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DETERMINATION OF FRACTAL DIMENSIONS FOR GEOMETRICAL MULTIFRACTALS

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Two independent approaches, the box counting and the sand box methods are used for the determination of the generalized dimensions (D_q) associated with the geometrical structure of growing deterministic fractals. We find that the multifractal nature of the geometry results in an unusually slow convergence of the numerically calculated D_q's to their true values. Our study demonstrates that the above-mentioned two methods are equivalent only if the sand box method is applied with an averaging over randomly selected centres. In this case the latter approach provides better estimates of the generalized dimensions.

Fractal measures characterized by an infinite hierarchy of exponents have attracted considerable interest recently (for reviews see refs. [1–6]). Objects with such multifractal properties have been shown to be relevant to a variety of physical processes, including turbulence [7, 8], chaos [9], conduction in random resistor networks [10], and growth of aggregates [11–13].

In many cases physical processes generate singular distribution of the corresponding measure on a support which is a fractal. Here we will be concerned with the case when the measure is uniformly distributed on the support, but the geometry of the latter exhibits multifractal behaviour. In fact, the question whether such multifractal structures exist in nature or they can be obtained only by mathematical methods has not been satisfactorily answered yet. A few very recent numerical results suggest that the mass distribution within clusters generated by growth models and laboratory experiments on

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diffusion-limited growth corresponds to a "monofractal" geometry rather than to a fractal measure [14]. On the other hand, in an analysis of sedimentary rock [15] the distribution of mass was described in terms of multifractality and similar results were found for aggregation models from a small cell renormalization group approach [16]. Anomalous behaviour of the number of boxes of size r containing a given number of particles was found in a study of the mass distribution within diffusion-limited aggregates [17]. Recent studies suggest, however, that the cluster sizes used in the numerical investigations were likely to be too small to see the true behaviour [18].

In order to clarify some of the controversial points about geometrical multifractality we shall study deterministic cluster growth models using the definitions and formalism published in ref. [19]. In ref. [19] we argued that anomalously large structures have to be generated to see the asymptotic regime. As an independent and quite surprising result we found that the fractal dimension of the above mentioned deterministic constructions determined from the usual expression $M(R) \sim R^D$ was inconsistent with the value obtained from self-similarity considerations. (Here $M(R)$ is the number of particles within a circle of radius R centered at the origin of the construction, and D is the fractal dimension.)

In general, two basic methods are used to determine fractal dimensions. For growing fractal structures the scaling of the mass within a region of size L is investigated as a function of L. Here we study an extension of this approach which is sometimes called "sand box method". In the case of fractals with a fixed size and infinitely small details, the dimension of the objects is determined from the scaling of the number of non-empty boxes of decreasing size. In this "box counting" method the structure is covered by a grid with a mesh size equal to l and a box or lattice unit is considered as occupied if its intersection with the fractal is larger than zero. In this paper we address the question of equivalence of these approaches in the case of geometrical (often called also mass) multifractals.

Let us define geometrical (mass) multifractality of growing clusters according to ref. [19]. We shall assume that the structure is defined on a lattice and its linear size and mass are L and M_0 respectively. Furthermore, a denotes the diameter of the particles the clusters are made of (the unit of the lattice on which the cluster is growing) and l is the lattice spacing (box size) of the grid which is put onto the cluster to determine its fractal dimension. Then one can define M_i as the mass (the number of particles) of the ith box ($i = 1, 2, \ldots$). Knowing the set of M_i values one can determine the quantity $N(M)$ which is the number of boxes with mass M. Assume that we plot $\ln N(M)$ versus $\ln M/M_0$ for various l. If these histograms fall onto the same universal (size independent) curve after rescaling both coordinates by a factor $\ln(l/L)$ [4], the structure is a geometrical multifractal [19]. Obviously, the above property should hold if

$$M \sim M_0\left(\frac{l}{L}\right)^{\alpha}, \tag{1}$$

and

$$N(\alpha) \sim \left(\frac{l}{L}\right)^{-f(\alpha)}, \tag{2}$$

where α is the mass index,

$$\alpha = \frac{\ln M/M_0}{\ln l/L}, \tag{3}$$

$N(\alpha)$ is the number of boxes with mass index α, and $(l/L) \rightarrow 0$. The D_q generalized dimensions [20] can be obtained from the scaling

$$\sum_i M_i^q \sim M_0^q \left(\frac{l}{L}\right)^{(q-1)D_q} \tag{4}$$

in the limit $l/L \rightarrow 0$.

