# CHARACTERISTIC EXPONENTS OF CHAOTIC REPELLERS AS EIGENVALUES

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A new type of eigenvalue problem is introduced whose solution provides a spectrum of characteristic exponents for chaotic repellers or semi-attractors. In explicitly tractable examples this spectrum coincides with that of the generalized dimensions. The eigenvalues form a continuum of fractal measures with a smooth density along unstable directions.

It has recently been observed that strange sets covered by a probability distribution possess in general a whole spectrum of characteristic exponents: generalized dimensions [1–4] or, equivalently, scaling indices [5–7]. A multifractal analysis reflecting this feature has already been applied to the problems of fully developed turbulence [5,6], diffusion limited aggregates [8], certain random systems [9] and chaotic attractors [7,10,11].

Besides chaotic attractors, however, there exist other important invariant strange objects in dynamical systems. These are the so-called chaotic repellers [12], or if they are partially attracting, semi-attractors [13], responsible for transient chaotic behaviour (see e.g. refs. [12–14] and references therein). The concept of transient chaos and of the related phase-space object is of great interest on its own right but it may also play a role in other fields of physics, like e.g. the theory of disordered systems, where invariant Cantor sets have been found to show up [15].

In this paper we point out, by extending the results of refs. [16] and [17], that characteristic exponents of chaotic repellers or semi-attractors appear as eigenvalues of certain linear equations. This fact may provide a simple method for determining the complete spectrum of the exponents.

Let us consider first chaotic repellers generated by one-dimensional maps $x'=f(x)$, where $f$ is a single humped function. It was shown in [16] that the equation

$$P_{n+1}(x')=\sum_{x\in f^{-1}(x')}\frac{P_n(x)}{|f'(x)|^{D_0}} \tag{1}$$

has a unique nontrivial limiting distribution $P(x)=\lim_{n\to\infty}P_n(x)$, only if $D_0$ is the fractal dimension of the repeller. $P(x)$ is then the stationary distribution on the coarse grained repeller determined by a finite resolution. Furthermore, it is known [17,18] that there exists another iteration scheme, namely

$$C_{n+1}(x')=q\sum_{x\in f^{-1}(x')}\frac{C_n(x)}{|f'(x)|}, \tag{2}$$

which converges to a finite $C(x)>0$ for $q=\exp(\alpha)$ only, where $\alpha$ is the escape rate. The latter quantity characterizes an exponential decay in the number of points which have not yet escaped a neighbourhood of the repeller [12]. The function $C(x)$ is the density of the so-called conditionally invariant measure [18].

Eqs. (1) and (2) suggest that there are two qualitatively different ways of compensating the escape from a repeller. Both equations arise by dividing the right-hand side of the Frobenius–Perron equation [19] (which describes the outflow of probability from a neighbourhood of the repeller) by certain factors. In the first case the local escape factor [16] $\exp[-\alpha(x)]\equiv|f'(x)|^{D_0-1}$ is to be applied, while in the second case the global escape factor $\exp(-\alpha)$. It seems to be plausible that a simultaneous compensation of local and global escape is also possible and leads to a set of new eigenvalue problems. In fact, as suggested by numerical investigations, for any positive number $R$ there exists one single exponent $E(R)$ so that the equation

$$\tilde{Q}^{(R)}_{n+1}(x') = R \sum_{x \in f^{-1}(x')} \frac{\tilde{Q}^{(R)}_n(x)}{|f'(x)|^{E(R)}} , \quad 0<R<\infty , \tag{3}$$

possesses a unique nontrivial limit solution $\tilde{Q}^{(R)}(x) = \lim_{n\to\infty} \tilde{Q}^{(R)}_n(x)$ for the class of smooth positive initial functions $\tilde{Q}^{(R)}_0(x)$. The limit $\tilde{Q}^{(R)}(x)$ will also be smooth. The existence of eq. (3) and of the spectrum $E(R)$ are the main results of this paper. As we shall see, analogous statements can be found for higher dimensional maps, too.

