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Wada dye boundaries in open hydrodynamical flows

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Abstract

Dyes of different colors advected by two-dimensional flows which are asymptotically simple can form a fractal boundary that coincides with a chaotic saddle's unstable manifold. We show that such dye boundaries can have the Wada property: every boundary point of a given color on this fractal set is on the boundary of at least two other colors. The condition for this is the nonempty intersection of the saddle's stable manifold with at least three differently colored domains in the asymptotic inflow region.

Coloring with dye certain regions of two-dimensional flows and monitoring the evolution of the boundaries separating different colors has been shown to be a useful tool in characterizing transport and mixing in open flows [1–4]. The dynamics of advected dye particles are responsible for the formation of such dye boundaries. By particle, we mean, a light granule of small extension. If it takes on the velocity of the flow very rapidly, i.e., inertial effects are negligible, we call the advection *passive* and the particle a passive tracer. In time-dependent flows, tracer particles exhibit chaotic motion even for simple time-periodic cases [5]. The dye bundaries between regions of different colors can be *fractals* of dimension less than two only if the flow is open [2,3] because

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for closed flows the boundary becomes plane filling in the infinite time limit. Fractal dye boundaries are the signatures of chaotic saddles underlying the tracer dynamics [2].

It has been discovered [6–10] that basin boundaries in dynamical systems can exhibit a rather surprising and peculiar feature, the *Wada property*. A boundary point is called Wada point if every neighborhood of it has nonempty intersections with at least three different basins. A basin boundary is called *Wada basin boundary* if the boundary is fractal and each of its points is a Wada point [10].

Our main objective here is to study the conditions for the existence of *Wada dye boundaries* in open hydrodynamical flows. It will be shown, that by using three or more different colors, the presence of such boundaries is generic in flows where a chaotic saddle governs the tracer dynamics.

We consider flows which are asymptotically simple, where the time dependence is restricted to a finite region (e.g., the wake of an obstacle, like the pillar of a bridge) and the flow in the far upstream and downstream regions is homogeneous. Therefore, the complex behavior (when it exists) is restricted to the time-dependent region, called the "mixing region". Outside of it, trajectories are simple curves. We shall focus our attention on time-periodic flows, that makes possible a convenient description of the dynamics in the form of snapshots of the tracers taken at integer multiples of the period T. The tracer dynamics is thus represented on a *stroboscopic map*.

The advection in this type of flow can be thought of as a *scattering* process. Particles enter the mixing region from the simple asymptotical region, spend some time there close to the chaotic saddle, and then exit downstream along the saddle's *unstable* manifold. If we take a tracer in the downstream region and follow its motion in the *time reversed* dynamics, a very similar situation occurs with the difference that it leaves the mixing region along the *stable* manifold of the chaotic saddle. Since it has recently been shown that Wada basin boundaries exist in chaotic scattering systems with multiple exit modes (at least three) [9], it is natural to look for the Wada property in open hydrodynamical flows too.

We consider open flows whose asymptotic inflow region is colored with a palette of n colors, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, as illustrated in Fig. 1. Because the flow is practically time-independent outside the mixing region (which is of *finite size*), this topology of coloring will be transported further with a slight distortion only to a close vicinity of the mixing region. The irregularly shaped region denotes the mixing region, while the horizontal arrows indicate the direction of the flow in the far upstream and downstream region. The boundaries between different colors far away upstream are perpendicular to the flow, and the bands need not be of equal width. For simplicity let us take the width of the block containing the full palette $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ upstream to correspond to an integer multiple, say k, of the period T, i.e., the width of the block is kv_0T, where v_0 denotes the asymptotic inflow velocity. Thus, the boundary between two colors is mapped onto another boundary between the same colors in k steps on the stroboscopic map. Dye bands from each block are transported towards the mixing region, where they are drastically deformed. An infinitely long time after "switching" on the coloring,