Before we discuss our results concerning the generalized dimensions of the growing asymmetric Cantor set we briefly review the definitions of the methods which will be used.

Box counting

As it was already mentioned, in this method the object is covered with a lattice of unit size l and the number of non-empty boxes $N(l)$ is determined. The fractal dimension is obtained from $N(\varepsilon) \sim \varepsilon^{-D_0}$ for $\varepsilon \rightarrow 0$, where $\varepsilon = l/L$. Let us introduce the box counting dimension

$$D_0^{\mathrm{bc}}(\varepsilon) = \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon} \tag{5}$$

for *arbitrary* $\varepsilon < 1$. Then we can investigate the conditions under which D_0^{bc} approximates the true D_0 well enough. We shall calculate the quantity D_q^{bc} corresponding to the generalized dimensions D_q from

$$D_q^{\mathrm{bc}}(\varepsilon) = \frac{\ln(\Sigma(M_i/M_0)^q)}{\ln \varepsilon} \frac{1}{q-1}. \tag{6}$$

The definition (4) is recovered in the $\varepsilon \rightarrow 0$ limit.

Sand box method

The main quantity used in this approach is the mass $M(R)$ (the number of particles) within a region of radius R centered on the fractal (typically on its origin). It is widely accepted that for growing fractals in the $R \to \infty$ limit the expression $M(R) \sim R^{D_0}$ determines the fractal dimension, where R is the actual length of the fractal, $R = L$, $M(L) = M_0$. However, as we shall show on a simple example, this statement does not hold, at least, for some geometrical multifractals. In fact, the generalized dimension defined by $M(L) \sim L^D$ exists, but its value is equal to $D_{-\infty}$, different from D_0 if the centre is the origin of the fractal.

If we chose an arbitrary point belonging to the fractal as the centre of the sandbox (instead of its origin), because of eq. (1) the quantity $\ln(M/M(R))$ for $R \ll L$ approaches the local mass index α characteristic for the position of the given centre. Both α and the actual form of the function $\ln M(R)$ versus $\ln R$ depend on the choice for the centre. In order to obtain well defined dimensions independent of the local behaviour, we shall study the *average* value of the masses $M(R)$ and their powers over *randomly distributed* centres on the fractal. Thus, we are interested in the question how $\langle M^q(R) \rangle$ scales with increasing R. Rewriting (4) we find

$$\sum_i \left(\frac{M_i}{M_0}\right)^{q-1} \frac{M_i}{M_0} \sim \left(\frac{l}{L}\right)^{(q-1)D_q} . \tag{7}$$

Since M_i/M_0 can be considered as a probability distribution on an approximating fractal, we get

$$\left\langle \left(\frac{M_i}{M_0}\right)^{q-1} \right\rangle \sim \left(\frac{l}{L}\right)^{(q-1)D_q} , \tag{8}$$

where the average is taken according to the distribution $P_i = M_i/M_0$. When instead of a grid of lattice unit l one uses randomly centred sand boxes of radius $R \ll L$, it is expected that

$$\left\langle \left(\frac{M(R)}{M_0}\right)^{q-1} \right\rangle \sim \left(\frac{R}{L}\right)^{(q-1)D_q} \tag{9}$$

holds if the averaging is made according to the same distribution. This means that the centres on the approximating fractal have to be chosen with a uniform distribution on it. We note that because of (9) $\langle M(R) \rangle \sim R^{D_2}$, and in general $D_2 \leq D_0$. Eq. (9) is a special case of a relation valid for fractal measures and often is used in connection with chaotic attractors [21]; up to our knowledge, however, it has not been applied to growing fractals.

We define the sand box dimension of order q as

$$D_q^{\mathrm{sb}}(R/L) = \frac{\ln\langle [M(R)/M_0]^{q-1} \rangle}{\ln(R/L)} \frac{1}{q-1}, \tag{10}$$

with the appropriate averaging discussed above, for any R between a and L. In the following we shall make use of a deterministic geometrical multifractal to investigate how D_q^{sb} is related to D_q and D_q^{bc}.

