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Noise-induced attractor explosions near tangent bifurcations

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Abstract

In dynamical systems with periodic attractors which have just emerged from a saddle-node bifurcation the addition of weak noise may induce chaotic behaviour. This is accompanied by two remarkable observable effects: noise-induced attractor explosion and noise-induced intermittency. The theory of quasipotentials is used to explain qualitative and universal aspects of these phenomena. The influence of the noise-distribution and its observational implications are discussed.

1. Introduction

The influence of small random perturbations on dynamical systems is most conspicuous near bifurcation values of a control parameter. While an unperturbed system changes its long-time behaviour exactly when the control parameter passes a bifurcation value, a similar change can already be induced before by random perturbations ("noise"). This phenomenon has been the subject of many investigations, beginning with Refs. [1,2].

One way of describing the effect of noise is to introduce the notion of a *noisy attractor* (see Section 2). As the strength of the noise is increased in a system shortly before a bifurcation, drastical changes in the structure of the noisy attractor may occur. This can be either a gradual melting of different attractors or different parts of a disconnected attractor, or – more spectacularly – an abrupt increase of the noisy attractors' volume. We call the later a *noise-induced attractor explosion*. This is due to the presence of a *nonattracting chaotic set* (chaotic saddle or repeller) outside the deterministic attractor(s). In the noise-free dynamics, the nonattracting set is responsible for *transient chaotic* behaviour [3] and does not influence at all the asymptotic motion of typical trajectories. In the presence of noise, however, trajectories can come close to the nonattracting chaotic set and, at sufficiently strong noise, it becomes embedded into the noisy attractor, which is the cause of the attractor explosion.

Despite the explosive enlargement of the noisy at-

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tractors, the neighbourhoods of the confined unperturbed attractors are still of special importance for the noisy system: The system leaves these regions only for occasional excursions through the rest of the noisy attractor. Thus, noise-induced attractor explosions imply *noise-induced intermittency*. The most importance observable connected with intermittent behaviour is the characteristic time τ between the occasional excursions.

Noise-induced attractor explosions typically occur for parameter values in periodic windows inside the chaotic regime where a nonattracting set exhibiting fractal properties is always present. A period-m window is closed at its lower edge by a saddle-node (tangent) bifurcation where a stable period-m orbit is created. When increasing the control parameter, this attractor undergoes a sequence of bifurcations which leads finally to the appearance of a multi-piece chaotic attractor. The window is closed at its upper edge by an "interior" crisis [4]: the chaotic attractor is suddenly enlarged resulting in a strange attractor consisting of one large piece. This crisis is mediated by the collision of the multi-piece attractor with the stable manifold of the nonattracting chaotic set.

The influence of noise on a system shortly *before a crisis* has been studied in Refs. [5,6], leading to the notion of a noise-induced crisis. Clearly, when studying the influence of noise, universal aspects which are valid for large classes of systems including realistic models of physical processes are particularly interesting. A universal scaling law for the characteristic time τ was established by Sommerer et al. in Ref. [6]. This scaling law is in agreement with experimental data on a periodically driven magnetoelastic ribbon [7].

Recently, it was proposed by Sommerer [8] to refine the scaling law found in Ref. [6] by taking into account results from Beale [9] about the asymptotics of τ for *weak* noise. Those results, which were obtained using the so-called principle of minimum escape energy, were conjectured in Ref. [9] to bear universal aspects, supported by former numerical investigations by Kautz [10]. In Section 2 we shall explain how the theory of quasipotentials or nonequilibrium potentials [11–18], which forms the framework of the principle of minimum escape energy, can be used to derive such universal aspects. In particular we shall substantiate the conjecture made by Beale [9].

In the present paper we shall put the study of noisy systems *near a saddle-node bifurcation* into the foreground in order to complement the existing literature on noise-induced crises. In addition, by the results presented here we hope to encourage experimental study similar to Refs. [7,8] for systems near a tangent bifurcation. In such cases the effect of noise-induced attractor explosions is more drastical than near crises: The explosion turns a *regular* period-m attractor into a *chaotic* one. This effect of noise has been discussed on a phenomenological basis in Refs. [19–21] among others under the label *noise-induced chaos*. Thus, the present paper deals with universal aspect of noise-induced chaos.