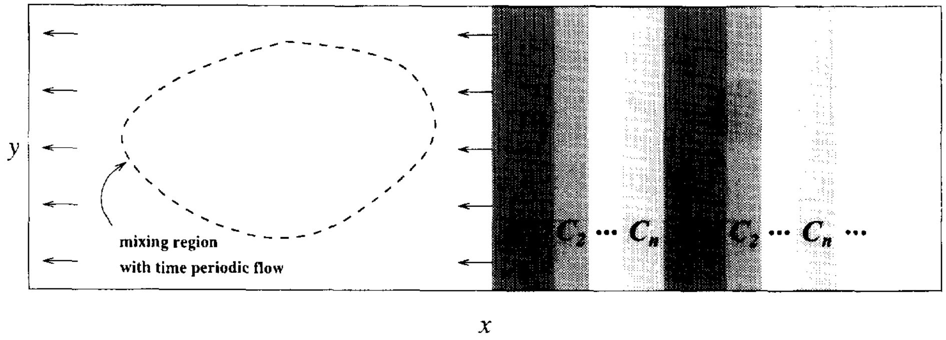

Fig. 1. The geometry of coloring the inflow region which leads to Wada boundaries if the number of colors is greater than 2 ($n \geqslant 3$)

a *stationary* pattern is generated on the stroboscopic map with rather interwoven dye boundaries in the mixing and downstream region.

In the case of flows colored by two dyes (not necessarily according to the geometry of Fig. 1), the boundary is known to be transported to the downstream region so that asymptotically it accumulates on the filaments of the saddle's unstable manifold [2]. This asymptotic accumulation takes place if the boundary has ever intersected the stable manifold. Based on this observation, we show that in the stationary pattern of our stroboscopic map the *unstable manifold* is a *Wada boundary if* there are at least 3 colors present in a block ($n \geqslant 3$). To see this, let us take a point M on the unstable manifold, and a small droplet $\mathscr{D}$ of radius ε around it. Consider now the set $\mathscr{D}_{-i}$ obtained from droplet $\mathscr{D}$ by taking its ith preimage. For $i \to \infty$ this is a complicated winding, very thin "ribbon" extending to ∞ along the positive x-axis. This is due to the existence of the chaotic saddle in the mixing region. Any point taken on the unstable manifold will come arbitrarily close to the saddle in the $t \to -\infty$ limit. However, a point not exactly on the unstable manifold (but still in the droplet) will escape the mixing region. The ones closer to the filaments of the unstable manifold will exit later, the ones further away, sooner. However, because the droplet is a connected set, its preimages $\mathscr{D}_{-i}$ have to be connected, too. Therefore, $\mathscr{D}_{-\infty}$ must continuously extend from the mixing region to plus infinity along the x-axis, as illustrated in Fig. 2.

Because some points of $\mathscr{D}_{-i}$ approach the chaotic set, the extension of $\mathscr{D}_{-\infty}$ to infinity in x takes place along the saddle's *stable* manifold. Consequently, there exists a finite number m such that $\mathscr{D}_{-m}$ intersects *all* the colors of the palette lying along the positive x-axis, at least once. Because $\mathscr{D}_{-m}$ maps onto $\mathscr{D}$ in the forward dynamics, after m steps all the color structure of $\mathscr{D}_{-m}$ is mapped into $\mathscr{D}$. Thus, the droplet $\mathscr{D}$ contains all the colors $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$. Since we did not make any assumption about the radius ε of the droplet, the statement is valid for *any* ε. This means that the point M on the unstable manifold has the Wada property. Furthermore, the point M can be anywhere on the unstable manifold, thus the *entire unstable manifold is a Wada*