Let us define the growing asymmetric Cantor set as shown in fig. 1. In the first step the (seed) structure is made of three particles placed at the first, third and the fourth sites, respectively. In the next step the twice enlarged version of the first configuration is added to the seed between the 9th and 16th sites. After the nth step of the construction the linear size of the structure is 4^n, and it is made of 3^n units. The fractal is obtained in the $n \to \infty$ limit.

The D_q spectrum can be determined for this fractal exactly. The general equation given in ref. [19] for our case can be solved explicitly leading to

$$D_q = \frac{1}{q-1} \left(q + \frac{\ln \frac{-1 + \sqrt{1 + 4(3/4)^q}}{2}}{\ln 2} \right). \tag{11}$$

From here

$$D_0 = \frac{\ln(\sqrt{5}+1)}{\ln 2} - 1 \simeq 0.6942, \qquad D_\infty = \frac{\ln(2/3)}{\ln(1/2)} \simeq 0.5849$$

and (12)

$$D_{-\infty} = \frac{\ln(1/3)}{\ln(1/4)} \simeq 0.7924.$$

It is easy to check that M_0 scales with L according to $D_{-\infty}$.

The results concerning D_q^{bc} and D_q^{sb} were obtained numerically for the asymmetric growing Cantor set. During the application of the *box counting method* we put a grid on the fractal and gradually decreased the grid size from

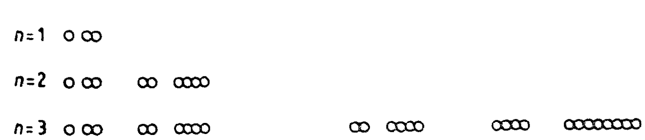


Fig. 1. The first three steps in the construction of the growing asymmetric Cantor set.

the largest possible size L to that of the particles. We determined the number of particles (M_i) in each box by an exact recursion relation. In this way it was possible to study relatively large systems (corresponding to $L = 4^{50}$) using a personal computer. Figs. 2a and b show $D_q^{bc}(\varepsilon)$ as a function of the box size ε. The q values were changed between -8 and 8. The curves we obtained have a region where they approach the value corresponding to the true D_q calculated from (11). The length of this region strongly depends on both q and the size of the object. For larger q values the numerical results fit the exact one better. On the other hand, for $q = -8$ there is only a relatively narrow region (between 2^{60} and 2^{80}), where the box counting method gives reasonable results. Fig. 2b demonstrates that for smaller objects the convergence to the exact values is considerably worse.

The *sand box method* was applied using many randomly chosen centres. In this case the number of particles was determined within a region of length R, where R was incremented from one particle size ($a = 1$) to W, the latter denoting the distance between a given centre and the most distant point belonging to the fractal. To understand the results obtained after averaging over the centres we first studied the $M(R)$ curves for a few fixed centre positions. In fig. 3, we plotted the quantity $M_0/M(R)$ versus L/R for $n = 8$. The small value of n is due to the fact that it was not possible to find a recursion relation for $M(R)$ similar to that we could use for the calculation of the M_i values.

(i) When the centre is at $x = 0$ the plot is made of two parts having different slopes (fig. 3a). On the interval $(2^n, 2^{2n})$ the slope is D_∞ which is just the mass index α_x corresponding to that point of the fractal. On the interval $(2^0, 2^n)$ the slope is trivial (equals to 1). (ii) For the centre position $x = L$ one obtains a stepwise function touching from above a straight line of slope $D_{-\infty}$ (fig. 3b). (iii) For a single randomly selected centre the behaviour is more complex. For example, if the centre is at $x = 3 \times 2^{2(n-1)}$ we obtain a function which on the interval $(2^0, 2^{2(n-1)})$ is similar to the case shown in fig. 3a, while on $(2^{2(n-1)}, 2^{2n})$ the plot becomes parallel to a straight line of slope $D_{-\infty}$ (fig. 3d). In general, for a single randomly selected centre one obtains a behaviour which is a mixture of various regimes.