Quasipotential methods, however, give only an asymptotic description, and so the results of Section 2 make predictions for numerical or real experiments only if the asymptotic regime of weak noise has been reached, which may be difficult to ascertain in practice. We discuss this question in Section 3. In addition, we comment on the relation of our results to a scaling law for τ near tangent bifurcations found by Eckmann et al. [22]. This relation is similar to the one between the quasipotential results for crises and the scaling law of Ref. [6], on which the above-mentioned refined scaling law proposed in Ref. [8] is founded. Therefore Section 3 also sheds additional light on that proposal.

2. Quasipotentials and noisy attractors

The discrete-time version of the so-called quasipotential method [13–18] deals with noise systems defined by the stochastic difference equation

$$X_{n+1}^{(\sigma,r)} = F(X_n^{(\sigma,r)}) + \xi_n^{(\sigma,r)} . \tag{1}$$

Here, the random sequence $(X_n^{(\sigma,r)})$ on some bounded d-dimensional state space is the outcome of the iteration of a function $F(x)$, describing the deterministic part of the dynamics, which in the nth time step is perturbed by the addition of a random variable $\xi_n^{(\sigma,r)}$ with zero mean and standard deviation σ (which can be used to measure the strength of the noise). The random variables are assumed to be independent for different time steps ("white noise"). Their probability density $\psi^{(\sigma,r)}(\xi)$ (possibly state dependent) has to fulfill the following asymptotic condition for weak noise, which involves a parameter $r>1$,

$$\psi^{(\sigma,r)}(\xi) \underset{\sigma\to 0}{\asymp} \exp(-|\xi|^r/r\sigma^r)\,, \tag{2}$$

where we introduced an abbreviated notation for asymptotic logarithmic equality,

$$A(\epsilon) \underset{\epsilon\to 0}{\asymp} B(\epsilon) \quad \text{means} \quad \lim_{\epsilon\to 0}\frac{\ln A(\epsilon)}{\ln B(\epsilon)}=1\,.$$

Condition (2) means that (even for small noise strength σ) the random perturbations can be arbitrarily large, but that the probability for large perturbations decreases exponentially.

The most important choice for r in (2) is $r=2$, because this corresponds to a Gaussian distribution of the perturbations. However, it is interesting to study the case $r\neq 2$ in order to reveal the influence of the shape of the noise distribution. The limit $r\to\infty$ is of special interest since it leads to a uniform box distribution of the random perturbations, which is often used in numerical experiments.

The stationary density $w^{(\sigma)}(x)$, which is forming under iteration of (1) starting from any initial distribution, is a basic characteristic of the noisy system's long-time behaviour. The so-called quasipotential $\Phi(x)$ gives (under rather mild conditions on $F(x)$, see Refs. [17,18]) the weak-noise asymptotics of $w^{(\sigma)}(x)$ on an exponential scale,

$$w^{(\sigma)}(x) \underset{\sigma\to 0}{\asymp} \exp[-\Phi(x)/\sigma^r]\,. \tag{3}$$

Conventionally, the quasipotential is normalized in such a way that its value is zero on the most stable attractor of the deterministic system.

The quasipotential satisfies an extremum principle, and the most instructive methods for determining quasipotentials make use of well-known ideas of Hamiltonian mechanics. This involves studying orbits of a canonical map $(p,q)\mapsto(P,Q)$ on a $2d$-dimensional phase space whose generating function is

$$\rho_r(q,Q)=\frac{1}{r}\,|Q-F(q)|^r\,, \tag{4}$$

i.e., $P=\partial_2\rho_r(q,Q)$ and $p=-\partial_1\rho_r(q,Q)$, where ∂_i denotes the derivative with respect to the ith argument. We refer to Refs. [15–18] for details, where also methods for the numerical computation of quasipotentials are given.

The quasipotential has local minima at the attractors of F, saddles at the saddles of F, and local maxima at the repellers of F. Saddles and repellers, however, are very often embedded in larger regions with constant (or scarcely varying) quasipotential, which we call *quasipotential plateaus*. In particular this is true for chaotic repellers or saddles with fractal structure. The qualitative reason is that the quasipotential must be constant on the entire nonattracting set, and it turns out that plateaus are then typically formed along unstable directions emanating from this set. For a discussion of examples we refer to Refs. [15–18], and further examples are given below. However, a detailed discussion of the formation of quasipotential plateaus lies beyond the scope of the present paper. But we shall explain how the existence of a quasipotential plateau can be used to predict a noise-induced attractor explosion.