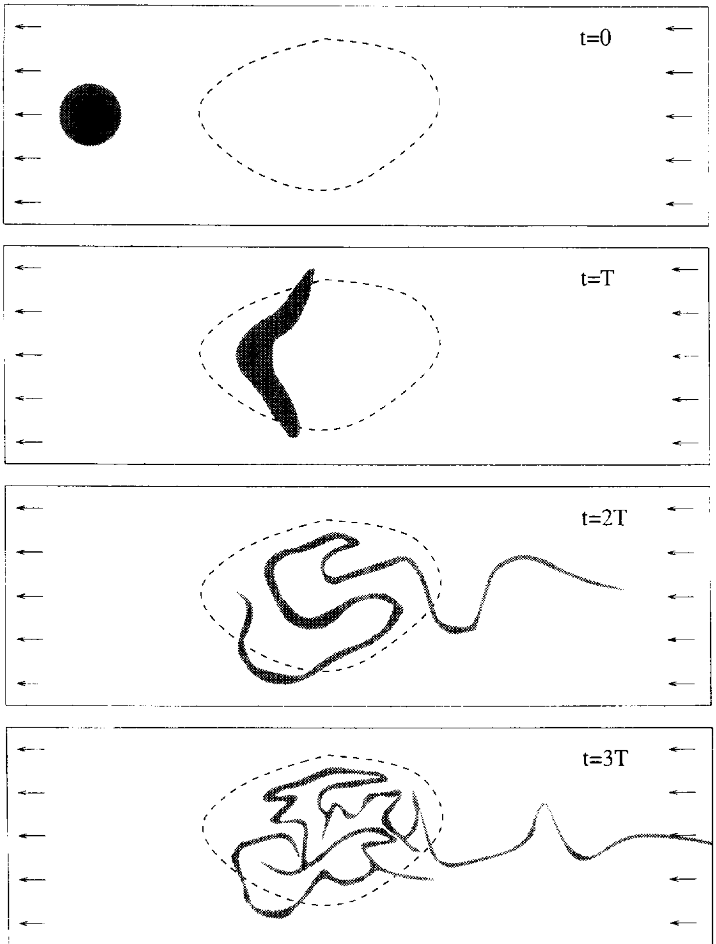

Fig. 2. Schematic diagram of the time reversed droplet dynamics. The little, circular droplet $\mathscr{D}$ (top) has preimages $\mathscr{D}_{-i}$, $i = 1, 2, 3, \ldots$. In other words, $\mathscr{D}$ has been developed from a very complicatedly winding ribbon accumulating around the chaotic saddle's stable manifold (bottom). The arrows define the negative x-direction.

boundary. Note that if the droplet $\mathscr{D}$ has no points *at all* on the unstable manifold, the preimages $\mathscr{D}_{-i}$ will be compactly shaped and of finite size. So the intersection with at least three colors inside a block is not guaranteed for any ε: the droplet center is not a Wada point. In conclusion, the Wada property of the unstable manifold is due to the existence of a chaotic saddle and to the fact that the stable manifold always extends to infinity and, thus, intersects the entire color block in the far inflow region.

To show how Wada dye boundaries look like, consider, as a first numerical example, the advection in the field of two pairs of ideal point vortices of same strength moving along the same symmetry axis (x-axis) [3]. This is a two-dimensional analog of the so-called "leapfrogging" motion of two smoke rings. If the rings have the same rotational direction and move along the same axis, the rear vortex ring attempts to pass through the front one. The leading ring then widens due to the mutual interaction and slows down. Simultaneously, the other ring shrinks, accelerates and penetrates the first one. This process is then repeated continuously with some period T. We use 3 different dyes for coloring the fluid in front of the vortex system: red, white and green (a "tricolor" palette), each band is of the same width $v_0 T$ that corresponds to a full period T of the flow generated by the leapfrogging motion. The velocity v_0 is now the average velocity of the vortex system. The entire block maps onto the next one after $k=3$ steps. In a reference frame co-moving with the center of mass of the two vortex pairs, the coloring corresponds to the one of Fig. 1. The global dye distribution obtained on the plane of the flow in the stationary state of the stroboscopic map is shown in Fig. 3(a). The blowup of a small region of the dye boundary can be seen in Fig. 3(b). The boundary is the unstable manifold of a chaotic saddle that was shown to be situated among the vortices but never reaching a close vicinity of any of the vortex centers [3]. The time-reversed droplet dynamics is illustrated in Fig. 4. Observe the quick extension along the x-axis and the shape of the stable manifold that reaches out to infinity. Thus, it is obvious that a droplet of any size will have, after a finite number of steps, an intersection with a block of all three colors upstream. This supports the visual impression of Fig. 3, and proves the Wada property of the unstable manifold.