As expected, averaging over many centres gives well defined scaling. In this case we plotted $\langle (M_0/M(R))^{q-1} \rangle /(q-1)$ versus L/R for various q (fig. 4). Our numerical test suggests that the slope of these curves is in good agreement with the D_q values given by (11). The error is less than $\pm 3\%$ even for $\pm q = 8$ and $n = 8$. Note that for a structure of the same size box counting gives worse results (fig. 2b).

In conclusion, we have demonstrated that for geometrical multifractals the the standard methods of determining fractal dimensions have to be applied

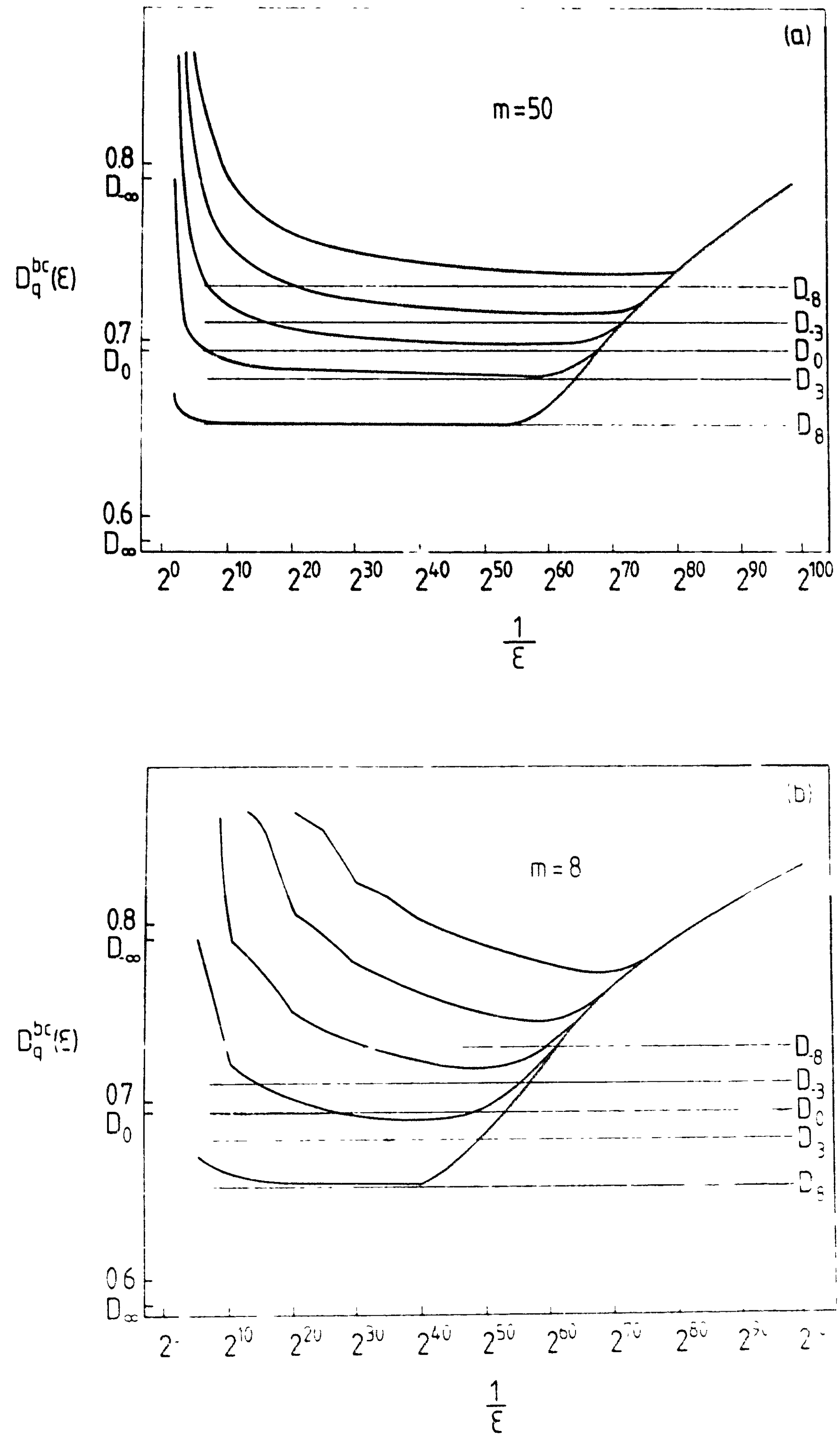


Fig. 2. The generalized dimension $D_q^{bc}(\varepsilon)$ determined from the box counting method versus the box size ε. (a) corresponds to the case $m = 50$ (i.e. to $L = 2^{100}$), while (b) shows the analogous results obtained for $m = 8$ ($L = 2^{16}$).

