ON THE DYNAMICS OF ENSEMBLE AVERAGES IN CHAOTIC MAPS

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The dynamics of mean values in two-dimensional Hénon-like maps and in their strongly dissipative limit is investigated by considering the direct product of n such maps represented in center-of-mass and relative coordinates. The existence of a well-defined mean value for $n \to \infty$ shows up by a nontrivial mechanism on the projection of the center-of-mass variable. Numerical simulations up to $n = 128$ suggest that the deviation between the actual mean value at large n and the limiting one obeys a gaussian distribution in the chaotic regime.

Deterministic dissipative systems exhibiting chaotic behaviour can be completely described only by statistical methods. One way of doing it is to investigate the time evolution of a certain initial probability distribution which for large times will approach a stationary density concentrated on the chaotic attractor [1,2]. If the dynamics is governed by a mapping (e.g. obtained from a Poincaré cross section) the probability distribution follows also a mapping [3]. Another, equivalent, description of such systems would require the knowledge of the mappings specifying the dynamics of all possible cumulants of the variables. The averaging should be taken over a large number of trajectories.

We shall investigate here the dynamics of one of the most important mean values, on the first cumulant, obtained after averaging over n trajectories. The relevance of this quantity is related to the fact that in a realistic experiment one always deals with averaged values deduced from a certain number of different measurements on the same system. Our aim is to describe the dynamics by means of a map containing the mean value explicitly and to study how the approach to a well-defined limiting value shows up in this set of recursions when the number of independent trajectories tends to infinity. We shall see that this map has common features with other multidimensional maps investigated in the literature.

As a prototype we consider the quadratic map [1, 2] for n trajectories:

$$x'^i = 1 - a(x^i)^2, \quad i = 1, 2, ..., n, \tag{1}$$

with a positive control parameter a not larger than 2. The mean value is defined by

$$x = \sum_{l=1}^{n} x^l / n. \tag{2}$$

By deriving a map for x one finds that, in addition to the center of mass variable (2), $n - 1$ independent relative coordinates are to be introduced in order to have a closed system. In the case of four trajectories, for example, a particularly convenient choice is:

$$v_0^1 = \tfrac{1}{2}(x^1 - x^2 + x^3 - x^4),$$

$$v_1^1 = x^1 - x^3, \quad v_1^2 = x^2 - x^4, \tag{3}$$

and the map itself reads:

$$x' = 1 - ax^2 - \tfrac{1}{4}a(v_0^1)^2 - \tfrac{1}{8}a[(v_1^1)^2 + (v_1^2)^2],$$

$$v_0'^1 = -2axv_0^1 + \tfrac{1}{4}a[(v_1^2)^2 - (v_1^1)^2],$$

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$$v'^{i}_{1} = -2axv^{i}_{1} + (-1)^{i}av^{1}_{0}v^{i}_{1}, \quad i = 1, 2. \tag{4}$$

This shows that the dynamics of the mean value is strongly coupled to that of the relative coordinates. Therefore, by investigating the average over n trajectories, an irreducible n-dimensional map must be considered. When constructing this map for arbitrary values of n one notices that it is convenient to study cases $n = 2^N$ with N positive integer. The hierarchy of the relative coordinates may then be given by

$$v^{i}_{j} = \sum_{l=0}^{2^{N-j}-1} (-1)^{l} x^{2^{j}l+i} 2^{j-N+1},$$
$$0 \leqslant j \leqslant N-1, \quad 0 < i \leqslant 2^{j}. \tag{5}$$

The map for x, v^{i}_{j} can be expressed in a rather compact form:

$$x' = 1 - ax^2 - a \sum_{m=0}^{N-1} 2^{-2-m} \sum_{l=1}^{2^m} (v^{l}_{m})^2, \tag{6a}$$

$$v'^{i}_{j} = av^{i}_{j}\left(-2x - \sum_{l=0}^{j-1} \overline{v}^{i}_{j}\right)$$
$$- a \sum_{m=1}^{N-1-j} 2^{-1-m} \sum_{l=0}^{2^m-1} (-1)^{l} (v^{2^{j}l+i}_{j+m})^2. \tag{6b}$$

