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Characterization of sign singular measures

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Abstract

Situations are considered in which a physical quantity varies rapidly in space from positive values to negative values down to some cutoff scale. Typically the cutoff scale is determined by some diffusive process (e.g., viscosity), and, as the cutoff scale decreases (e.g., viscosity decreases), the physical quantity as a function of the spatial coordinate varies more and more rapidly, approaching a generalized function as the cutoff scale approaches zero. As one examines such a generalized function on smaller and smaller scale, it is shown that a certain suitably defined quantity sensitive to the intensity of spatial oscillations from positive to negative can exhibit an algebraic divergence. The exponent of this algebraic divergence with scale size is one characterization of this situation, and is called the *cancellation exponent*. In the spirit of the multifractal formalism, we introduce a spectrum of cancellation exponents yielding a more complete characterization. Partition function and thermodynamic formalisms are employed. We also discuss how to associate D_q spectra with these signed generalized functions and discuss the relationship of D_q to the cancellation exponent spectrum. Illustrative examples are presented.

1. Introduction and review

Many physical processes have been shown to be described by multifractal *probability measures*. For a probability measure μ_p on the collection of all subsets of a set X, $\mu_p(\sum_i S_i) = \sum_i \mu_p(S_i)$ if the subsets $S_i \subset X$ are countable and do not overlap, $\mu_p(S_i) \geq 0$, and $\mu_p(X) = 1$. A probability measure is commonly said to be multifractal if its generalized dimension spectrum [1,2] D_q varies with q. Equivalently, Halsey et al. used the singularity spectrum $f(\alpha)$ to characterize the multifractal nature of a probability measure [2c].

In this paper we will consider what we shall call *sign singular measures*. In contrast to a probability measure, a sign singular measure of a set can be either positive or negative. Recently, Ott et al. showed that sign singular measures arise in many physical situations [3]. To describe what is meant by a sign-singular measure, consider a mapping μ from subsets of a finite interval X of the x axis to the real numbers. Let $A \subset X$ be an x interval such that $\mu(A) \neq 0$. We say μ is a sign singular measure if, for any such interval A (no matter how small), there is an interval B contained in A such that $\mu(B)$ has the opposite sign from $\mu(A)$. Thus μ everywhere changes sign on arbitrarily fine scale.

As a first example of a physical sign singular measure, we consider the fast kinematic dynamo problem. The fast kinematic dynamo problem can be stated as follows: Will a small seed magnetic field in an unmagnetized electrically-

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conducting flowing fluid grow exponentially with time? Combining Maxwell's equations, Ohm's law and the incompressibility condition for the flow, this linear instability problem leads to the following equation for the magnetic field $\boldsymbol{B}$,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{v} + \frac{1}{R_m} \nabla^2 \boldsymbol{B}, \qquad (1)$$

where R_m is the normalized electrical conductivity of the fluid and is called the "magnetic Reynolds number". The velocity $\boldsymbol{v}$ is presumed given and incompressible. The dynamo is said to be "fast" if exponential growth persists in the limit of infinitely large R_m. Fast dynamos are thought to be relevant to explaining the presence of magnetic fields in astrophysical objects, since these objects typically have very large magnetic Reynolds number (i.e., $R_m > 10^8$ on the surface of the Sun). Fig. 1 (from Ref. [3]) shows the y-component of the magnetic field as a function of the coordinate x for $R_m = 10^7$. (The magnetic field is generated by a simple baker's map dynamo model which we shall not describe here.) The main point is that the magnetic field oscillates rapidly from positive to negative values, and this situation becomes more and more extreme as R_m increases. In fact in Ref. [3] it is shown that, in the $R_m \to \infty$ limit, the integral of the magnetic field over a set A yields a sign singular measure $\mu(A)$.

A second physical example is provided by the vorticity in fully developed high Reynolds number turbulent flows. Experimental results in [3] show that the vorticity yields a sign singular measure in a manner similar to that for the magnetic field of a dynamo in the limit $R_m \to \infty$ limit.

To characterize this type of singularity, we consider the "cancellation exponent" [3-6]. Again let μ be defined on subsets of a finite interval X of the x axis. Cover X with disjoint intervals of equal length ϵ. Then define the cancellation exponent κ by

$$\kappa = \limsup_{\epsilon \to 0} \frac{\ln \chi(\epsilon)}{\ln(1/\epsilon)}, \qquad (2)$$

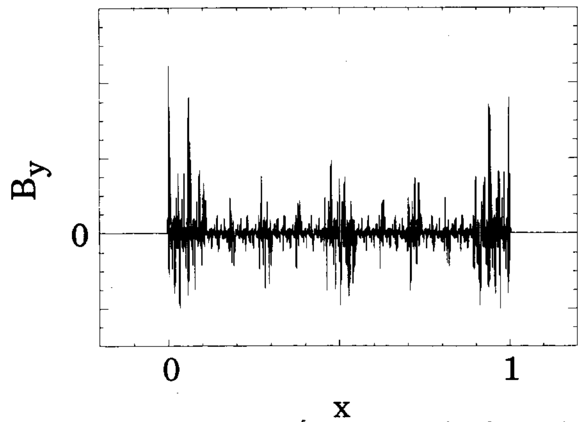

Fig. 1. B_y versus x for the dynamo model of Ref. [3] with $R_m = 10^7$.

where

$$\chi(\epsilon) = \sum_i^{N(\epsilon)} |\mu(I_i)|, \qquad (3)$$

with I_i denoting the ith ϵ-length interval and $N(\epsilon)$ denoting the total number of the ϵ-intervals[1]. For a probability measure $\bar{\mu}$, $|\bar{\mu}(I_i)| \equiv \bar{\mu}(I_i)$, so that $\chi(\epsilon) = 1$, and thus κ is trivially zero. For a measure which has a *smooth* bounded density $\rho(x)$ and can be allowed to change sign with x, we have $\mu(S) \equiv \int_S \rho(x)\, dx$. Thus, as $\epsilon \to 0$, we have $\chi(\epsilon)$ approaches a constant, $\int_S |\rho(x)|\, dx$, leading to $\kappa = 0$. Signed measures [7], in general, have $\kappa = 0$. This follows because the countable additivity condition required of a signed measure allows it to be proven of that a signed measure μ_s can be expressed as a linear combination of two probability measures $\bar{\mu}_1$ and $\bar{\mu}_2$, one with a positive coefficient a and one with a negative coefficient b; i.e., $\mu_s = a\bar{\mu}_1 - |b|\bar{\mu}_2$. Thus the sum in Eq. (3) satisfies