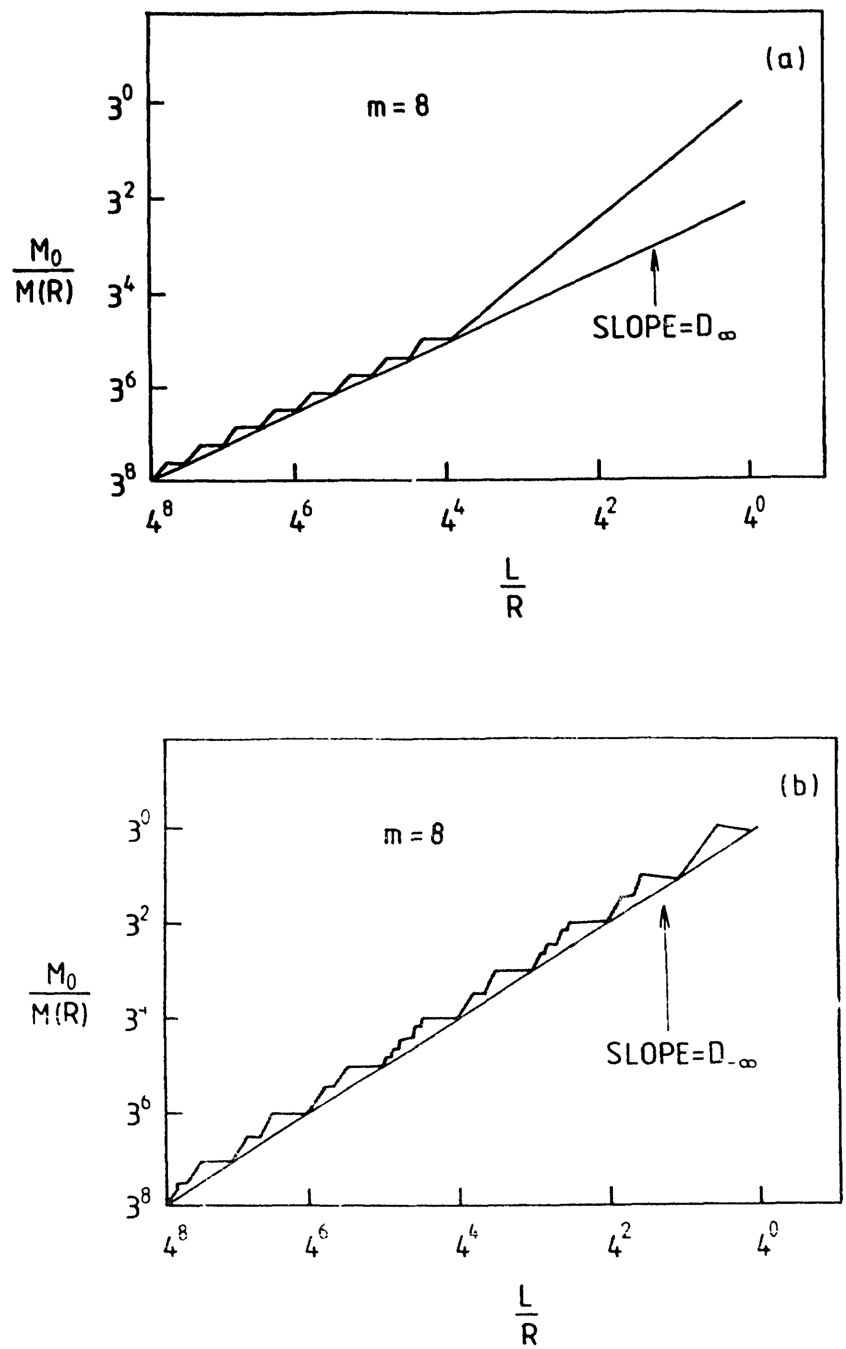
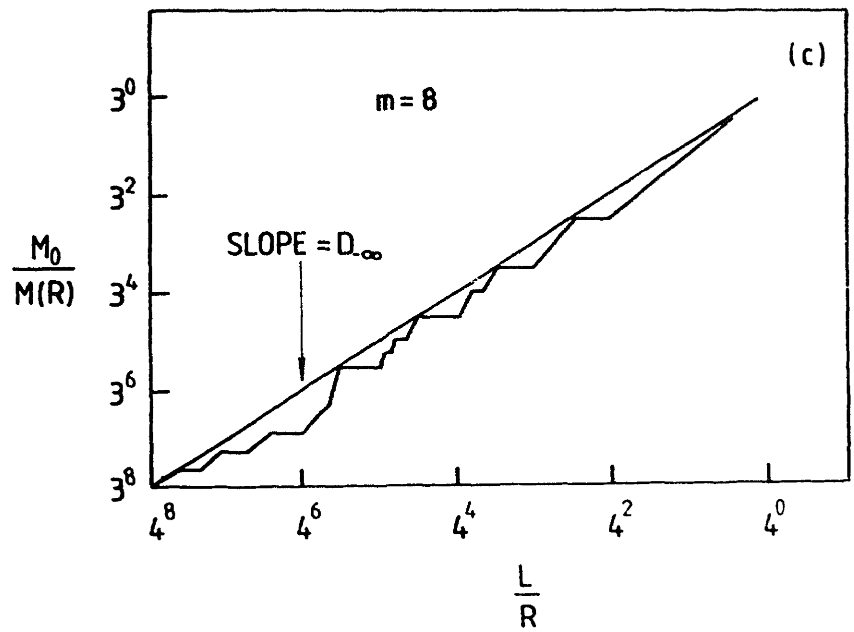


