ON THE CONSTRUCTION OF INVARIANT CURVES OF PERIOD-TWO POINTS IN TWO-DIMENSIONAL MAPS

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We present an equation describing invariant curves associated with periodic points of period two in a wide class of two-dimensional invertible maps. Several branches of the unstable manifolds for the map $x_{n+1} = 1 - a|x_n| + bz_n$, $z_{n+1} = x_n$ are constructed in a situation when they are related to a two-piece strange attractor.

Stable and unstable manifolds play an essential role in non-integrable systems [1,2]. For chaotic behaviour, however, not only the invariant curves associated with fixed points are important, but the invariant curves of the periodic points as well [3]. By varying one parameter in a dissipative system, for example, it may happen that homoclinic points cease to exist along the invariant manifolds of the hyperbolic fixed points. The strange attractor may then be related to the unstable manifold of hyperbolic points of period two [3,4] and, therefore, it consists of two parts.

Here, we introduce an equation for the invariant curves associated with periodic points of period two in two-dimensional invertible maps and present a method of solution which, by specifying a piece of the curve, generates further parts of it. It is an extension of the method we proposed in ref. [5] for solving the equation for the invariant manifolds of the fixed points. For the piecewise linear map [6,7] the method allows explicit constructions of the invariant curves (see also ref. [5]).

We consider the class of two-dimensional invertible maps with a constant jacobian defined by the recursion relations

$$x_{n+1} = f(x_n) + bz_n\ , \quad z_{n+1} = x_n\ , \tag{1}$$

where f denotes a single-humped symmetric function and b is assumed to be positive. For $f(x) = 1 - ax^2$ we recover Hénon's map [8], while the choice

$$f(x) = 1 - a|x|\ , \tag{2}$$

which we shall use as a particular example, corresponds to the piecewise linear map introduced by Lozi [6].

When deducing an equation for the invariant curves associated with the fixed points of (1), one assumes that each of the curves is described by a continuous (generally multi-valued) function, $x = f^*(z)$, in the x,z plane. From the invariance property of the curve it follows that f^* fulfills the relation $f^*(z) = f(z) + bf^{*-1}(z)$. Starting from the inverted map, a similar equation can be obtained: $\tilde{f}^*(z) = f(z)/b + f^{*-1}(z)/b$, where $x = \tilde{f}^*(z)$ represents an invariant curve after performing the coordinate change $x \leftrightarrow -z$. The first of these equations has been used by Bridges and Rowlands [9] to obtain an approximate expression for the shape of the Hénon attractor, and latter by Daido [10] to approximate other invariant manifolds in the same model. Simó used a similar equation in his calculation [3].

Turning now to the invariant curves associated with points of period two, we note that they would be one-piece objects (consisting of a single continuous line) in the second iterated map. In the original map (1), however, they consist of two parts mapped into each other. Their description, therefore, requires two continuous functions f_1^*, f_2^* with the property that

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the first image of the curve given by $x = f_1^*(z)$ is $x = f_2^*(z)$ and vice versa. That is, f_1^* and f_2^* fulfill the relation

$$f_1^*(z) = f(z) + b f_2^{*-1}(z)\,, \qquad (3a)$$

$$f_2^*(z) = f(z) + b f_1^{*-1}(z)\,. \qquad (3b)$$

A similar relation follows from the inverted map. The generalization to invariant curves associated with points of higher periodicity is straightforward.

We illustrate the solution of eq. (3) with the example of the piecewise linear map defined by (2), where analytic calculations are possible. In more general cases, e.g. for the Hénon model, these equations can be used to obtain approximate expressions for the invariant curves of the points of period two in a similar way as for those of the fixed points in refs. [9, 10].

The periodic points F_1, F_2 of period two are the images of each other. Therefore, it follows from the second equation of (1) that the coordinates of F_2 are z_1^*, x_1^* provided those of F_1 are x_1^*, z_1^*. Furthermore, x_1^* and z_1^* satisfy the equations

$$x_1^* = f(z_1^*) + b x_1^*\,, \qquad z_1^* = f(x_1^*) + b z_1^*\,. \qquad (4)$$

In the case of the piecewise linear map the only solution differing from the fixed points is specified by

$$x_1^* = \frac{a+1-b}{a^2+(1-b)^2}\,, \quad z_1^* = -\frac{a-(1-b)}{a^2+(1-b)^2} \qquad (5)$$

for typical values of the parameters $x_1^* > 0, z_1^* < 0$.

Proceeding as in the case of the invariant manifolds of the fixed points, we start by determining single branches of the invariant manifolds of F_1 and F_2. Assuming, they are given by linear functions in a finite neighbourhood of the periodic points we write