The first step is to put in concrete terms the idea of noisy attractors as the sets of points to which the long-time behaviour of the system (1) is confined. If the stationary density $w^{(\sigma)}(x)$ were zero on large regions of the state space, it would be clear that the support of $w^{(\sigma)}(x)$ should be considered as the noisy attractor. However, the type of noise we are dealing with according to condition (2) leads to a nowhere vanishing stationary density. But we can use the argument that any observation of the stationary density has a *finite threshold of resolution* χ, so that we define, depending on that threshold, the noisy attractor of the system (1) as

$$\mathrm{A}_{\sigma,\chi}=\{x\colon w^{(\sigma)}(x)\geqslant\chi\}\,. \tag{5}$$

A different definition of noisy attractors has been proposed in Ref. [23] but it is more complicated and less appropriate for the quasipotential approach.

If the quasipotential of a system is known, but not its complete stationary density, the noisy attractor for sufficiently weak noise may be approximated by the idealised noisy attractor, defined as

$$\bar{\mathrm{A}}_{\sigma,\chi}=\{x\colon \Phi(x)\leqslant\sigma^r\ln\chi^{-1}\}\,. \tag{6}$$

The right-hand side of (6) would be equivalent to that of (5) if the asymptotic equality in (3) were an exact equality. Since this is not true, predictions derived on the level of idealised noisy attractors can deviate from those based on definition (5). We shall analyse this point in section 3, but for the moment adhere to the idealised viewpoint.

If there is a quasipotential plateau of height Φ_P, then there is, obviously, an abrupt increase in $\tilde{A}_{\sigma,\chi}$ as soon as the noise strength σ passes the critical value

$$\sigma_\chi^{cr} = \sqrt{\Phi_P/\ln \chi^{-1}}\,, \qquad (7)$$

because then the total region of constant quasipotential is added at once to the noisy attractor. So, from the idealised point of view, one expects a noise-induced attractor explosion at noise strength σ^{cr}. In order to be able to compare our statements below with other results, we mention that Φ_P coincides with the quantity called "minimum escape energy" in Refs. [10,9] for reasons connected with (17).

In Fig. 1 we give an example showing that the expectation of noise-induced attractor explosions in systems with quasipotential plateaus is qualitatively correct. This example is a two-dimensional map of the type described by McDonald et al. [24] with regard to disconnected fractal basin boundaries. The explicit expression of the map is not important here. We just remark that the map has two attracting fixed points at (0, −0.9) and (0.5, 0.9) and a fractal saddle with the typical Cantor set structure arising from stretching and folding. The unstable manifold is smooth, roughly S-shaped, but its cross-sections parallel to the y-axis exhibit fractal features. Fig. 1a shows that there is a quasipotential plateau which follows the S-shape of the unstable manifold, and also the fractal fine structure can be guessed at, despite the low numerical precision. Figs. 1b and 1c are the results of a numerical simulation of the noisy system and show that at some critical noise strength there is a noise-induced attractor explosion: Whereas in Fig. 1b the system remains confined to the neighbourhood of the attracting fixed points, there is a significant probability to find the noisy system of Fig. 1c at any point in the region of the quasipotential plateau, too.

Another, even simpler appearance of the same phenomenon can be found in the period-3 window of the family of logistic maps

$$F_\mu = 1 - \mu x^2\,, \qquad (8)$$

which extends over the parameter range $\mu_t = 1.75 < \mu < \mu_c = 1.7903\ldots$. As an example we show in