Fig. 3. Results obtained for $M_0/M(R)$ as a function of L/R from the sand box method applied without averaging to the structure corresponding to $n = 8$. Depending on the position of the centre different behaviours could be observed. (a) $x = 0$, (b) $x = L$, (c) $x = 2^8$, and (d) $x = 3 \times 2^{14}$.


(c)
m = 8
3^0
3^2
3^4
3^6
3^8
$\frac{M_0}{M(R)}$
SLOPE = D_∞
4^8
4^6
4^4
4^2
4^0
$\frac{L}{R}$

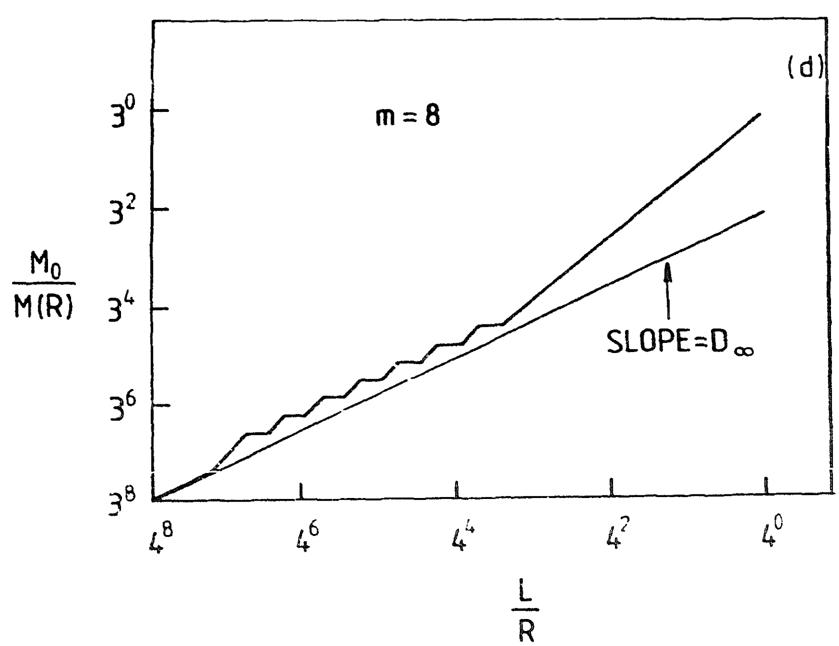
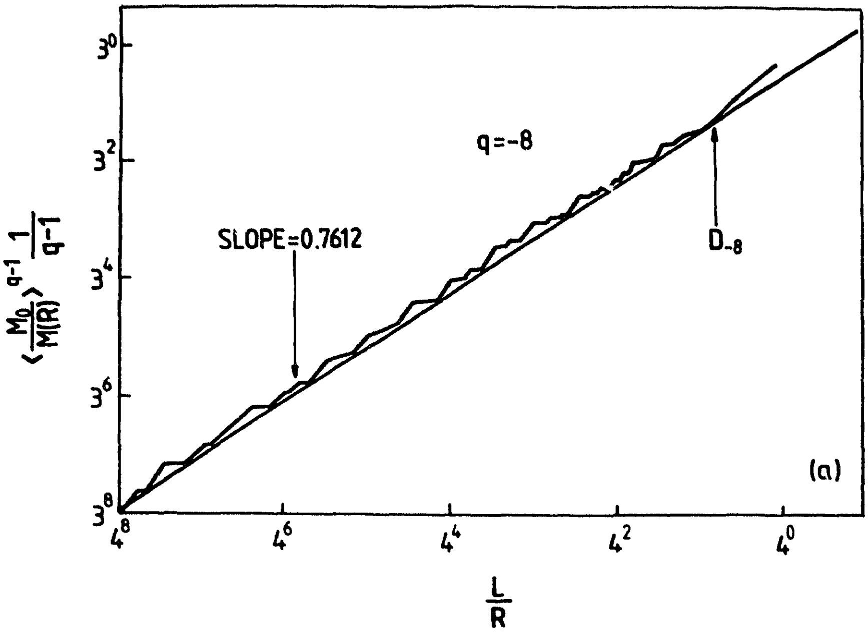