In order to make the notation clearer we have introduced in the first term of the right-hand side of (6b) the 2^{N-1} component vectors $\overline{v}_j$ ($j = 0, 1, ..., N-1$). They are constructed from v_j by repeating the elements v^{i}_{j} with an appropriate sign rule:

$$\overline{v}^{i+2^{j}k}_{j} = (-1)^{k} v^{i}_{j},$$
$$k = 0, 1, ..., 2^{N-j-1} - 1, \quad 0 < i \leqslant 2^{j}. \tag{7}$$

Note that the recursion for the last block of variables v^{i}_{N-1} does not contain inhomogeneous terms. Therefore, by choosing p of them to be zero, they remain unchanged describing a case where p of the n trajectories coincide. We shall always use initial conditions for which the motion remains bounded.

The map (6) is identical to the direct product of 2^N quadratic maps (1) represented in the collective coordinates defined by (2) and (5). The motivation for using the form (6) is the fact that we are mainly interested in the dynamics of the mean value x and perhaps of a few other "macroscopic" relative coordinates, like v^{1}_{0}, which can be considered to be averaged values, too. Therefore, we investigate only certain projections of the product map. Owing to this restriction there is a loss of information and our aim is just to deduce the behaviour which can be observed on these projections.

We have simulated the map (6) at a fixed value of a with n up to $n = 128$. Figs. 1–3 show two projections of the chaotic attractor of this multivariable map obtained at different values of n. In order to be able to interpret the map as a limit of a more complicated one, like that associated with Hénon's map [4], we have introduced also an auxiliary variable y^i via the dynamics $y'^{i} = x^{i}$. Figs. 1a–3a show the projection on the x, y plane where y is the mean value of y^i, i.e., $y = \Sigma^{n}_{l=0} y^{l}/n$, and figs. 1b–3b give the projection on the (x, v^{1}_{0}) plane.

Fig. 1 displays the strange attractor obtained for $n = 2$ at $a = 1.9$. Both projections cover a piece of the plane since the direct product of two logistic maps has a strange attractor of dimensionality 2. The system is hyperchaotic [5] possessing two positive Lyapunov exponents. The dots in fig. 1a represent the centers of mass of different pairs of trajectories from the chaotic attractor of the quadratic map. Therefore, the upper borderline of the object in fig. 1a is an arch of the parabola $x = 1 - ay^2$ for $1 - a \leqslant y \leqslant 1$, while the lower border consists of two similar arches, reduced by a factor two. The borderlines of the projection shown in fig. 1b are curves specifying the largest possible difference v^{1}_{0} between two points from the interval $[1 - a, 1]$ having the mean value x. The results obtained for averaging over $n = 16$ trajectories can be seen in fig. 2. We notice the significant difference in the density of dots for $n = 2$ and 16 on the projections of the strange attractor. The parabola arch mentioned above is shown in fig. 2a as well in order to indicate that this segment of curve is the upper borderline of the projection in the present case, too, although it is being approached quite rarely. (The parabola is in fact an invariant curve of the (x, y) plane associated with n identical trajectories.) The lower borderline now consists of 16 small parabola arches. The tendency observed in fig. 2 becomes more pronounced for $n = 128$ shown in fig. 3. The dots obtained with typical initial conditions are here concentrated on small spots in both projections inspite of