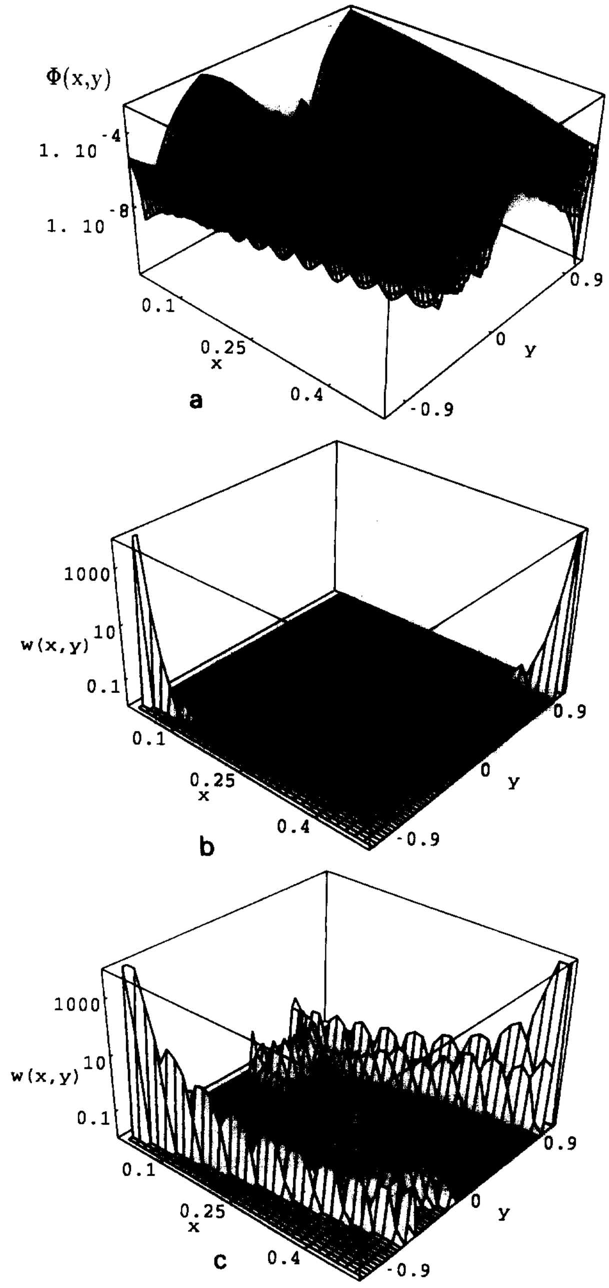

Fig. 1. (a) Quasipotential $\Phi(x, y)$ (for $r=2$) of a two-dimensional system with two fixed points and a Cantor saddle. Φ_P is approximately 4×10^{-4}. (b) Invariant density $w(x, y)$ of that system, perturbed by Gaussian noise with variance $\sigma^2=2\times10^{-6}$. (c) Same as (b), but with $\sigma^2=2\times10^{-5}$. In (b) and (c) the threshold value is $\chi=0.01$. The asymptotic region of weak noise has not yet been reached so that the idealised critical noise strength as given by (7) only leads to an order of magnitude estimate for the explosion.

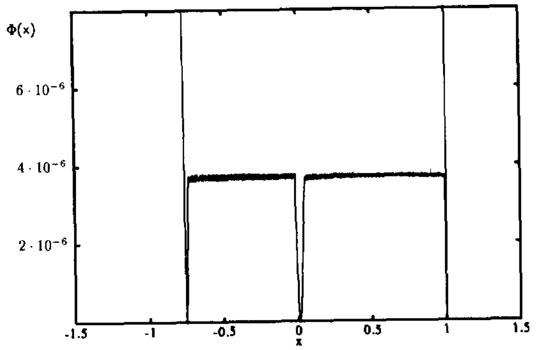

Fig. 2. Quasipotential for the logistic map (8) with $\mu=1.752$ and $r=2$.

Fig. 2 the numerically computed quasipotential for (8) with $\mu=1.752$, and $r=2$. The plateaus belong to the two shortest intervals that cover the repeller lying between the components of the period-3 attractor. The critical noise strength can be fairly well estimated via (7) by using the plateau value $\Phi_P \approx 3.7\times10^{-6}$.