[1] For some purposes a more satisfactory definition of κ is obtained by replacing (3) by $\chi(\epsilon) = \int |\int G_\epsilon(x - x')\widetilde{B}(x')\, dx'|\, dx$, where G_ϵ is a smooth positive function with integral one (e.g., $G_\epsilon(x) = (\pi\epsilon)^{-1/2}\exp(-x^2/2)$ and $\widetilde{B}$ is a generalized function whose integral gives the measure, $\mu[J] = \int_J \widetilde{B}(x)\, dx$. See [4] for discussion of this definition. In this paper we use the simpler definition, (2) and (3) throughout.

$$\sum_i |\mu_s(I_i)| \le \sum a\bar{\mu}_1(I_i) + |b|\bar{\mu}_2(I_i) = a + |b|.$$

Hence, by (2) we have $\kappa = 0$. (Thus for the situations of most interest here (i.e., $\kappa > 0$), the spatial variation is not describable by a signed measure.) In order to have positive κ, we must require $\chi(\epsilon)$ to continue increasing as ϵ gets smaller. Since $\chi(\epsilon)$ increases only because cancellation of positive and negative contributions is reduced with decreasing ϵ, it follows that $\kappa > 0$ is an indication of oscillation in sign on arbitrarily fine scale [3], what we call a sign singular measure.

This paper is organized as follows. In Section 2 we consider three examples illustrating the occurance of a positive cancellation exponent. The first two examples are analytical and the third example is experimental. In Section 3 we associate sign singular measures with probability measures and study the dimensions of these probability measures. In Section 4 we generalize the cancellation exponent to a cancellation exponent spectrum κ_q. In Section 5 we discuss a partition function formalism for sign-singular measures, and in Section 6 we discuss the analogous thermodynamics.

Throughout this paper we will be discussing sign singular measures on an interval X of the x axis. We note, however, that there are obvious generalizations to the consideration of measures in higher dimensional spaces.

Finally, we emphasize that we assume throughout that all limits exists and that the definition (2) makes sense. This is so for the physical cases so far examined (dynamos and fluid turbulence), thus justifying the introduction of definition (2). We note, however, that it is possible to construct mathematical examples of μ's for which definition (2) either yields a "peculiar" result or else the limit does not behave. With respect to peculiar results, a very analogous situation arises for the familiar box counting definition of the capacity: That definition yields [2] that the set of points given by $1/n$ with n positive integers has dimension $D_0 = 1/2$, and this is somewhat unsatisfactory since we would not usually think of a countable set of points as a fractal. An example yielding a similar difficulty for (2) is discussed in the Appendix. In the case of the set $\{1/n\}$, one can associate weights with the points so that the total measure is positive and normalized. The introduction of the generalized dimensions D_q [2] then correctly reflects the low dimensionality of the set since one finds $D_q \equiv 0$ for q larger than some $q_c > 0$. In an analogous manner, we find in our example that the generalized cancellation exponents κ_q become identically zero in a range $q > q_c > 0$.

For a recent example of sign-singular behavior in a fast dynamo, see Ref. [8].

2. Examples

2.1. Self-similiar 3-strip generation process

As our first analytical example we consider the limiting measure generated by repeated applications of the process shown in Fig. 2. Fig. 2a shows an initial upward-directed field uniformly distributed over an x-interval (0, 1]. The total measure is one. After the first operation, the measure is redistributed uniformly to the intervals (0, 1/3], (1/3, 2/3] and (2/3, 1] with the measures $p_1 > 1/2$, $p_2 < 0$ and p_1 respectively as shown in Fig. 2b. Since $2p_1 + p_2 = 1$, the total measure in (0, 1] is still one. Applying the operation a second time generates Fig. 2c. The resulting nine intervals have equal length of 1/9. Four intervals have a measure p_1^2. Four intervals have a measure $p_1 p_2$. One interval has a measure p_2^2. Note that if the distribution on the interval (0, 1/3] in Fig. 2c is stretched horizontally by a factor of three and vertically by a factor of $1/p_1$, it becomes a precise replica of the Fig. 2b. Similar statements apply to the intervals (1/3, 2/3] and (2/3, 1]. Thus the graph generated by the re-

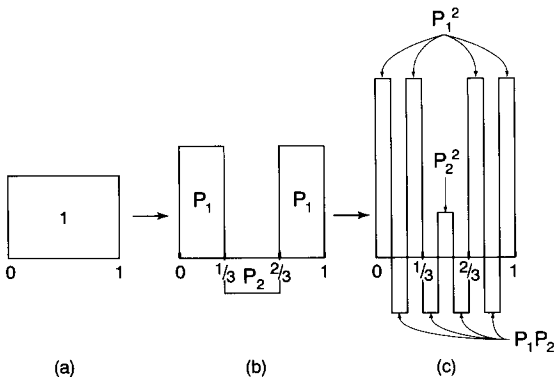

Fig. 2. Self-similar 3-strip generation process.

peated operations shown in Fig. 2 is self-similar. If one applies the operation again and again, the measure eventually changes sign on arbitrarily fine scale.

To calculate the cancellation exponent of the limiting measure, consider the situation after the nth iteration, where n is a large integer. At the nth iteration there will be 3^n intervals of width $(1/3)^n$, and each such interval will have a measure which is one of the values of $p_1^l p_2^m p_1^{n-l-m}$, where l and m are integers, $0 \leq (l, m) \leq n$. The number of intervals with a measure of $p_1^l p_2^m p_1^{n-l-m} = p_1^{n-m} p_2^m$ is $C_n^l C_{n-l}^m$ where C_n^l denotes the binomial coefficient. Choosing $\epsilon = (1/3)^n$, we have,

$$\chi(\epsilon) = \sum_l \sum_m C_n^l C_{n-l}^m |p_1^{n-m} p_2^m| = (2|p_1| + |p_2|)^n.$$

Hence

$$\kappa = \lim_{n\to\infty} \frac{\ln(2|p_1| + |p_2|)^n}{\ln(1/3)^n} = \frac{\ln(2|p_1| + |p_2|)}{\ln(1/3)}, \tag{4}$$

Since $2p_1 + p_2 = 1$ and $p_2 < 0$, $2|p_1| + |p_2| > 1$, and thus $\kappa > 0$ indicating that the measure is sign-singular. If p_1 and p_2 are both positive, then the measure is simply a probability measure, and since $2|p_1| + |p_2| = 2p_1 + p_2 = 1$ we obtain the result $\kappa = 0$.