$$f_1^*\colon \quad x = x_1^* + \lambda_1(z - z_1^*)\,, \qquad (6)$$

$$f_2^*\colon \quad x = z_1^* + \lambda_2(z - x_1^*)\,. \qquad (7)$$

Substituting (6) and (7) into (3) and using (5) one obtains

$$\lambda_{1,2}^{u} = \pm[a + (a^2 - 4b)^{1/2}]/2\,, \qquad (8)$$

$$\lambda_{1,2}^{s} = \pm[a - (a^2 - 4b)^{1/2}]/2\,. \qquad (9)$$

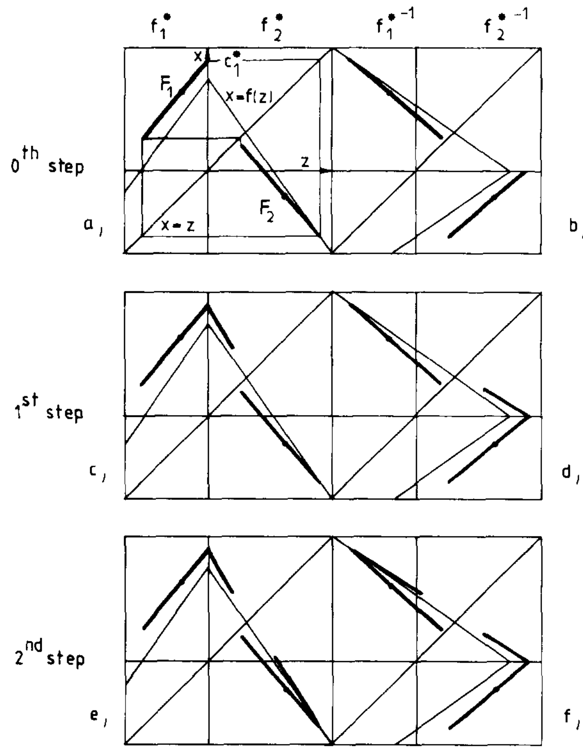

Fig. 1. The first steps in constructing the solution of eq. (3) with f as given by (2). Fat lines denote the branches of the unstable manifolds of F_1 and F_2 (a = 1.44, b = 0.3).

It is to be noted that $\lambda_1\lambda_2$ is just the eigenvalue of the second iterated map at F_1 and F_2. In the region of interest: $b \leqslant 1$, $1 < a < 2$ the modulus of $\lambda_{1,2}^{s}$ is less than unity, while that of $\lambda_{1,2}^{u}$ is greater than unity, provided $a > 1 + b$. The latter inequality is the criterion for the existence of hyperbolic points of period two in the model. Here we restrict our investigations to this region.

Eq. (3) specifies the range of validity of the branches (6) and (7), too. Let us consider the unstable manifolds characterized by slopes given by (8). Assuming f_1^* is described by (6) for negative values of z, it follows from eq. (3b) that f_2^* can be defined through (7) for $z \leqslant c_1^*$ only, where $c_1^* = x_1^* - \lambda_1^u z_1^*$ is the maximum value along the first branch of the unstable manifold emanating from F_1 (see figs. 1a and 1b). The restriction in the range of f_2^* then leads to a confinement of f_1^* into the interval $[1 - a c_1^*, 0]$ by eq. (3a) as the minimum value of f_2^* at c_1^* is just $f(c_1^*)$. Finally, the minimum value of f_1^* yields a lower bound in the range of f_2^* (fig. 1a).

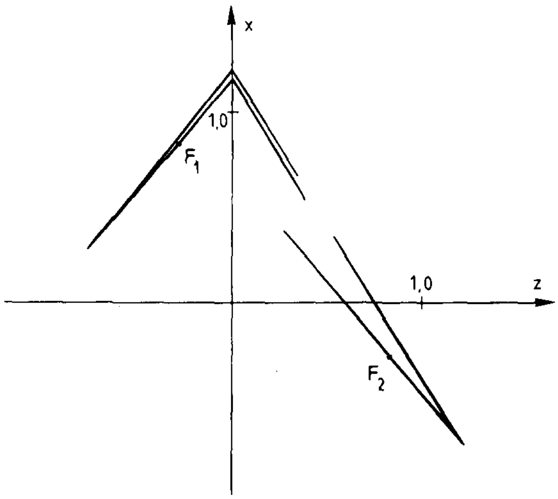

Fig. 2. Unstable manifolds of the periodic points F_1 and F_2 as obtained after nine steps of construction at parameter values a = 1.44, b = 0.3. These values are relatively close to the critical point characterized by homoclinic tangents between the invariant manifolds of the fixed point, therefore, the two pieces are relatively close to each other.

One cannot stop here, however, as the inverse of f_2^* (fig. 1b) is defined in a longer interval than f_1^* itself. Eq. (3a) then specifies a new branch in f_1^* which is again a piece of a straight line (fig. 1c). Considering now the inverse of f_1^*, the new branch of it (fig. 1d) generates through eq. (3b) a new branch in f_2^* (fig. 1c), etc. The self-generating procedure obtained by this way converges in the sense that the new branches come close to the previous ones and the neighbouring lines become soon indistinguishable due to the finite thickness of the drawing facility.

The stable manifolds, similarly as in the case of invariant curves of the fixed points [5], can be constructed most conveniently by means of the equation extracted from the inverted map.

As an example, we show the result for the unstable manifolds of F_1 and F_2 (fig. 2) at parameter values where no homoclinic points exist along the unstable manifolds of F_1 and F_2; therefore, the closure of the latter set is expected to give the (two-piece) strange attractor [3]. Fig. 2 agrees well with the strange attractor we obtained in a computer simulation.

The method presented here makes it possible to follow the modifications of the invariant manifolds of the periodic points of period two up to the critical situation characterized by homoclinic tangents, beyond which a four-piece strange attractor would appear. On the other hand, the procedure is not restricted for small values of b, therefore it is appropriate for studying the conservative limit (b = 1) where the invariant curves of F_1 and F_2 become invariant tori. We hope to return to a more detailed discussion of these questions soon.

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