In order to go beyond the realms of qualitative statements we address the question of how the conditions of a noise-induced attractor explosion change if a system approaches a saddle-node bifurcation. Consider a generic one-parameter family of d-dimensional maps $F_\lambda(x)$ in which a saddle-node pair of period-m orbits is created at $\lambda=0$, and in which the unstable periodic orbit is embedded in a fractal saddle, giving rise to a quasipotential plateau. It is a well-known fact [25] that locally the map $(\lambda, x)\mapsto(\lambda, F_\lambda^m(x))$, restricted to its two-dimensional centre manifold, is topologically equivalent to the map

$$(\lambda, y)\mapsto(\lambda, y+\lambda-y^2)\,, \tag{9}$$

where y is a real variable, so that we are dealing with a problem which is effectively one-dimensional. This fact enables us to use results of Ref. [17] (especially its equations (3.11) and (3.12)) for estimating the increase of the quasipotential between the node and the saddle which exist for $\lambda>0$. This leads to

$$\Phi_\lambda(\Delta x) \underset{|\Delta x|\to 0}{\sim} C_r(\lambda)\,|\Delta x|^r \tag{10}$$

if the value of the quasipotential is set to zero at the node; $|\Delta x|$ means the distance from the node in direction towards the saddle, and "$\sim$" denotes asymptotic equality. Thus, the quasipotential is locally parabolic with the same exponent r as the noise-distribution (2). The leading order of the λ-dependence of the coefficient C_r is determined by (9) according to the above-mentioned results of Ref. [17] (for details see Ref. [18]),

$$C_r(\lambda) \underset{\lambda\to 0}{\sim} c_r\lambda^{(r-1)/2}\,, \tag{11}$$

where the coefficient c_r depends on the specific family F_λ. From (9) follows further that the distance between saddle and node is proportional in leading order to $\sqrt{\lambda}$. Inserting this for $|\Delta x|$ in (10), together with (11), we obtain that the plateau height $\Phi_P(\lambda)$ scales with the bifurcation parameter λ as

$$\Phi_P(\lambda)\sim\tilde{c}_r\lambda^{r-1/2}\,. \tag{12}$$

This means that for all generic one-parameter families undergoing a tangent bifurcation there is a universal leading power in the λ-dependence of $\Phi_P(\lambda)$, whereas the corresponding coefficient depends on the specific family. In particular, this establishes that Beale's conjecture of a universal exponent 3/2 for Gaussian noise (i.e. $r=2$), based on the numerically and analytically treated examples in Refs. [10,9], is correct.

We tested the parameter dependence of the quasipotential plateau height (see Fig. 2) numerically for the family (8) with $\lambda=\mu-\mu_t$ and found

$$r=2:\quad \Phi_P(\mu)\sim4.1\times10^{-2}(\mu-\mu_t)^{1.5} \tag{13}$$

and

$$r=5:\quad \Phi_P(\mu)\sim5.4\times10^{-4}(\mu-\mu_t)^{4.5}\,, \tag{14}$$

with a numerical uncertainty of about 5% in the values of the coefficients.

We note in passing that (10) can still be applied in the vicinity of crises. However, there are two important changes: The coefficient C_r does not vanish at the critical parameter value, and the distance between attractor and quasipotential plateau is proportional to λ, generically. Thus, near a crisis we obtain in place of (12)

$$\Phi_P(\lambda)\sim\bar{c}_r\lambda^r\,, \tag{15}$$

again in accordance with Beale's conjecture for $r=2$ [9].

The result (12), together with (7), implies

$$\sigma_X^{cr}(\lambda)\sim\hat{c}_{r,X}\lambda^{1-1/2r}\,. \tag{16}$$

Thus, from the idealised point of view, a universal power law holds for the critical noise strength for an attractor explosion near a tangent bifurcation if the threshold χ is fixed. The critical noise strength for attractor explosions is an observable which is suited for experimental studies. One has to bear in mind, however, that the adequacy of the idealised point of view may diminish as λ approaches zero. It is in this limit when definitions (5) and (6) drastically differ. Section 3 deals with a different formulation of that problem.

We now turn to a more traditional observable: the characteristic time $\tau^{(\sigma)}$, introduced in Section 1. The weak noise asymptotics of $\tau^{(\sigma)}$ is entirely accounted for by the asymptotics of the mean escape time from the node to the saddle, which on an exponential scale is determined by Φ_P [9,15,17],

$$\tau^{(\sigma)} \underset{\sigma\to 0}{\asymp} \exp(\Phi_P/\sigma^r) . \quad (17)$$

Thus, we learn from (12) that

$$\lim_{\sigma\to 0} \sigma^r \ln \tau_\lambda^{(\sigma)} \underset{\lambda\to 0}{\sim} \tilde{c}_r \lambda^{r-1/2} . \quad (18)$$

3. Predictions and computer experiments concerning $\tau_\lambda^{(\sigma)}$

Besides (18) there exists another statement relevant for the weak noise limit of $\tau_\lambda^{(\sigma)}$. The arguments and numerical studies of Ref. [22] seem to suggest the existence of a scaling function $f(\sigma')$ such that for all $\sigma'>0$

$$\lim_{\lambda\to 0} \sqrt{\lambda}\, \tau_\lambda^{(\lambda^{3/4}\sigma')} = f(\sigma') \quad (19)$$

and that the scaling function is independent of the noise distribution.