As another numerical example, we consider the flow of a viscous fluid around a cylinder of radius R. If the inflow velocity v_0 is in an intermediate range so that the Reynolds number $Re = v_0 R/\nu$ is of the order of 100 (ν denotes the kinematic viscosity), a periodic separation of vortices takes place. They detach from the upper and lower rear of the cylinder in an alternating fashion and lead to a flow pattern called the von Karman vortex street which repeats itself with a period T. It has been shown [11–13,2] that the tracer dynamics is governed by a chaotic saddle existing in the wake of the cylinder. In coloring this flow we choose 4 dyes: red, white, green and yellow arranged upstream in bands of width $v_0 T$. The dye distribution around the cylinder is shown in Fig. 5. The actual computation was carried out with a model of the flow introduced in Ref. [12]. Dyes accumulate on a fractal pattern in the wake of the cylinder that is the chaotic saddle's unstable manifold. The time reversed droplet dynamics is very similar to that of the previous case and therefore we do not show it. The extension of the droplet and of the stable manifold to infinity ensures the Wada property of the dye boundary.

We conclude with a brief discussion on the robustness of the Wada property in open flows. The saddle's stable manifold must have an intersection with a block of more than 2 colors, but the particular geometry of the coloring can deviate from that of Fig. 1. The upstream and downstream flow need not be homogeneous, any smooth stationary flow field is allowed. It is also worth mentioning that we do not have to make any assumptions about the *compressibility* of the flow. The flow can also be compressible as long as there is a chaotic saddle in the system. The condition that the length of a block be integer multiples of the flow's period times v_0 is necessary only