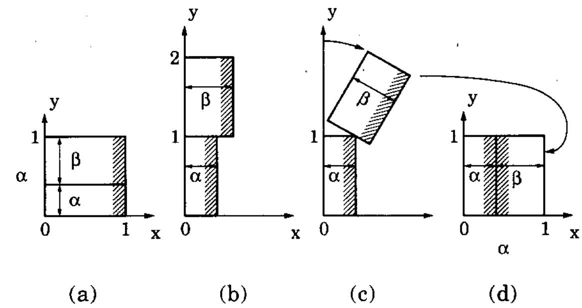

Fig. 3. Phase-shift 2-strip baker's map

2.2. Phase-shift 2-strip Baker's map

As a second analytical example [4] we consider a phase-shift 2-strip baker's map which has previously been shown to be a simple model illustrating aspects of fast kinematic dynamos. At time $t_n = nT$ (n is an integer), there are rapid incompressible fluid motions which map the coordinates (x, y) as follows:

$$x_{n+1} = \begin{cases} \alpha x_n, & \text{for } y_n < \alpha, \\ \beta(1 - x_n) + \alpha, & \text{for } y_n \geq \alpha, \end{cases} \tag{5a}$$

$$y_{n+1} = \begin{cases} y_n/\alpha, & \text{for } y_n < \alpha, \\ (1 - y_n)/\beta, & \text{for } y_n \geq \alpha, \end{cases} \tag{5b}$$

where $\alpha + \beta = 1$. The action of this map on the unit square $0 < (x, y) < 1$ is illustrated in Fig. 3. Between these impulsive mappings, there is a stepwise z-directioned shear flow which operates for a time duration T, where $v_z = 0$ for $x_n < \alpha$ and $v_z = v_{z0}$ for $x > \alpha$.

Imagine that the whole space is occupied by perfectly conducting fluid and there is an initial seed magnetic field in the unit square shown in Fig. 3a. Furthermore, assume that the magnetic field $\boldsymbol{B}_n$ at the end of nth application of the shear flow is independent of y and takes the form,

$$\boldsymbol{B} = \mathrm{Re}\{B_n(x) \exp(ikz) \boldsymbol{y}_0\}.$$

Since the fluid is perfectly conducting, the flux is "frozen-in" while the mapping of Fig. 3 occurs. Thus the vertical flux per unit length in z through each of the two strips in Fig. 3b is $\widehat{\Phi}_n = \mathrm{Re}\{\Phi_n e^{ikz}\}$, where $\Phi_n = \int_0^1 B_n(x)\,dx$. The contribution from the interval $(0, \alpha)$ (the "α strip") to Φ_{n+1} is then Φ_n and the contribution from the interval $(\alpha, 1)$ (the "β strip") is $\Phi_n e^{i\theta}$, where $\theta = kv_{z0}T + \pi$. Hence,

$$\Phi_{n+1} = (1 + e^{i\theta})\Phi_n.$$

Thus, we obtain the flux multiplication rate λ as

$$\lambda = 1 + e^{i\theta},$$

where λ is, in general, a complex number $\lambda = |\lambda| e^{i\phi}$.

Let

$$\chi(\epsilon,\xi) = \sum_j^{N(\epsilon)} \left|\mathrm{Re}\left\{e^{i\xi}\tilde{\mu}(I_j)\right\}\right|, \tag{6}$$

where ξ is a phase parameter that we introduce to facilitate the analysis, I_j denotes the jth x-interval of length ϵ, and we define the complexified measure $\tilde{\mu}$ as

$$\tilde{\mu}(I_j) = \lim_{n\to\infty} \frac{\int_{I_j} B_n(x)\,dx}{\int_0^1 B_n(x)\,dx}$$

(we assume the limit exists). Taking the real part of $\tilde{\mu}(I_j)$ as the measure of the interval I_j and comparing the definitions of $\chi(\epsilon)$ (Eq. (3)) and $\chi(\epsilon,\xi)$ (Eq. (6)), we have

$$\chi(\epsilon) = \chi(\epsilon, 0).$$

Denote the contribution of the α strip to $\chi(\epsilon,\xi)$ as $\chi_\alpha(\epsilon,\xi)$, i.e.,

$$\chi_\alpha(\epsilon,\xi) = \sum_j^{N_\alpha(\epsilon)} \left|\mathrm{Re}\left\{e^{i\xi}\tilde{\mu}(I_j)\right\}\right|,$$

where the summation is over the ϵ-intervals in the α strip. $\chi_\beta(\epsilon,\xi)$ is similarly defined. By the self-similarity inherent in the map of Fig. 3, we can express $\chi_\alpha(\epsilon,\xi)$ as a sum over all intervals I_l of length ϵ/α in the entire interval $(0, 1)$

$$\chi_\alpha(\epsilon,\xi) = \sum_j^{N(\epsilon/\alpha)} \left|\mathrm{Re}\left\{\lambda^{-1}e^{i\xi}\tilde{\mu}(I_j)\right\}\right|.$$

Thus,

$$\chi_\alpha(\epsilon,\xi) = |\lambda|^{-1}\chi(\epsilon/\alpha, \xi - \phi),$$

and similarly

$$\chi_\beta(\epsilon,\xi) = |\lambda|^{-1}\chi(\epsilon/\beta, \xi - \phi + \theta).$$

Hence,

$$\chi(\epsilon,\xi) = |\lambda|^{-1}[\chi(\epsilon/\alpha, \xi-\phi) + \chi(\epsilon/\beta, \xi - \phi + \theta)]. \tag{7}$$

Expanding $\chi(\epsilon,\xi)$ in a Fourier series in ξ, we have

$$\chi(\epsilon,\xi) = \sum_m \chi_m(\epsilon) e^{im\xi},$$

which when substituted in Eq. (7) yields

$$|\lambda|\chi_m(\epsilon)e^{im\phi} = \chi_m(\epsilon/\alpha) + \chi_m(\epsilon/\beta)e^{im\theta}. \tag{8}$$

To solve Eq. (8), we assume solutions of the form $\chi_m(\epsilon) = K_m \epsilon^{-\kappa_m}$ (where K_m and κ_m can both be complex). (With regard to the assumed form, $\chi_m(\epsilon) = K_m\epsilon^{-\kappa_m}$, see also the last paragraph in Section 4.1 of Ref. [8].) Using this form in Eq. (8), we obtain an equation for the exponent κ_m,

$$|\lambda| e^{im\phi} = \alpha^{\kappa_m} + \beta^{\kappa_m} e^{im\theta}. \tag{9}$$

