$$\langle \psi_{1,k,\omega}' \tilde{\psi}_{1,-k,-\omega}' \rangle = \frac{2\mathrm{i}(-\mathrm{i}\omega + \overline{s_k^+})}{|s_k^+|^2 - |s_k^-|^2 - \omega^2 - 2\mathrm{i}\omega s_k^{+(1)}}. \tag{B.6}$$

The autocorrelation function of ψ_1' vanishes.

In the case of the usual TDGL model the correlation function of the order parameter and of the response fields is simply related to the response function of the system[21]). It is natural to ask what is the corresponding relation for the generalized TDGL model. In order to answer this question in general let us consider eq. (2.1) and let us formally introduce an infinitesimal complex external field $h_j(x, t)$ on the right-hand side of the equation of motion of the jth component. After constructing the path probability distribution for this process and calculating the average of $\phi_j(x, t)$ by means of it we obtain

$$\langle \phi_{j,k,\omega} \rangle = G_j^+(k, \omega) h_{j,k,\omega} + G_j^-(k, \omega) \overline{h_{j,-k,-\omega}}, \tag{B.7}$$

where $h_{j,k,\omega}$ denote the Fourier transform of $h_j(x, t)$ and

$$G_j^+(k, \omega) = \left(\frac{-\mathrm{i}}{2}\right) \langle \phi_{j,k,\omega} \tilde{\phi}_{j,-k,-\omega} \rangle, \tag{B.8}$$

$$G_j^-(k, \omega) = \left(\frac{-\mathrm{i}}{2}\right) \langle \phi_{j,k,\omega} \overline{\tilde{\phi}_{j,k,\omega}} \rangle. \tag{B.9}$$

Next we discuss the Gaussian fluctuations of fields in the wave-number, time representation. For the longitudinal correlation functions we obtain for $t \geqslant 0 (\lambda \geqslant \lambda_0)$

$$\begin{aligned} C_{\mathrm{L}}^{(0)}(k, t) &\equiv \langle \overline{\psi_{1,k}'(t)} \psi_{1,k}'(0) \rangle \\ &= \exp(-s_k^{+(1)} t)[C_{\mathrm{L}}^{(0)}(k) \operatorname{ch} \alpha_k t - (D_{\mathrm{L}}^{(0)}(k) \overline{s_k^-} - \mathrm{i} C_{\mathrm{L}}^{(0)}(k) s_k^{+(2)})(\operatorname{sh} \alpha_k t)/\alpha_k], \end{aligned} \tag{B.10}$$

$$D_{\mathrm{L}}^{(0)}(k,t) \equiv \langle \psi'_{1,-k}(t)\psi'_{1,k}(0)\rangle$$

$$= \exp(-s_k^{+(1)}t)[D_{\mathrm{L}}^{(0)}(k) \operatorname{ch} \alpha_k t - (C_{\mathrm{L}}^{(0)}(k)s_k^- + \mathrm{i}D_{\mathrm{L}}^{(0)}(k)s_k^{+(2)})(\operatorname{sh} \alpha_k t)/\alpha_k], \tag{B.11}$$

where

$$\alpha_k = [|s_k^-|^2 - s_k^{+(2)2}]^{1/2}, \tag{B.12}$$

and the equal-time correlation functions $C_{\mathrm{L}}^{(0)}(k)$ and $D_{\mathrm{L}}^{(0)}(k)$ have been defined by (4.16) and (4.17), respectively.

It is a peculiar feature of (B.10), (B.11) that it depends on the value of k (and λ) whether $C_{\mathrm{L}}^{(0)}(k,t)$ and $D_{\mathrm{L}}^{(0)}(k,t)$ exhibit temporal oscillations or not. Oscillations appear only if α_k is imaginary, i.e. if

$$k^2 > (\lambda - \lambda_0)(|u_4| - u_4^{(2)})/(u_4^{(1)}a^{(2)}), \quad \lambda \geqslant \lambda_0. \tag{B.13}$$

As for the transverse correlation function it is most illuminating to consider its expression in the (x,t) representation. For a fixed space coordinate the correlation function falls off as a power law in time: in a d-dimensional system for $t \to \infty$, $C_{\mathrm{T}}^{(0)}(x,t) \to t^{1-d/2}$ which is a Goldstone-like singularity. In a higher order approximation also the longitudinal correlation function is expected to decay for large t according to a power law for a fixed value of x. (See for a discussion of these properties in context of equilibrium phase transitions ref. 28.)

Finally we mention, as can be easily checked, that all the correlation functions discussed here and in the main text obey dynamical scaling hypothesis[29,30]) as generalized for hard mode instabilities[15]) with classical exponents: $\eta = 0$, $z = 2$.

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