For a Gaussian noise distribution ($r=2$) the statements (18) and (19) are consistent: From (18) follows the asymptotic form of the scaling function,

$$f(\sigma') \underset{\sigma'\to 0}{\asymp} \exp(\tilde{c}^2/\sigma'^2) , \quad (20)$$

where $\tilde{c}_2$ is the coefficient introduced in (12).

As a side-remark we note that combining (18) and (19) in order to conclude (20) is similar to the description of the situation near crises in Ref. [8], referred to in the introduction, which reads ($r=2$)

$$\lim_{\lambda\to 0} \lambda^\gamma \tau_\lambda^{(\lambda\sigma')} = h(\sigma') ,$$

where the exponent γ depends on the deterministic dynamics, and

$$h(\sigma') \underset{\sigma'\to 0}{\asymp} \exp(\tilde{c}_2/\sigma'^2) .$$

Regarding the latter relation, Ref. [8] did not include the warning that only asymptotic logarithmic equality is implied for $h(\sigma')$.

Fig. 3 shows the results of computer simulations of (1) for the maps (8) with Gaussian noise. The quantities σ and $\lambda=\mu-\mu_t$ are varying between 5×10^{-4} and 9×10^{-4} and 10^{-3} and 2.5×10^{-3}, respectively, so that $\sigma'=\sigma\lambda^{-3/4}$ changes over several decades. The measured values for the escape times $\tau_\lambda^{(\sigma)}$ are compared with the asymptotic prediction of (20), which is plotted as a broken line in Fig. 3, using the value of $\tilde{c}_2$ obtained in (13). We find excellent agreement in the whole σ'-interval accessible to our simulations.

Now we return to the above remark about $f(\sigma')$ putatively being independent of the noise distribution, which would not be compatible with our result (18). Fig. 4 shows results of simulations analogous to those shown in Fig. 3, but this time the noise distribution satisfies (2) with $r=5$. Remarkably, we find agreement with the scaling function for Gaussian noise for those measurements which result in mod-

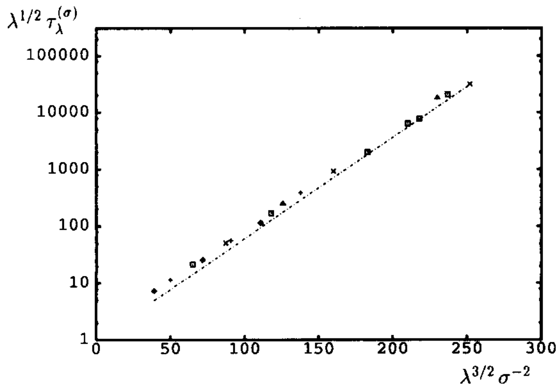

Fig. 3. Data obtained by simulation of (8) near $\mu_t=1.75$, perturbed by Gaussian noise (i.e., $r=2$). Different symbols denote different noise strengths σ. The sampling error in the escape time τ is 5%. The broken line shows the behaviour as predicted by (19), (20), and (13).

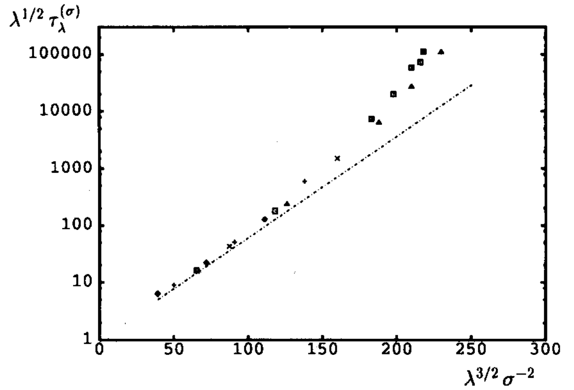

Fig. 4. Same as Fig. 3, but now with $r=5$ for the shape of the noise distribution.