Since $\chi(\epsilon) = \sum \chi_m(\epsilon) = \sum K_m \epsilon^{-\kappa_m}$, for small ϵ the quantity will be dominated by the root of Eq. (9) with the largest real part of κ_m. Taking magnitudes of both sides of Eq. (9) and noting that $|a + b| \le |a| + |b|$, we have

$$|\lambda| \le \alpha^{\kappa_m^{(r)}} + \beta^{\kappa_m^{(r)}}, \tag{10}$$

where $\kappa_m^{(r)} = \mathrm{Re}\{\kappa_m\}$. Since α and β are both less than one, Eq. (10) implies an upper bound on $\kappa_m^{(r)}$,

$$\kappa_m^{(r)} \le \bar{\kappa}, \tag{11}$$

where $\bar{\kappa}$ is the unique real positive root of $|\lambda| = \alpha^{\bar{\kappa}} + \beta^{\bar{\kappa}}$. We now note that this upper bound is actually attained: setting $m = 0$, we see that Eq. (9) reduces to the equation for $\bar{\kappa}$. Thus $\chi \sim$

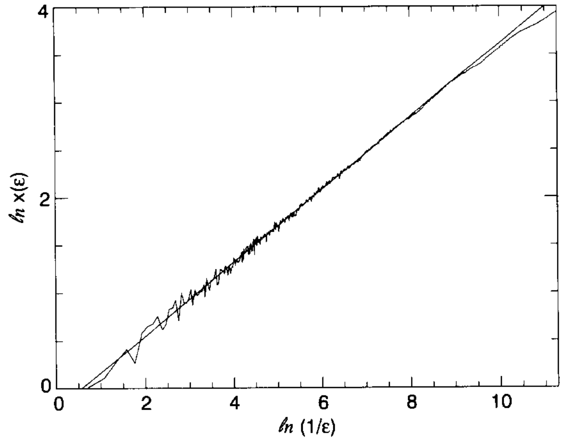

Fig. 4. $\ln\chi(\epsilon)$ versus $\ln(1/\epsilon)$ for phase-shift 2-strip baker's map with $\alpha = 0.4$, $\beta = 0.6$, and $\theta = 1.4$. The superposed straight line is the linear fit, whose slope yields $\kappa = 0.36$.

$\epsilon^{-\bar{\kappa}}$, implying that $\bar{\kappa}$ is in fact the cancellation exponent κ, determined by

$$|\lambda| = \alpha^{\kappa} + \beta^{\kappa}. \tag{12}$$

In terms of α, β and θ, one can express the cancellation exponent as

$$\alpha^{\kappa} + \beta^{\kappa} = 2|\cos(\theta/2)|. \tag{13}$$

We now numerically verify Eq. (13). We take the initial magnetic field at $n = 0$ to be $B_0 \equiv 1$ and choose $\alpha = 0.4$, $\beta = 0.6$ and $\theta = 1.4$. Eq. (13) gives $\kappa = 0.3798\ldots$. We calculate the magnetic field at time step $n = 15$ at 10^7 points uniformly distributed on the x-interval (0, 1). Then we obtain $\chi(\epsilon)$ and plot $\ln\chi(\epsilon)$ versus $\ln(1/\epsilon)$ as shown in Fig. 4. We see that the plot is fairly linear for the range $\ln(1/\epsilon) < 8$. For smaller ϵ (or larger $\ln(1/\epsilon)$), the plot deviates from the superposed fitted straight line since we iterate the map only for a finite number of times. The slope of the fitted line yields $\kappa = 0.36$, which is close to the value obtained for the given values of α, β and θ from Eq. (13), $\kappa = 0.3798\ldots$.

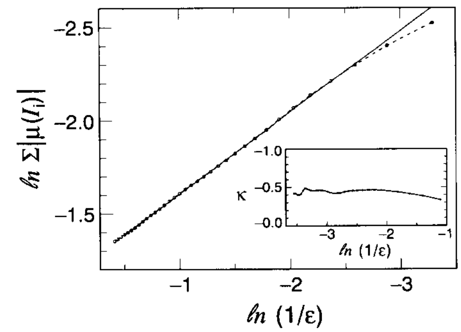

Fig. 5. $\ln\sum|\omega_\epsilon|$ versus $\ln(1/\epsilon)$ for the high Reynolds number turbulent wake behind a cylinder. The superposed straight line is the linear fit, whose slope yield $\kappa = 0.45$.

2.3. *fluid turbulence*

As an experimental example, we now consider the vorticity field in high Reynolds number fluid turbulence [3]. The y component of the vorticity ω_y was obtained from data for the wake behind a cylinder, where x is the streamwise direction and y is perpendicular to the axis of the cylinder. The cylinder Reynolds number is 1,100. Defining $\omega_\epsilon = \int_\epsilon \omega_y\, dx$ as the measure of the ϵ-interval, the ϵ scaling of $\sum|\omega_\epsilon|$ was examined. Fig. 5 shows that a decent scaling exists, and yields a cancellation exponent of about 0.45. The cutoff at small ϵ is due to finite Reynolds number.

3. Cancellation exponent spectrum

Consider a sign singular measure μ on an x-interval X. Divide X into many x-intervals of length ϵ. In the spirit of the multifractal formalism for probability measures define the cancellation exponent spectrum κ_q as

$$\kappa_q = \limsup_{\epsilon\to 0} \frac{\ln\chi(q,\epsilon)}{\ln(1/\epsilon)}, \tag{14}$$

where

$$\chi(q,\epsilon) = \sum_i^{N(\epsilon)} |\mu(I_i)|^q. \tag{15}$$

When $q = 1$, κ_q becomes the cancellation exponent κ.