In what follows, another representation of the spectrum will be used which makes the relation between eqs. (3) and (2) more explicit. Namely, we write

$$R \equiv \exp(\alpha r) , \quad E(R) \equiv r + (1-r) D(r) , \tag{4}$$

which defines the characteristic exponents $D(r)$ for $-\infty < r < \infty$. The corresponding functions appearing in the eigenvalue problem (3) will then be denoted by $Q^{(r)}_n(x)$. It is obvious that $D(0) = D_0$, and $Q^{(0)} \equiv P$, $Q^{(1)} \equiv C$.

As an explicit example, we consider the map

$$\begin{aligned} f(x) &= 1 - a_1 x , \quad x > 0 , \\ &= 1 + a_2 x , \quad x < 0 , \end{aligned} \tag{5}$$

where $a_1$, $a_2 > 1$ and $a_1^{-1} + a_2^{-1} < 1$ so that a chaotic repeller exists. The evolution of a linear initial function $Q^{(r)}_0(x) = \gamma^{(r)}_0 x + \beta^{(r)}_0$ can be followed exactly. The result is of the same type: $Q^{(r)}_n(x) = \gamma^{(r)}_n x + \beta^{(r)}_n$. A nontrivial limiting function for $n \to \infty$ can be reached only if $\beta^{(r)}_n \to 0$ which requires the exponent $D(r)$ to be the solution of the equation

$$e^{\alpha r} \left( a_1^{-r+(r-1)D(r)} + a_2^{-r+(r-1)D(r)} \right) = 1 , \quad -\infty < r < \infty . \tag{6}$$

It is easy to see then that all $\gamma^{(r)}_n$ tend to zero (exponentially fast). Consequently, the $Q^{(r)}(x)$ are constant. The stability of this solution in the space of nonlinear smooth functions can be checked numerically. For $r=0$ and $r=1$ the results for the fractal dimension [16] and for the escape rate [17], respectively, are recovered from (6). Moreover, since the stationary distribution $P(x)$ is constant, the spectrum of generalized dimensions $D_q$ [2,3] can be directly calculated for the coarse grained repeller and one finds that $D_{q=r}$ coincides, for all $r$, with the eigenvalue $D(r)$ of eqs. (3), (4).

At present, it is an open question whether the spectra $D(r)$ and $D_{q=r}$ are identical also in more general cases. The following preliminary result at least does not contradict this possibility. We have investigated the quadratic map $x' = 1 - \alpha x^2$ in the region $a > 2$. By using the data obtained in ref. [17] for the escape rate and considering $D(2)$ to be a free parameter, eq. (3) was iterated for $r=2$ with a constant initial function. As long as $D(2)$ was too small (large) $Q^{(2)}_n(x)$ monotonously decreased (increased) with $n$ at a fixed $x$. If, however, $D(2)$ was appropriately chosen, a rapid convergence was found (just like in the previous example). We used this fact to obtain a lower and upper bound for $D(2)$ as the values where $|Q^{(2)}_6(x) - Q^{(2)}_5(x)| < \epsilon$ with a small $\epsilon$. A comparison of the result obtained for $D(2)$ with the correlation exponent $D_2$ [2,3,20] calculated by a direct numerical method [21] shows agreement within computational error (see table 1). Furthermore, it has been found that $D(r)$, as well as $D_{q=r}$, varies extremely slowly with $r$ in the region $0 \leqslant r \leqslant 2$.

We now investigate invertible maps of the plane $\boldsymbol{x}' = T(\boldsymbol{x})$ producing chaotic transients. At nearly all points of chaotic trajectories a stable and an unstable direction exists [22,23]. Let $\lambda_1(\boldsymbol{x})$ denote the local coefficient of expansion [24] along the unstable direction. This quantity plays the same role now as $f'(x)$ in one-dimensional cases. The generalization of eqs. (3), (4) can be most conveniently formulated in terms of certain measures, $\mu^{(r)}$, which we call $r$-measure (the analogue of $\int^x Q^{(r)}_n(\bar{x})\, d\bar{x}$), and has the form

$$(\mu^{(r)}_j)' = e^{\alpha r} T\left(\mu^{(r)}_j |\lambda_1(\boldsymbol{x}_j)|^{(1-r)(1-D^{(1)}(r))}\right) , \quad -\infty < r < \infty . \tag{7}$$