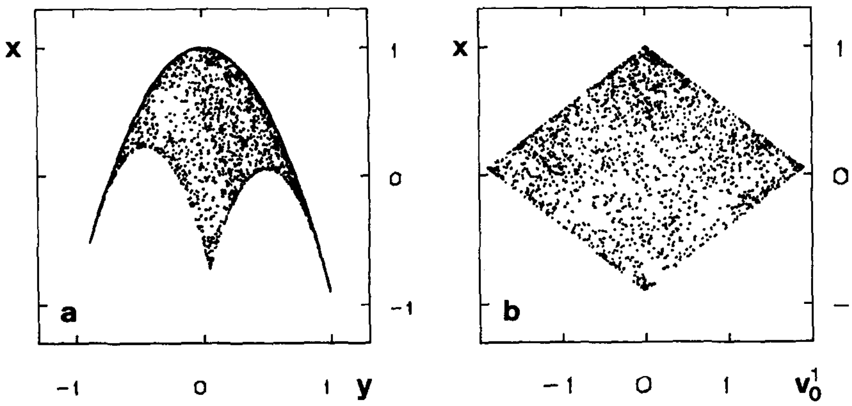

Fig. 1. Projections of the strange attractor of the map (6) with $a = 1.9$, $n = 2$ obtained by plotting dots between the 500th and 2500th iterates. Initial data: $x = 0.8$, $v_0^1 = 10^{-3}$.

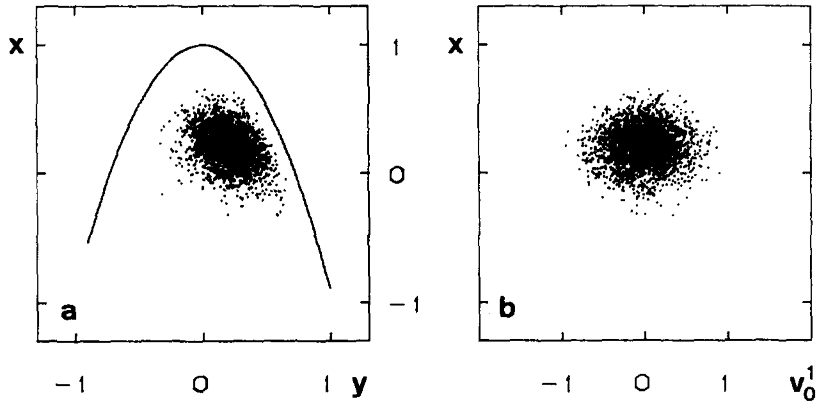

Fig. 2. The same as fig. 1 with $n = 16$. Initial data $x = 0.8$, $v_j^i = (j + 1) \times 10^{-3}/i$.

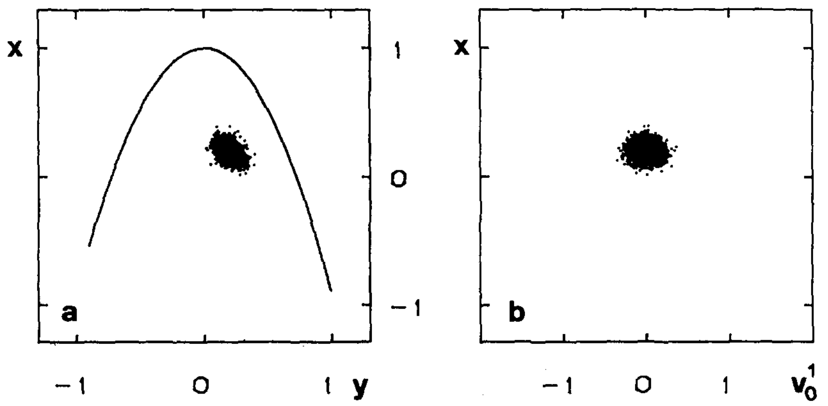

Fig. 3. The same as fig. 2 with $n = 128$.

the fact that there exists an extended 128-dimensional strange attractor in the complete phase space. Note that with untypical initial conditions other strange attractors can be reached which may show up with different shapes on the (x, y) and (x, v_0^1) planes. By choosing, for example, several of the coordinates v_{N-1}^i to be zero much larger spots would appear on the projections or, with the extreme case of 128 identical trajectories the parabola arch shown in fig. 3 would be reached. These objects, however, are attractive on certain hypersurfaces only. Typical initial points approach the most compact attractor on the (x, y) plane.