For the 3-strip process considered in Section 2.1, we have

$$\sum_i |\mu_i|^q = \sum_l \sum_m C_n^l C_{n-l}^m |p_1^{n-m} p_2^m|^q$$
$$= (2|p_1|^q + |p_2|^q)^n.$$

Hence

$$\kappa_q = \frac{\ln(2|p_1|^q + |p_2|^q)}{\ln(1/3)}. \tag{16}$$

For the phase-shift 2-strip baker's map considered in Section 2.2, we define

$$\chi(q,\epsilon,\xi,n) = \sum_j^{N(\epsilon)} \left|\mathrm{Re}\left\{e^{i\xi}\tilde{\mu}(I_j)\right\}\right|^q. \tag{17}$$

Thus, by the similarity already noted in Section 2.2, we have

$$\chi(q,\epsilon,\xi) = |\lambda|^{-q}\Big[\chi(q,\epsilon/\alpha,\xi-\phi)$$
$$+\chi(q,\epsilon/\beta,\xi-\phi+\theta)\Big]. \tag{18}$$

Hence, following the procedure in Section 2.2, the cancellation exponent spectrum is obtained,

$$\alpha^{\kappa_q} + \beta^{\kappa_q} = |2\cos(\theta/2)|^q. \tag{19}$$

In Fig. 6, we show the prediction of Eq. (19). We also numerically calculate the cancellation exponent spectrum κ_q for several values of q and plot the numerical data as small circles in Fig. 6. We see that the data agree well with the curve predicted by Eq. (19).

After having defined the generalized cancellation exponents κ_q, it is worth introducing a spectrum of local exponent γ. Define a *crowding index* or *Hölder exponent* γ for the quantities $|\mu(I_i)|$ exactly in the same spirit as for multifractal probability measures,

$$|\mu(I_i)| \sim \epsilon^\gamma. \tag{20}$$

This expresses the fact the modulus of the signed measure tends to zero with the interval size in

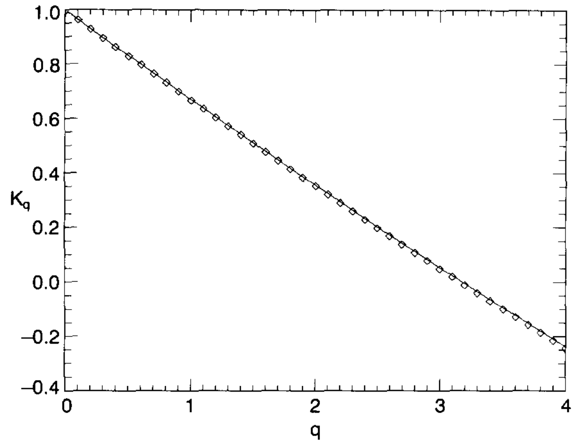

Fig. 6. The cancellation exponent spectrum κ_q versus q. The solid line is the prediction based on Eq. (19). The diamonds in the figure are the results of the numerical calculations using the box-counting method (i.e., Fig. 4).

an algebraic fashion and determines crowding indices lying in some finite range $\gamma_{min} \le \gamma \le \gamma_{max}$.

Obviously, there can be several ϵ-intervals characterized by the same crowding index. It is natural to assume that in the limit $\epsilon \to 0$, intervals belonging to a given range γ to $\gamma + d\gamma$ form a fractal subset of some dimension $f(\gamma) \le 1$. Therefore, the number of such intervals $N(\gamma)d\gamma$ increases with ϵ according to a power law

$$N(\gamma) \sim \epsilon^{-f(\gamma)}. \tag{21}$$

The quantity $f(\gamma)$ will be called the *multifractal spectrum of cancellation exponents.*

By rewriting the sum $\chi(q,\epsilon)$ defined by Eq. (18) as an integral over γ,

$$\chi(q,\epsilon) \sim \int N(\gamma)\epsilon^{q\gamma}d\gamma \sim \int \epsilon^{-f(\gamma)+q\gamma}d\gamma, \tag{22}$$

and evaluating it via the saddle point method, we obtain

$$\kappa_q = f(\gamma_q) - q\gamma_q, \tag{23}$$

where

$$f'(\gamma_q) = q. \tag{24}$$

This means that κ_q is the Legendre transform of $f(\gamma)$ just as the generalized dimensions are Legendre transforms of the multifractal spectrum of probability measures. Because of the lack of normalization of $|\mu(I_i)|$ in our case, however, the graph of $f(\gamma)$ does not touch the diagonal $f = \gamma$ but rather, as we shall see, intersects it in two points.

It is easy to explicitly determine the $f(\gamma)$ spectrum in the example of Section 2.1. Assuming that $p_1 > |p_2|$, we obtain

$$f(\gamma_q) = \frac{1}{\ln 3}\left[-\left(\frac{\gamma_{max}-\gamma}{\Delta}\right)\ln\left(\frac{\gamma_{max}-\gamma}{2\Delta}\right) - \left(\frac{\gamma-\gamma_{min}}{\Delta}\right)\ln\left(\frac{\gamma-\gamma_{min}}{\Delta}\right)\right], \quad (25)$$

with

$$\Delta = \gamma_{max} - \gamma_{min}, \quad (26)$$

where $\gamma_{max} = (\ln|p_2|)/\ln(1/3)$ and $\gamma_{min} = (\ln p_1)/\ln(1/3)$.

Finally we comment that the crowding index can be defined at a point x by

$$\gamma(x) = \limsup_{\epsilon\to 0}\{\ln|\mu_\epsilon(x)|\}/\ln\epsilon,$$

where $\mu_\epsilon(x) = \mu([x-(\epsilon/2)],[x+(\epsilon/2)])$. The need for using the limit supremum (rather than a simple limit) can be appreciated from the example in Section 2.1. Taking $x = 1/2$, we have $\mu_1(1/2) = 1$ and $\mu_{1/3}(1/2) = -p_2 < 0$. Since $\mu_\epsilon(x)$ is continuous in ϵ, we have $\mu_\epsilon(1/2) = 0$ for some ϵ in $[1/3, 1]$. Similarly $\mu_\epsilon(1/2) = 0$ for some ϵ value in $[(1/3)^{m+1}, (1/3)^m]$ for $m = 0, 1, 2, \ldots$. Thus, in general, $\mu_\epsilon(x)$ can have zeros as a function of ϵ, accumulating on $\epsilon = 0$, and the simple limit does not exist while the limit supremum does.