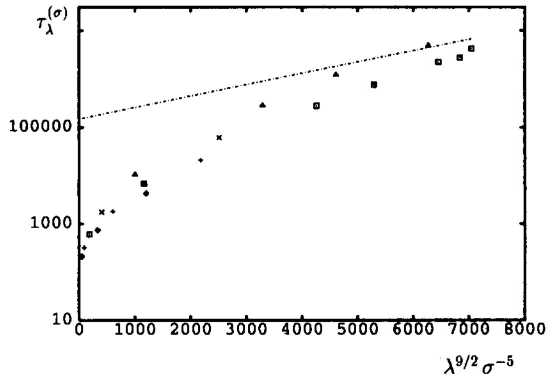

Fig. 5. Same data as in Fig. 4, but now compared with the prediction of (18) and (14) (broken line).

erate escape times. But clearly, the behaviour of processes with very long escape times is not properly described by (19), (20), and in this regime a cross-over to another scaling law occurs.

Fig. 5 contains the same data as Fig. 4, but now in a different presentation in order to test (18). According to that statement, a logarithmic plot of $\tau_\lambda^{(\sigma)}$ versus $\lambda^{4.5}/\sigma^5$ should result in a curve whose slope approaches the value $\tilde{c}_5$ (which can be taken from (14)) as $\lambda^{4.5}/\sigma^5$ grows. Our measurements tend to confirm this statement, although it is obviously only in the domain of very long escape times that the exponential asymptotics (18) becomes apparent, which makes it difficult to observe.

The fact that deviations from (19) can only be seen for very large escape rates is the reason why the numerical experiments of Ref. [22] did not suggest a dependence on the noise distribution. We can only sketchily indicate, why neither in the mathematical arguments of Ref. [22] such a dependence showed up. The scaling behaviour (19) is derived by comparing the escape time of (1) with the escape time of a continuous-time process for which the scaling is obvious. However, the arguments used to establish the relation between both escape times implicitly assume that, for fixed value of σ', the escape time of (1) does not depend exponentially on λ. But this is only true for $r=2$ (cf. (18)). Nevertheless, with a few modifications it should be possible to use the arguments of Ref. [22] to give a rigorous explanation of the observation of Fig. 4, which suggested that there is independence of the noise distribution as long as the escape time does not exceed a certain bound.

The following heuristic considerations may be helpful to understand the existence of two different domains in the behaviour of $\tau_\lambda^{(\sigma)}$. There are two small parameters involved: σ, characterising the weak random influence, and λ, also characterising a weak influence, namely the nontrivial influence of the deterministic dynamics (as opposed to the trivial dynamics for $\lambda=0$ that is locally given by the identity map). So the question is reasonable whether we should, in the situation at hand, regard (1) as deterministic dynamics with small random perturbations or as a sum of random variables which would be independent if F were the identity, but whose independence is slightly perturbed for small $\lambda>0$. What is appropriate depends on the way how the laboratory or numerical experiment is carried out, in particular, on what the ratio of the small parameters is.

The quasipotential method calls for the first point of view – the deterministic influence must be dominant compared to the random influence. This is the reason for the order of limits in (18): The asymptotics for $\lambda\to 0$ is accessible only after the limit $\sigma\to 0$. This implies that, given a fixed finite value of σ, the results of the quasipotential method are relevant only if λ exceeds some minimal value, as it was observed in Fig. 5.

If, in contrast, the second point of view is appropriate, then one should expect that the long-term effect of adding up the random contributions essentially amounts to what the central limit theorem predicts: a Gaussian distribution of the random influences irrespective of the distribution of the single contributions. This explains why we found correspondence with the Gaussian behaviour in Fig. 4 as long as λ is sufficiently small for fixed σ.

These considerations show that a Gaussian distribution of the random perturbations has the unique property of combining smoothly the two domains, as demonstrated in Fig. 3. Characteristic scaling laws like (19), (20) suggest to make use of controlled small random perturbations for probing bifurcating systems. Our findings strongly recommend the use of Gaussian noise for that purpose (and not, for instance, uniformly distributed noise) since then the scaling laws have the largest range of validity.

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