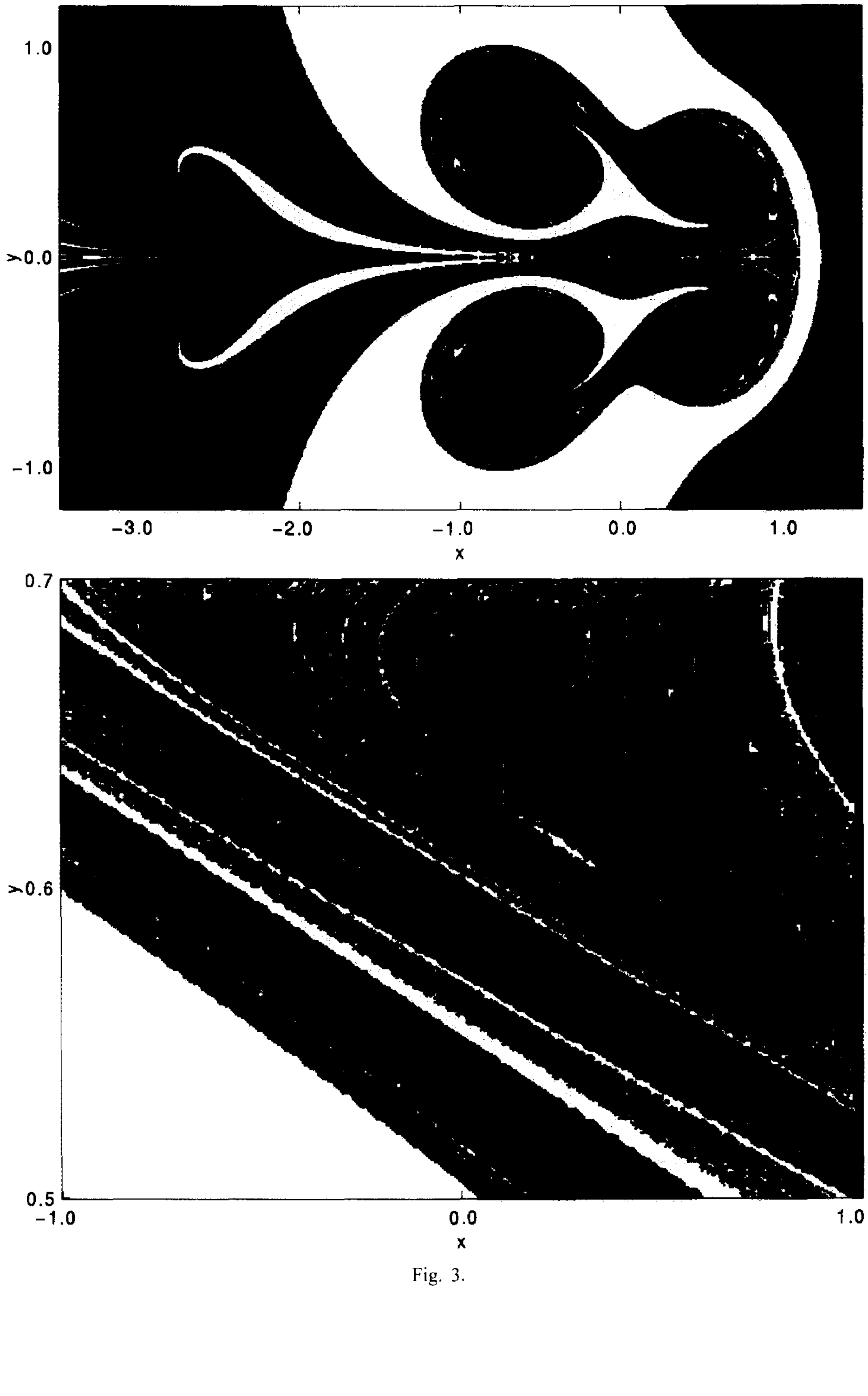

Fig. 3.