(d)
m = 8
3^0
3^2
3^4
3^6
3^8
$\frac{M_0}{M(R)}$
SLOPE = D_∞
4^8
4^6
4^4
4^2
4^0
$\frac{L}{R}$

Fig. 3 (cont.).



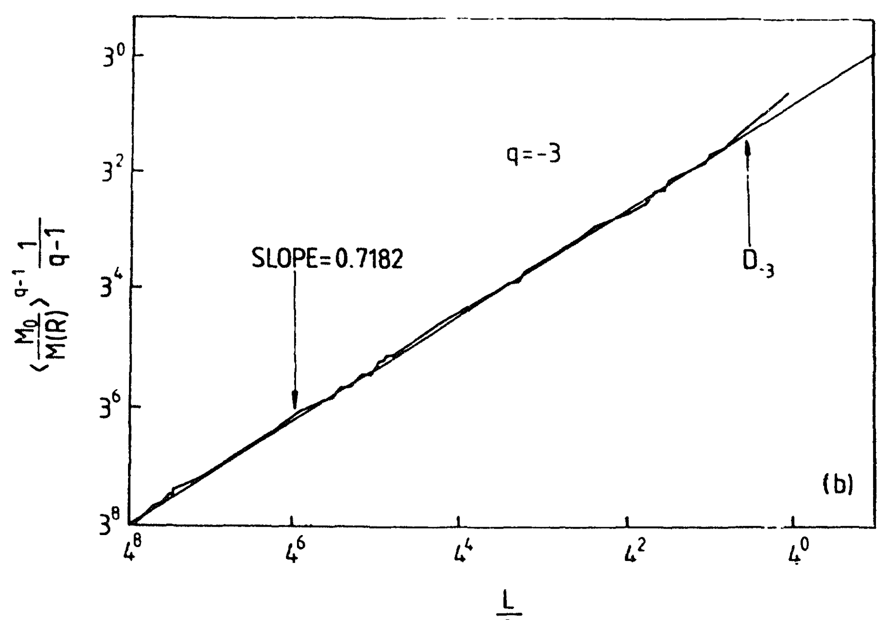


Fig. 4. Estimation of the generalized dimensions D_q for $q = -8, -3, 3$ and 8 ((a), (b), (c) and (d), respectively) from the slopes of the plots $\langle [M_0/M(R)]^{q-1} \rangle /(q-1)$ versus L/R on a double logarithmic scale. The averaging was made over 75 randomly positioned centres.