4. Partition function

In the case of sign singular measures possessing a hierarchic organization with *a level structure*, another definition of the generalized cancellation exponents can also be given. In such a case one finds a natural partitioning of the unit interval X into subintervals $I_i^{(n)}$ at each level $n = 1, 2, \ldots$. Note, however, that these intervals will be, in general, of different length. Let $l_i^{(n)}$ and $\mu_i^{(n)}$ denote the length and the measure of the partitioning interval $I_i^{(n)}$, respectively. Let us define a *partition function* Γ by taking different powers of the length scales and the moduli of the measure at a given level n:

$$\Gamma(q,\beta,n) = \sum_i |\mu_i^{(n)}|^q l_i^{(n)\beta}, \quad (27)$$

where q and β can be any real numbers. We claim that the generalized cancellation exponent κ_q has the property that when $\beta = \kappa_q$ the partition sum becomes asymptotically independent of n, i.e.

$$\Gamma(q,\kappa_q,n) \sim 1. \quad (28)$$

In particular, for $q = 1$ we obtain

$$\sum_i |\mu_i^{(n)}| l_i^{(n)\kappa}, \quad (29)$$

expressing a special property of the cancellation exponent κ.

The equivalence of this definition with Eq. (17) can best be seen on the example of a self-similar N strip process (in Section 2.1, $N = 3$). At level-1 the distribution is defined on N strips of the unit interval. Strips of width $w_k, k = 1, \ldots, N$ carry measures $p_k, k = 1, \ldots, N$ where the p_k's can be of different sign but their total contribution is unity: $\sum_{k=1}^N p_k = 1$. A self-similar repetition of this construction in each subinterval up to infinity defines a sign singular measure. One easily sees that

$$\Gamma(q,\beta,n) = \left(\sum_{k=1}^N |p_k|^q w_k^\beta\right)^n. \quad (30)$$

Let us now use definition (17) to this measure. One can decompose the quantity $\chi(q,\epsilon)$ into N terms

$$\chi(q,\epsilon) = \sum_{k=1}^{N} \chi^{(k)}(q,\epsilon), \tag{31}$$

where $\chi^{(k)}$ denotes the contribution from strip k. Because of selfsimilarity, $\chi^{(k)}$ can be obtained from χ by measuring the length in units of w_k and multiplying the total weight by p_k^q:

$$\chi^{(k)}(q,\epsilon) = p_k^q\ \chi(q,\epsilon/w_k). \tag{32}$$

By recalling the scaling relation

$$\chi(q,\epsilon) \sim \epsilon^{-\kappa_q}, \tag{33}$$

we find from Eq. (32) that

$$\sum_{k=1}^{N} |\ p_k\ |^q w_k^{\kappa_q} \sim 1, \tag{34}$$

in agreement with Eqs. (28) and (30). In the special case of equal strip widths $w_k = 1/N$, we obtain

$$\kappa_q = \frac{\ln\left(\sum_{k=1}^{N} |\ p_k\ |^q\right)}{\ln(1/N)}. \tag{35}$$

This contains the result derived for the three strip process in Section 2.1.

An interesting consequence of definition (28) is that the generalized cancellation exponent κ_q *vanishes* at some value $q = \sigma > 1$. To see this, we recall that a vanishing κ_σ implies that

$$\sum_i |\ \mu_i^{(n)}\ |^\sigma \sim 1. \tag{36}$$

As known from the theory of fractals, the dimension D_0 of a set covered hierarchically by lengths scales $l_i^{(n)}$ follows from relation

$$\sum_i l_i^{(n)D_0} \sim 1. \tag{37}$$

By comparing the last two formulas, we conclude that σ is a kind of fractal dimension, namely the dimension of a set constructed with $|\ \mu_i^{(n)}\ |$ as its length scales. Since, however, $\sum_i |\ \mu_i^{(n)}\ | > 1$, such a set cannot be embedded in an interval and its fractal dimension must be bigger than unity.

5. Association of probability measures with sign singular measures

Consider a physical quantity ϱ that varies rapidly but is ultimately smooth on a sufficiently small scale, e.g. in the fluid turbulence example, ϱ might be a component of the vorticity vector and the short smoothing scale is the Kolmogorov cut-off. Consider the variation of ϱ in an interval X along the x-axis, $\varrho = \varrho(x)$, for scales larger than the smoothing scale.

We consider two situations:

(i) *Truncation at given stage.* The quantity $\varrho(x)$ results from a finite number of applications of a generating process. For example, in Sections 2.1 and 2.2 we can consider the density at some particular large value n_* of n. Note that it is possible that the generated range of small scales might be exponentially large in n_*. For the example of in Section 2.2 the generated small scales range from α^{n_*} to β^{n_*}; the generated interval sizes at level n_* are $\alpha^l\beta^{n_*-l}$, for $l = 1,2,...,n_*$. (For the example of Section 2.1 the generated intervals at level $n = n_*$ have the same length 3^{-n_*}).

(ii) *Truncation by smoothing.* The quantity results from a physical process determining a cutoff scale ϵ_*. Examples are the Kolmogorov scale for the vorticity measure (Section 2.3), and in the magnetic dynamo case the scale $\epsilon_* \sim R_m{}^{-1/2}$ where R_m is the magnetic Reynolds number. Note that this case is qualitatively different from that resulting from a truncation at a given stage in that there is necessarily a single smallest scale ϵ_* and not an exponentially large range of small truncation scales.

We now associate with $\rho(x)$ a family of probability measures $\bar{\mu}_p$ with densities

$$\rho_p(x) = \frac{|\rho(x)|^p}{\int_X |\rho(x)|^p\,dx}. \tag{38}$$

Divide X into many x-interval of length ϵ. We then define the dimension spectrum $D_{q,p}$ for the associated probability measures by the following scaling in the range $\epsilon > \epsilon_c$ (for truncation by smoothing $\epsilon_c = \epsilon_*$),

$$J_p(q,\epsilon) \sim \epsilon^{(q-1)D_{q,p}}, \tag{39}$$

where

$$J_p(q,\epsilon) = \sum_i [\bar{\mu}_p(I_i)]^q,$$

$$\bar{\mu}_p(I_i) = \int_{I_i} \rho_p(x)\,dx. \tag{40}$$

In the case of truncation by smoothing, knowledge of the spectrum of fractal dimensions for the case $p = 1$, implies knowledge of the fractal dimension spectrum of the measure $\bar{\mu}_p$ for any p. In particular, it can be shown for this case that [5],

$$\tau_{q,p} = \tau_{pq,1} - q\tau_{p,1}, \tag{41}$$

where $\tau_{q,p} = (q-1)D_{q,p}$. Thus $D_{q,p}$ as a function of q is determined from $D_{q,1}$ as a function of q.

From now on we shall only consider the case $p = 1$ and the subscript "p" on $\bar{\mu}$ and D_q will be dropped with the understanding that it is one.