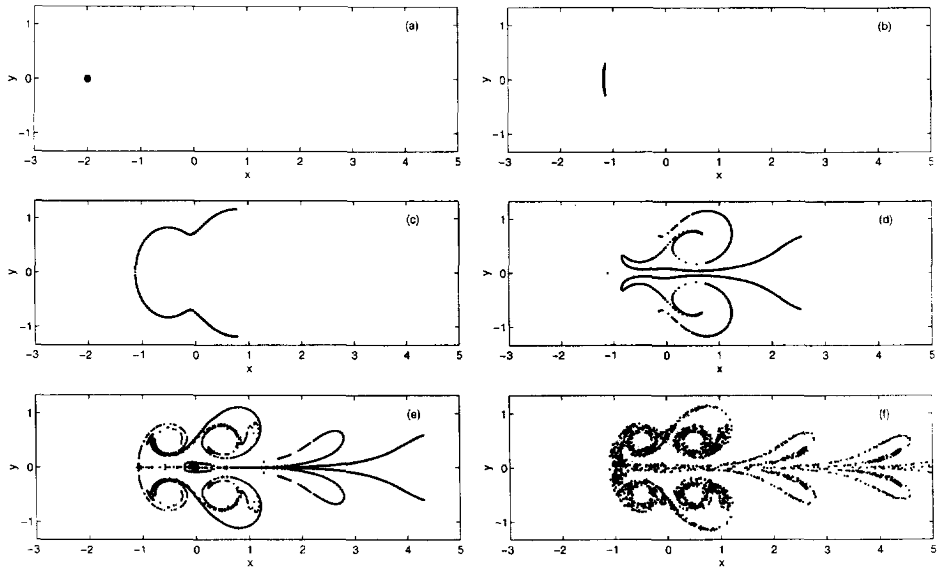

Fig. 4. The time reversed droplet dynamics in the leapfrogging flow in a reference frame co-moving with the vortex pairs. At $t = 0$ a circular droplet of radius 0.1 was taken at $x = -2$, $y = 0$ so that it overlaps with the unstable manifold (a). The preimages of this at times T, $2T$, $3T$, and $5T$ are shown in (b), (c), (d), and (e), respectively. Part (f) exhibits the stable manifold which the droplet converges to in the time reversed dynamics.

for the stationarity of the stroboscopic image, and not for the Wada property. If we use an infinite number of colors, and no repeated blocks, Wada-like boundaries will still show up, however, the colors will not be repeated.

The algorithm of coloring used above corresponds to marking particles according to their initial conditions. It is also possible to color tracers depending on their way of leaving the system. In particular, we can define vertical bands of different colors in the far downstream region and color all the other points of the fluid depending on which band they fall when reaching this downstream region (cf. Experiment B of Ref. [2]). Because this is the time-reversed version of the method discussed above, the *stable manifold is then a Wada boundary* if three or more colors are used. The analogy with the scattering problem of Poon et al. [9] is quite close now, but our way of coloring does not correspond to physically different exit modes. Rather, it reflects different *phases* of the exit.

Fig. 3. Dye boundaries of a tricolor palette ($\mathscr{C}_1$ = green, $\mathscr{C}_2$ = white, and $\mathscr{C}_3$ = red) in a reference frame co-moving with the leapfrogging vortex pairs. (a) A global picture of the mixing region. The regions around the vortices where no points of the chaotic saddle are situated, and where no dye particle can enter from outside can be considered as the vortex cores. They are colored by black. (b) A blow-up of a small region around $x = 0$, $y = 0.6$ of size $x \in [-0.1, 0.1]$, $y \in [0.5, 0.7]$. The accumulation of all three colors can clearly be seen on a fractal-like pattern. The snapshots are taken at $t = \mathrm{mod}(T)$ when both the width and the distance between vortex pairs is 1 in dimensionless units [3]. (Erratum: (b) displays a wrong scale on the x-axis. The correct scale is $[-0.1, 0.1]$.)