Our numerical observations for cases between $n = 16$ and 128 suggest that the linear size of the most densely filled region, reached with typical initial conditions, decreases approximately like $n^{-1/2}$. Thus, by assuming ergodicity, the deviation between the mean value x taken over n ($\gg 1$) trajectories and the limiting one, x^* obtained for $n \to \infty$, can be characterized by a gaussian distribution after transients have died out. This is, in fact, supported by numerical simulations of (6) (see fig. 4) illustrating that the central limit theorem may be valid for independent variables with chaotic dynamics. The variance, b^2, of this distribution is given by $b^2 = C_0/n$, where C_0 denotes the second cumulant of the one-dimensional dynamics in the stable stationary state. With the data of fig. 4, for instance, $b = 0.054$ and therefore $C_0 = 0.37$. Note, that for $a = 2$ an exact result is known: $C_0 = 0.5$ [6]. Moreover, we may conjecture that eq. (6a) with typical initial conditions can be replaced after many iterations and for large n by an equation describing the relaxation toward the limiting value x^* with gaussian fluctuations. This specifies then the reduced dynamics for the macroscopic variable x in analogy with a Langevin type description of thermodynamic systems. With a further increase of n the intensity of the deterministic noise decreases in this equation. The fluctuations, of course, vanish at control parameter values where the quadratic dynamics (1) exhibits nonchaotic behaviour. For the dynamics of other macroscopic variables, like v_0^1, similar properties are expected as those just discussed for the mean value.

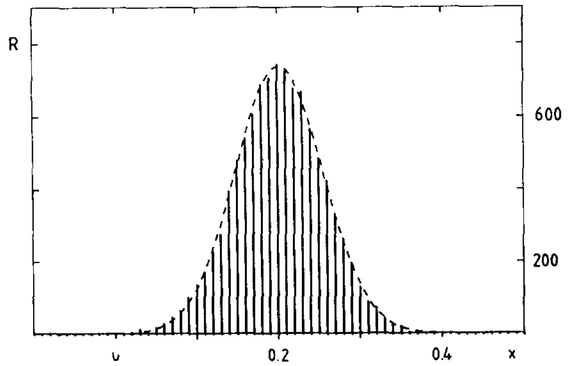

Fig. 4. Histogram showing the number R of dots falling in cells of size $\Delta x = 1/100$ for iterates between 100 and 10500 of the map (6); $a = 1.9$, $n = 128$, initial data as in fig. 3. The dashed line represents a gaussian curve fitted to the histogram.

It is also of interest to investigate regions where the one-dimensional map possesses a multi-piece chaotic attractor or a nonchaotic periodic attractor. The general features can be best understood on the example of the stable period-two orbit. In the $n = 2^N$-dimensional product map (6) then 2^{n-1} different attracting trajectories are present. However, on the (x, y) plane only $n/2 + 1$ different attractors can be observed. One of them is realized by n identical one-dimensional trajectories. The corresponding relative variables vanish in this case. In general, the mth period-two attractor on the (x, y) plane of the n-dimensional map can be constructed by $n - m$ one-dimensional trajectories starting at one of the periodic points and m other ones at the other periodic point. By increasing m, the value of the corresponding relative coordinates typically increases and the difference between the x-values decreases. The attractor with $m = n/2$ realized by $n/2$ one-dimensional trajectories at both periodic points appear then as a fixed point on the (x, y) plane. The "stength of attraction" of these objects may be estimated by the number of different ways they can be constructed from one-dimensional trajectories. This is obviously given by a binomial distribution with a maximum at $m = n/2$. Therefore, the fixed point should have the largest basin of attraction in the (x, y) plane for large n. Similarly, at a control parameter value where the one-dimensional map possesses a multi-piece chaotic attractor (or stable higher periodic orbits) may coexisting multi-piece chaotic attractors (or periodic orbits) appear on the (x, y) plane of the map (6) at any finite value of n. For large n, however, the most symmetric attractor, realized by the largest number of one-dimensional trajectories, dominates. This is a one-piece object on the (x, y) plane in accordance with the fact that there exists a single mean value x^* at any control parameter value for $n \to \infty$. The deviation between the actual mean value, obtained at a fixed large n, and x^* obeys in the chaotic regime again a gaussian distribution the variance of which typically decreases by decreasing the control parameter a.