Here, $\mu^{(r)}_j$ stands for the $r$-measure of a tiny region $B$ around a point $\boldsymbol{x}_j$ lying in the neighbourhood of the semi-attractor, and the induced map in the space of measures has been denoted by $T$, too. The new $r$-measure $(\mu^{(r)}_j)'$ belongs to the region $T(B)$ around $\boldsymbol{x}'_j$. The transformation is then to be repeated after

Table 1
The exponent $D(2)$ as obtained from the eigenvalue problem (3), (4) in comparison with the correlation dimension $D_2$ calculated numerically in ref. [21]. The error for both types of data is: $\pm 5\times 10^{-3}$.

| $a$ | 2.01 | 2.02 | 2.03 | 2.04 | 2.05 | 2.06 | 2.07 | 2.08 | 2.09 | 2.10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $D(2)$ | 0.960 | 0.942 | 0.926 | 0.915 | 0.905 | 0.894 | 0.885 | 0.877 | 0.870 | 0.863 |
| $D_2$ | 0.955 | 0.939 | 0.926 | 0.911 | 0.905 | 0.890 | 0.885 | 0.876 | 0.869 | 0.863 |

dividing the support of the new $r$-measure into areas where $\lambda_1(x)$ can be regarded as constant. A nontrivial limit will be approached only at a single value of $D^{(1)}(r)$ for a given $r$. The superscript 1 is to express the fact that the spectrum belongs to the unstable direction along which escape occurs. In the special case $r=0$, $\mu^{(0)*}$ obtained for $n\to\infty$ is the natural measure defining the stationary distribution on the coarse grained semi-attractor, and $D^{(1)}(0)$ becomes the partial fractal dimension along the unstable direction [16]. The iteration for $r=1$ leads to the conditionally invariant measure [17,18].

In order to give a nontrivial example we consider a generalised version of the Baker transformation [22] introduced in ref. [16]. The dynamics is defined by

$$\begin{aligned} y' &= sy\,, & y&<c\,,\\ &= -t(1-y)\,, & y&>c\,,\\ x' &= ax\,, & y&<c\,,\\ &= 1/2+bx\,, & y&>c\,, \end{aligned} \tag{8}$$

where $0<a$, $b$, $c<1/2$ and $sc$, $t(1-c)>1$. The latter condition ensures escape along the unstable $y$ direction. Starting with the Lebesque measure on the unit square an application of eq. (7) yields the $r$-measures

$$\sigma(r)=\mathrm{e}^{\alpha r}s^{-r+(r-1)D^{(1)}(r)}$$

and

$$\tau(r)=\mathrm{e}^{\alpha r}t^{-r+(r-1)D^{(1)}(r)} \tag{9}$$

on the strips $0<x<a$, $0<y<1$ and $1/2<x<1/2+b$, $0<y<1$, respectively. After $n$ steps the total $r$-measure on the unit square is then $[\sigma(r)+\tau(r)]^n$. The condition for a nontrivial limit is therefore

$$\sigma(r)+\tau(r)=1\,, \tag{10}$$

which specifies the spectrum $D^{(1)}(r)$, $-\infty<r<\infty$. The limiting measures obtained for $n\to\infty$ all have a smooth, constant, density along the $y$ direction. Among them the natural measure $\mu^{(0)*}$ is of special importance since its restriction to the coarse grained repeller yields the stationary distribution on this strange set. Calculating the generalized dimensions of this stationary distribution, we find that $D^{(1)}(r)=D^{(1)}_{q=r}$, where $D^{(1)}_q$ is the partial order-$q$ dimension [23] along the unstable direction.

Furthermore, all limiting measures $\mu^{(r)*}$ possess their own spectrum of generalized dimensions along the stable $x$ direction. So, to any $r$ belongs a set of partial dimensions $D^{(2)}_q(r)\, -\infty<q<\infty$. They are found to be the solutions of

$$[\sigma(r)]^q a^{(1-q)D^{(2)}_q(r)}+[\tau(r)]^q b^{(1-q)D^{(2)}_q(r)}=1\,. \tag{11}$$

Note that $D^{(2)}_q(r)$ depends strongly on $r$, and implicity (cf. eq. (10)) also on $D^{(1)}(r)$.

This example illustrates that there exists a whole range of measures related to chaotic semi-attractors. They appear as eigenfunctions of the eigenvalue problem (7). All measures are smooth along the unstable direction but show fractal features along the stable one.

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