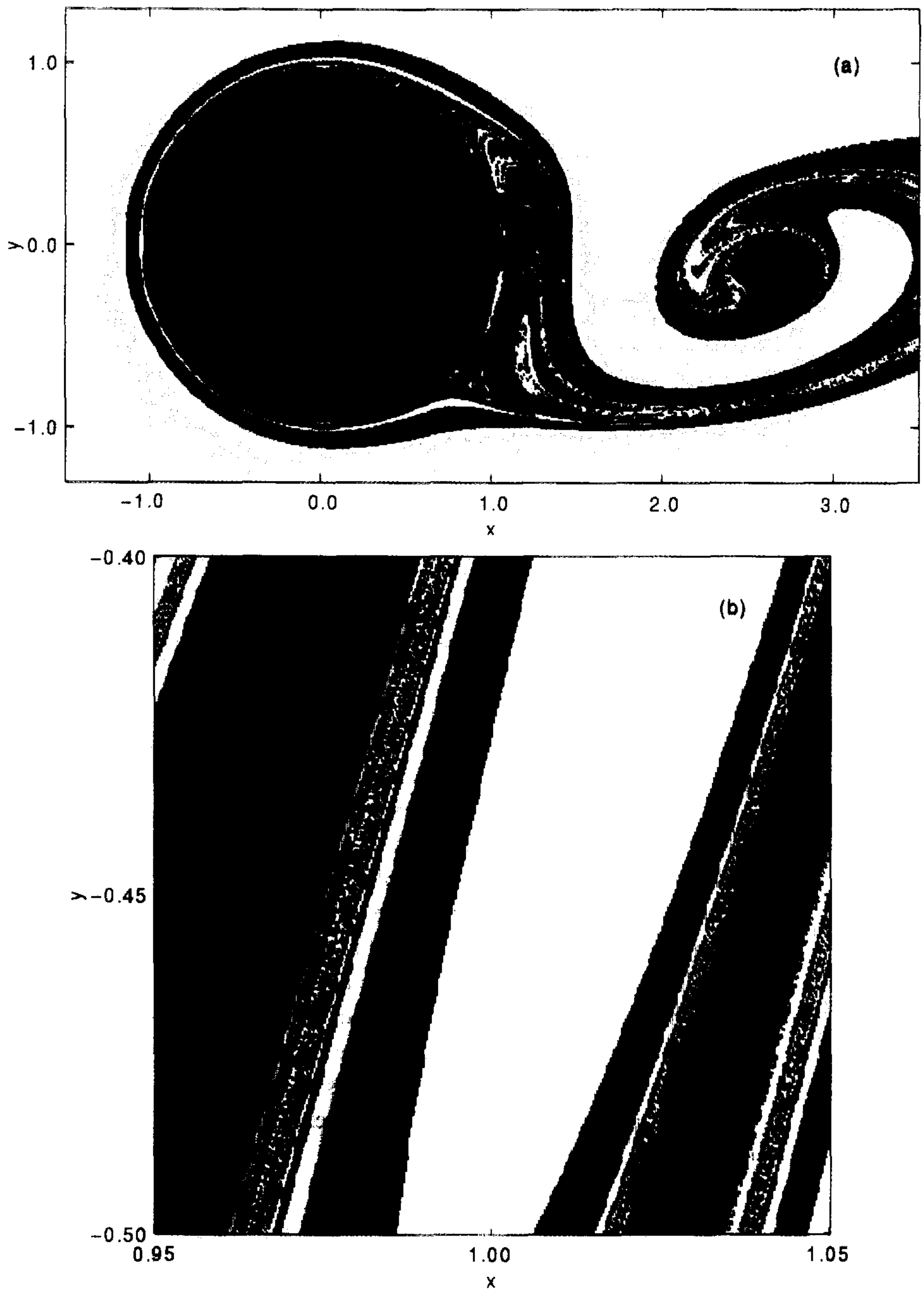

Fig. 5.

Finally, we underline the fact, that the Wada property *cannot* be extracted from the structure of the invariant sets (the saddle and its manifolds) alone. It reflects properties of the coloring procedure chosen, and of how the dye regions evolve either under the forward or the backward dynamics. Being so, it contains a certain arbitrariness which in turn can be used as a *tool* for exploring different aspects of the dynamics taking place *around* the invariant sets.

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Fig. 5. Dye boundaries of a $n = 4$-color palette ($\mathscr{C}_1$ = red, $\mathscr{C}_2$ = white, $\mathscr{C}_3$ = green, and $\mathscr{C}_4$ = yellow) for the von Karman vortex street problem in the wake of a cylinder. Note that the flow direction is now from left to right unlike the previous case. (a) A global picture in the mixing region. The cylinder is colored by black. (b) A magnification of the region $x \in [0.95, 1.05]$, $y \in [-0.5, -0.4]$ in the wake of the cylinder. The accumulation of all colors around the cylinder surface and on the fractal unstable manifold in the wake is clear. The model flow used for numerical computations is identical to the one described in Refs. [2,12]. The scanty appearance of the fractal part when compared to Fig. 3(b) is due to the fact that the fractal dimension of the unstable manifold (d_0 = 1.65) is smaller than in the case of the leapfrogging vortices (d_0 = 1.82). The parameters of the model flow used for numerical computation are identical with those considered in Refs. [12,2]. The snapshots are taken at time $t = \mathrm{mod}(T)$ relative to the flow.