One can easily include the effect of weak random noise in the map (6). We start then with a noisy quadratic recursion having a gaussian delta-correlated fluctuating force on the right-hand side of (1) with intensity $\sigma^2 \ll 1$. According to the definition of the collective coordinates (2), (5) new additive noise terms show up in the equations for x' and v'^i_j. It follows immediately that these fluctuating forces are mutually independent and that the noise intensity for x' is given by $\sigma^2 2^{-N}$ and that for v'^i_j by $\sigma^2 2^{2+j-N}$ if the map describes independent trajectories. Consequently, for large N the noise plays an explicit role for the last blocks of variables, for $j \gg 1$, only. Since the intensity is weak, an appropriate decorrelation of higher order cumulants becomes possible and the resulting map obtained for the mean values averaged over the noise behaves then in the (x, y) and (x, v_0^1) planes similarly to that shown in figs. 1–3. Noise terms assure that the second and eventually higher cumulants of v^i_j are bounded from below by $\sigma^2 2^{2+j-N}$. Moreover, the presence of noise obviously reduces the size of the basin of attraction for extended attractors on the (x, y) plane. The reason is that these objects are characterized by small values of the relative coordinates which cannot be maintained in a noisy system.

A cumulant expansion can be performed also for one single trajectory. In the presence of weak noise one may keep the recursion for the mean value (taken over the noise) and the second cumulant [7,8]. This method is valid in a time interval in which the deviation between trajectories with and without noise remains small. It may, therefore, successfully simulate [8] the noisy bifurcation diagram [9]. For describing the approach to the ensemble average it is most convenient to take an additional averaging over the initial conditions which would lead to a map similar to that we just discussed.

Let us now shortly mention possible generalizations. For the sake of simplicity we turn back to the deterministic case. Maps for the mean value of higher powers of the variable or for its correlation function can be constructed and would lead to similar conclusions for the behaviour at large n as those obtained for (6). The investigation of the map $x'^i = f(x^i)$ where f is a polynomial is also possible along similar lines. As a next step, one may include the effect of a second variable y^i by considering a class of maps in the form:

$$x'^i = f(x^i) + by^i, \quad y'^i = x^i, \quad i = 1, 2, ..., n. \tag{8}$$

Besides the mean value y, relative coordinates z^i_j of the variables y^i are to be introduced in complete analogy with (5). Since y^i appears linearly in the recursion for x^i the map for the collective coordinates will read:

$$x' = (x')_0 + by, \quad y' = x,$$

$$v'^i_j = (v'^i_j)_0 + bz^i_j, \quad z'^i_j = v^i_j, \tag{9}$$

where quantities with subscript zero denote the recursions associated with the one-dimensional map obtained in the limit of extremely strong dissipation ($b = 0$). Eqs. (9) describe, for example, the dynamics of the mean values x and y in the Hénon map if $(x')_0$, and $(v'^i_j)_0$ are given by the right-hand side of (6a) and (6b), respectively. The borderlines of the strange attractor in the (x, y) plane for $n = 2$ will then be related to certain branches of the Hénon attractor. Apart from changes like this, however, the conclusion concerning the approach to a fixed point for large n is identical to those discussed for the case $b = 0$. Therefore we have limited ourselves to illustrate only the latter case having in mind clearness rather than completeness of the presentation.

Finally, we emphasize that the map (6) may be considered as a starting point when considering coupled logistic maps. The case of two such maps, which has been investigated extensively [10–15], is a perturbation of (6) with $n = 2$. In the study of a ring of coupled logistic maps [16] the Fourier components play a role similar to our collective coordinates. Therefore, some properties of these maps, like high dimensionality of the strange attractor or the coexistence and multistability of attractors, can be traced back to the properties of the product map. More generally, when dealing with dissipative maps of several variables, they may turn out to be close to the direct product of simple low-dimensional maps. By this we mean that the coefficients of the map only slightly differ from those of the product map represented in certain reference frame. The knowledge of typical features of the product maps may thus be also useful in studying the behaviour of high-dimensional mappings.

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