6. Thermodynamic formalism

In the spirit of the thermodynamic formalism of strange sets [2,9], one expects in the large-n limit an exponential scaling of the partition function with n,

$$\Gamma(q,\beta,n) = \sum_i |\mu_i^{(n)}|^q l_i^{(n)\beta} \sim \exp(F(q,\beta)n).$$

This defines a *bivariate free energy* $F(q,\beta)$ which provides the most general characterization of sign singular measures. A comparison with definition (28) of the generalized cancellation exponent immediately yields that the free energy vanishes at $\beta = \kappa_q$,

$$F(q,\kappa_q) = 0. \tag{42}$$

The framework of the thermodynamical formalism provides a good opportunity for relating the cancellation exponents of sign singular measures to generalized dimensions of some associated probability measures. We recall that the generalized dimensions D_q of a measure which is concentrated at level n on boxes of linear size $l_i^{(n)}$ carrying probabilities $P_i^{(n)}$ follows from the implicit relation [2]

$$\sum_i P_i^{(n)q} l_i^{(n)-\tau(q)} \sim 1, \tag{43}$$

where $\tau(q) = (q-1)D_q$.

Next, we address the question whether the generalized dimensions of the associated probability measures can be related to the spectrum of cancellations exponents. Situations with different truncation procedures will be treated separately.

(i) *Truncation at given stage.* Let $\overline{\mu}^{(0)}$ denote the associated probability measure, where the superscript "0" is used to distinguish the case of truncation at a given stage. Say the truncation level is denoted n_*. Now consider the probability measure $\bar{\mu}_j^{(0)(n)}$ of an interval $I_j^{(n)}$ at some level $n < n_*$,

$$\bar{\mu}_j^{(0)(n)} = \frac{\sum_i{}^j |\mu_i^{(n_*)}|}{\sum_i |\mu_i^{(n_*)}|},$$

where $\mu_i^{(n_*)} = \mu(I_i^{(n_*)})$, $I_i^{(n_*)}$ denotes an interval generated at the cutoff level n_*, and the numerator sum $\sum_i{}^j$ is only over those values of i for which $I_i^{(n_*)} \subset I_j^{(n)}$. From the thermodynamic relation (42) taken at $q = 1$, $\beta = 0$, the denominator in the above relation scales as $\exp(F(1,0)n_*)$. We expect that the numerator is proportional to the magnitude of the sign singular measure $\bar{\mu}_i^{(0)(n)}$ of the interval $I_j^{(n)}$ and the proportionality coefficient is a function of $n_* - n$ only:

$$\sum_i{}^j |\mu_i^{(n_*)}| = \bar{\mu}_j^{(0)(n)} f(n_* - n). \tag{44}$$

Obviously $f(m)$ must be unity for $m = 0$. Take a level n which is large but much below n_*: $1 \ll n \ll n_*$. Assuming effective self-similarity, the sum on the left hand side grows like the partition sum (42) taken at $q = 1, \beta = 0$, i.e., $f(m) \sim \exp[F(1,0)m]$. Thus, the probability measure $\bar{\mu}_j^{(0)(n)}$ of subinterval i at level n $(1 \ll n \ll n_*)$ is

$$\bar{\mu}_j^{(0)(n)} \sim \exp[-F(1,0)n] \mid \mu_j{}^{(n)} \mid, \tag{45}$$

where the quantity $\exp(-F(1,0)n)$ provides the necessary normalization for a probability measure.

Substituting this into relation (43), we find that the quantity $\tau^{(0)}(q)$ of measure $\bar{\mu}^{(0)}$ fulfills the implicit equation

$$F(q, \beta = -\tau^{(0)}(q)) = qF(1,0). \tag{46}$$

Thus, $\tau^{(0)}$ can only be computed with full knowledge of the bivariate free energy and does not follow from κ_q alone.

(ii) *Truncation by smoothing.* The associated probability measure will be denoted by a superscript "1", that is $\overline{\mu}^{(1)}$. According to the definition of κ, Eq. (2),

$$\sum_i |\mu(I_i)| \simeq |\mu(X)|(l_0/\epsilon)^{\kappa}, \tag{47}$$

where l_0 is the length of the interval X and ϵ is the size of the intervals I_i. Consider now the probability measure $\bar{\mu}_j^{(1)(n)}$ of an interval $I_j^{(n)}$ at some level $n < n_*$, such that all the interval lengths $l_j^{(n)}$ are greater than ϵ_*. By definition,

$$\bar{\mu}_j^{(1)(n)} = \frac{\sum_i{}^j |\mu(I_i^*)|}{\sum_i |\mu(I_i^*)|}, \tag{48}$$

where the numerator sum $\sum_i{}^j$ is only over those i for which $I_i^* \subset I_j^{(n)}$. We now write (48) as

$$\bar{\mu}_j^{(1)(n)} = \left\{ \frac{|\mu_j^{(n)}|}{\sum_i |\mu(I_i^*)|} \right\} \left\{ \frac{\sum_i{}^j |\mu(I_i^*)|}{|\mu_j^{(n)}|} \right\}, \tag{49}$$

where $\mu_j^{(n)} \equiv \mu(I_j^{(n)})$. The denominator in the first ratio in (49) can be estimated directly from (47),

$$\sum_i |\mu(I_i^*)| \simeq |\mu(X)|(l_0/\epsilon_*)^{\kappa}. \tag{50}$$

Assuming that the sign-singular measure is effectively self-similar, the numerator of the second ratio is (again from (47))

$$\sum_i{}^j |\mu(I_i^*)| \simeq |\mu_j^{(n)}|(l_j^{(n)}/\epsilon_*)^{\kappa}. \tag{51}$$

Combining (49)–(51), we have

$$\bar{\mu}_j^{(1)(n)} \simeq \frac{|\mu_j^{(n)}|}{|\mu(X)|} \left(\frac{l_j^{(n)}}{l_0} \right)^{\kappa} \sim |\mu_j^{(n)}| \, (l_j^{(n)})^{\kappa}, \tag{52}$$

where, in the last relation, we have dropped the constant factors $|\mu(X)|$ and l_0. Thus, truncation by smoothing leads to taking the modulus of the sign singular interval measures and ensuring normalization by multiplying this with power κ of the interval's length (cf. Eq. (29)).