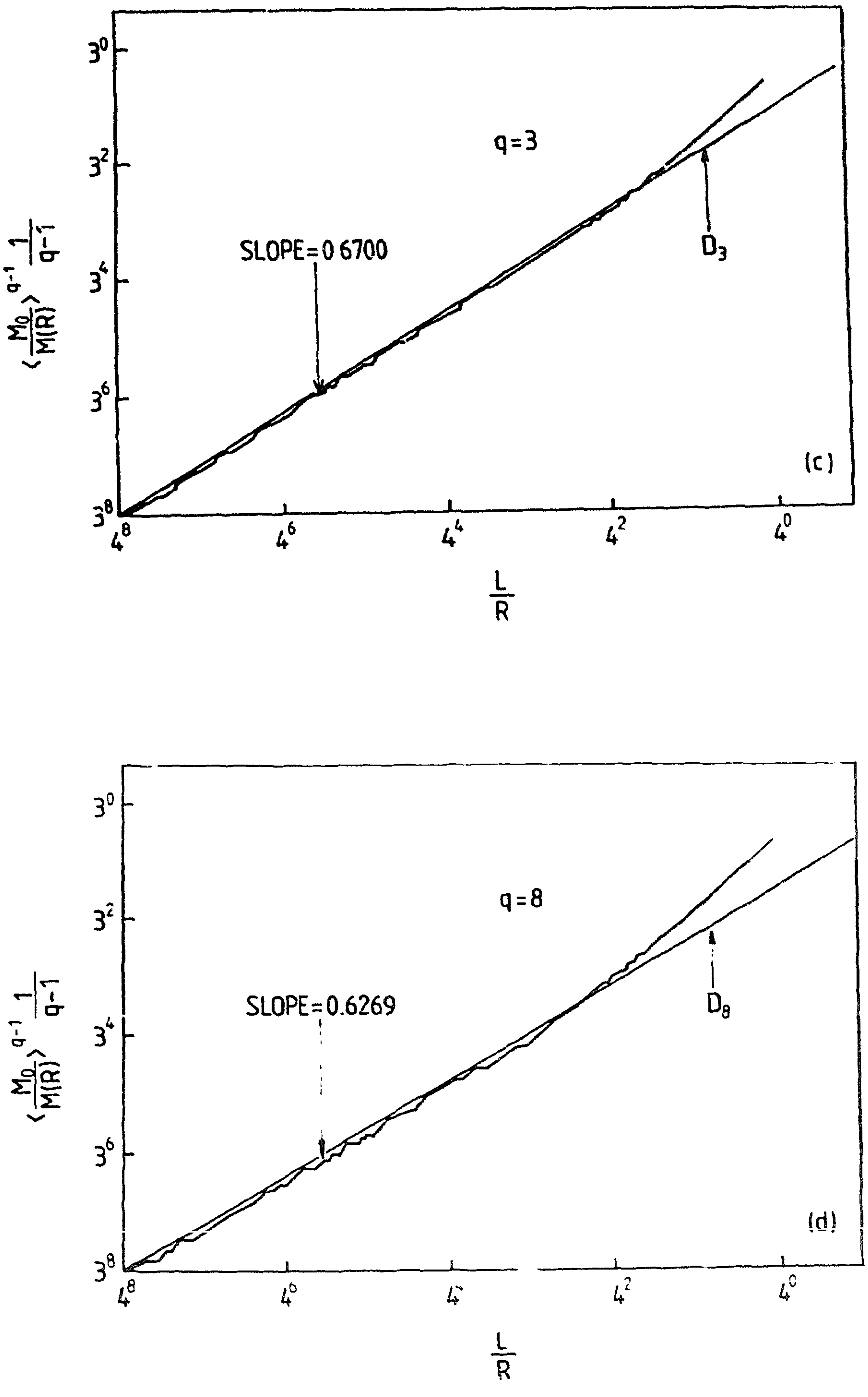


Fig. 4 (cont.).

with some precaution. In particular, the box counting algorithm gives reliable results only for cluster sizes *much larger* than those usually obtained in numerical simulations. During the application of the sand box algorithm it is essential to *average* over many randomly selected centres. Our results indicate that for geometrical multifractals the sand box method provides better estimates of the generalized dimensions, however, because of the necessary averaging this method requires considerably more computing time.

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