From Eq. (43) it follows that the quantity $\tau^{(1)}(q)$ characterizing this measure fulfills

$$F(q, \beta = q\kappa - \tau^{(1)}(q)) = 0. \tag{53}$$

Since, however, $F(q, \kappa_q) = 0$, this can only hold if

$$\kappa_q = q\kappa - \tau^{(1)}(q). \tag{54}$$

We note that the same result also follows for sign singular measures without any obvious level structure. One can nevertheless associate a probability measure $\bar{\mu}^{(1)}(I_j)$ to any ϵ length interval of size longer than some cut-off scale ϵ_* via

$$\tilde{\mu}^{(1)}(I_j) = | \mu(I_j) | \epsilon^{\kappa},$$

and

$$\bar{\mu}^{(1)}(I_j) = \tilde{\mu}^{(1)}(I_j) / \sum_j \tilde{\mu}^{(1)}(I_j).$$

Note $\sum_j \tilde{\mu}^{(1)}(I_j)$ remains a finite number as ϵ goes to zero because of definition (2) of the cancellation exponent. By substituting this into the formula $\sum \bar{\mu}^{(1)}(I_j)^q \sim \epsilon^{\tau^{(1)}(q)}$, Eq. (55) is recovered.

In cases with truncations be smoothing, the generalized cancellation exponents can thus be expressed by the generalized dimensions of the associated probability measure $\overline{\mu}^{(1)}$. As a consequence, the multifractal spectrum $f(\gamma)$ of cancellation exponents is just a shifted version of the multifractal spectrum $f^{(1)}(\alpha)$ of probability measure $\overline{\mu}^{(1)}$:

$$f(\gamma) = f^{(1)}(\alpha)_{|\alpha=\gamma-\kappa}. \tag{55}$$

Since $f^{(1)}$ is shifted to the left by an amount of κ, the graph of $f(\gamma)$ intersects the diagonal.

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Appendix A. An example that is singular at a point

Let

$$\rho(x) = \sum_{n=2}^{\infty} f_n \delta(x - x_n), \tag{A.1}$$

where

$$f_n = (-1)^n n^{-a}, \quad a > 0, \tag{A.2}$$

$$x_n = n^{-b}, \quad b > 0, \tag{A.3}$$

and $\delta(x)$ denotes the delta function. We consider the quantity $\mu(S) = \int_S \rho(x)\,dx$ for subsets S of the interval (0, 1). For large n, the separation between two successive values of x_n is

$$\begin{aligned}\Delta x_n &= x_n - x_{n+1}\\ &\cong -dx_n/dn = bn^{-(b+1)}.\end{aligned} \tag{A.4}$$

Now divide the interval (0, 1) into subintervals $I_i = (i\epsilon, (i+1)\epsilon)$ and consider the quantity $\chi(\epsilon)$ into Eq. (3). We have from (A.4) and (A.3) that $\Delta x_n \gtrsim \epsilon$, if

$$n < n_\epsilon \equiv (b/\epsilon)^{(b+1)^{-1}}, \tag{A.5}$$

or

$$x > x_\epsilon \equiv n_\epsilon^{-b} = (\epsilon/b)^{b/(b+1)}. \tag{A.6}$$

We can write $\chi(\epsilon)$ in (3) as

$$\chi(\epsilon) = \sum_i{}' |\mu(I_i)| + \sum_i{}'' |\mu(I_i)|, \tag{A.7}$$

where the first sum is over intervals lying in $x > x_\epsilon$, and the second sum is over the remaining intervals. Noting that $\Delta x_n > \epsilon$ implies that at most one delta function falls in an interval, we have that

$$\begin{aligned}\sum_i{}' |\mu(I_i)| &= \sum_{n=2}^{n_\epsilon} |f_n| = \sum_{n=2}^{n_\epsilon} n^{-a}\\ &\sim \int_2^{n_\epsilon} \frac{dn}{n^a} = (1-a)\,[n_\epsilon^{(1-a)} - 2^{(1-a)}].\end{aligned}$$

Hence, for small ϵ,

$$\sum_i{}' |\mu(I_i)| \sim \begin{cases} n_\epsilon^{(1-a)} \sim \epsilon^{-\eta}, & \text{if } a < 1,\\ O(1), & \text{if } a > 1,\end{cases} \tag{A.8}$$

where $\eta \equiv (1-a)/(b+1)$. A similar, but slightly more elaborate, calculation shows that the second summation in (A.7) never exceed the estimates on the right hand side of (A.8). Hence (A.8) determining the ϵ scaling of $\chi(\epsilon)$,

$$\chi(\epsilon) \sim \begin{cases} \epsilon^{-\eta}, & \text{for } a < 1,\\ O(1), & \text{for } a > 1,\end{cases} \tag{A.9}$$

and the cancellation exponent is

$$\kappa = \begin{cases} (1-a)/(b+1), & \text{for } a < 1,\\ 0, & \text{for } a > 1.\end{cases} \tag{A.10}$$

Thus we see that $\kappa > 0$ if $a < 1$, even though $\rho(x)$ can be said to change sign on arbitrarily fine scale only at a single point, namely $x = 0$.

An analogous computation for the cancellation exponent spectrum κ_q yields

$$\kappa_q = \begin{cases} (1-aq)/(b+1), & \text{for } aq < 1, \\ 0, & \text{for } aq > 1. \end{cases} \quad \text{(A.11)}$$

Thus, for large enough q values ($q > q_c \equiv 1/a$) the generalized cancellation exponents become zero correctly reflecting that the singularity is at one single point only.

Eq. (A.1) involves delta functions. An essentially equivalent example using a function that is completely smooth in $x > 0$ is provided by

$$\rho(x) = \frac{1}{x^c} \cos \frac{\pi}{x^d}. \quad \text{(A.12)}$$

We can regard (A.12) as a sequence of positive and negative wiggles, analogous to the sequence of positive and negative delta functions in (A.1). The location of the center of the nth wiggle is $\pi x_n^{-d} = n\pi$, or

$$x_n = n^{-1/d}. \quad \text{(A.13)}$$

The width of the wiggle centered at x_n is $\Delta x_n \sim |dx_n/dn| = n^{-[(d+1)/d]}$. The amplitude of the wiggle centered at $x = x_n$ is $x_n^{-c} = n^{c/d}$. Thus the wiggle strength (i.e., amplitude multiplied by width) is

$$s_n \sim n^{-e}, \quad e \equiv (d+1-c)/d. \quad \text{(A.14)}$$

Identifying (A.13) with (A.3) and (A.14) with (A.2), we see that (A.10) also applies to this case, if we replace a by e and b by $(1/d)$.

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