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Integrability of Hamiltonians associated with Fokker-Planck equations

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Stochastic dynamical models described by Fokker-Planck equations, in the limit of weak noise, can be formally associated with Hamiltonian dynamical systems. Of special interest are "Fokker-Planck Hamiltonians" with a certain smooth separatrix at zero energy, since a differentiable macroscopic potential was shown to exist in this case. In the present paper, integrability of Fokker-Planck Hamiltonians with two degrees of freedom is investigated, with the aim of identifying cases in which a smooth potential exists and cases in which the eigenvalue problem of the Fokker-Planck operator becomes separable. Additional first integrals of polynomial and nonpolynomial form and also explicitly time-dependent conserved quantities are obtained. The singular-point analysis testing for the Painlevé or the weak Painlevé property and polynomial conserved quantities in dynamical systems is applied. This method is found to be useful for the purpose of identifying solvable special cases in classes of Fokker-Planck models with free parameters. However, the utility of this method in a search for systems with a smooth potential is found to be severely limited because it turns out that complete integrability with polynomial conserved quantities is a much stronger requirement than the existence of a smooth separatrix at zero energy.

I. INTRODUCTION

A useful formulation of equilibrium thermodynamics is provided by a Fokker-Planck equation for the probability density of a complete set of macroscopic variables which undergo a continuous Markov process due to small thermal fluctuations.[1] The form of the drift and diffusion coefficients of the Fokker-Planck equation for thermodynamics ensures that the time-independent probability density is given by the Boltzmann-Einstein formula[2]

$$P(q)\sim\exp[S(q)/k_B]\ , \qquad (1.1)$$

where q denotes some chosen values for the complete set of macroscopic variables, $S(q)$ is the total (constrained or "coarse-grained") entropy of the system (including all reservoirs to which it may be coupled) for the complete set of macroscopic variables constrained to assume the values q, and k_B is Boltzmann's constant which serves as a formal measure of the strength of fluctuations on a macroscopic scale. By the well-known methods of equilibrium thermodynamics the entropy $S(q)$ may be expressed entirely in terms of q and the fixed intensive parameters of the reservoirs. In this way $S(q)$ becomes the thermodynamic potential which is appropriate for the particular constraints provided by the reservoirs. For example, in the case of a heat reservoir with temperature T, $S(q)$ can be expressed as $-F(q,T)/T$, where F is the free energy of the subsystem.

The basic concept of a complete set of relevant variables which undergo a Fokker-Planck process as a result of weak fluctuations has proved to be very useful also in cases far from thermodynamic equilibrium.[3–8] In such cases the possible form of the drift and diffusion coefficients of the corresponding Fokker-Planck models is much less constrained than in the case of thermodynamic equilibrium.[9] An important question, which appears in this more general case, is the existence or nonexistence of a nonequilibrium potential generalizing the coarse-grained entropy $S(q)$ in Eq. (1.1). More precisely, if η is a parameter which characterizes the strength of the fluctuations, and if the Fokker-Planck equation has the unique time-independent solution $P(q,\eta)$, is there a potential $\phi(q)$, with some specified smoothness properties like first-order differentiability, which is defined by the limit

$$\phi(q)=-\lim_{\eta\to 0}[\eta\ln P(q,\eta)]\ ? \qquad (1.2)$$

In recent papers by two of us[10,11] this question was analyzed by studying the weak-noise limit of the Fokker-Planck equation[10] and its path integral solution.[11] In Ref. 10 it was shown that the Fokker-Planck equation in the weak-noise limit becomes equivalent to the Hamilton-Jacobi equation of a certain Hamiltonian dynamical system in which the potential $\phi(q)$ assumes the role of the action on a separatrix at energy zero. The smoothness of the potential, therefore, requires the smoothness of the corresponding separatrix. It is well known, that smoothness of separatrices is nongeneric in Hamiltonian systems, but is guaranteed by the integrability of the latter. Therefore, the existence of a smooth potential $\phi(q)$ is exceptional in nonequilibrium steady states[10] and a smooth $\phi(q)$, if present in a special case, is not structurally stable against arbitrary perturbations of the Fokker-Planck equation.[12] It was shown in Ref. 11 that in the general nonintegrable case a nondifferentiable potential $\phi(q)$ is defined by Eq. (1.2) which is associated with the minimum of the action on a certain "wild separatrix" of the nonintegrable Hamiltonian system. These results indicate that among the Hamiltonian systems associated with Fokker-Planck models in the weak-noise limit those which are integrable, eventually at energy zero only, play a special role because they provide examples of physical systems which have smooth nonequilibrium potentials.

In the present paper it is, therefore, our goal to investigate by the available methods the integrability properties of Hamiltonians corresponding to Fokker-Planck models in the weak-noise limit, which are shortly denoted as "Fokker-Planck Hamiltonians" in the following. Section II summarizes the framework in which these Hamiltonians appear. The questions which we then ask and answer in the subsequent sections are as follows.

(1) What is the relationship, if any, between the smooth potential and the second conserved phase-space functions in obviously integrable (e.g., symmetric) cases (Sec. III)?

(2) What are the conserved phase-space functions in other obviously solvable cases like the Ornstein-Uhlenbeck process or the case of Brownian motion (Sec. III)?

(3) Can one apply to the Fokker-Planck Hamiltonians the singular point analysis which has been developed[13] to test for the Painlevé or weak Painlevé property of ordinary differential equations, and which has in conservative systems so far been applied to motion in potential wells; and how useful is this method in practical cases for discovering completely integrable Fokker-Planck Hamiltonians and models with a smooth potential? What are the consequences of complete integrability for the eigenvalue problem of the Fokker-Planck equation (Sec. IV)?

(4) Finally, what is the relationship between integrability and the existence of a smooth separatrix. In particular, under which structural perturbations of the Fokker-Planck drift is the smooth separatrix of an integrable Fokker-Planck Hamiltonian preserved, while otherwise the property of integrability is destroyed (Sec. V)?

Even disregarding the particular application to Fokker-Planck models which originally motivated our work, the results we obtain are of interest because they extend considerably the class of Hamiltonian models which have up to now been investigated from the point of view of integrability. In particular, the Hamiltonians we consider include contributions from vector potentials. Generally speaking, for the class of models we consider we find a richer spectrum of possibilities than has been encountered up to now in the investigation of motion in scalar potentials. For instance, we find systems which are integrable only on a particular energy hypersurface ($E=0$), while lacking integrability for all other energies. In other cases we find that a smooth separatrix at $E=0$ exists, although the system is nonintegrable on this energy hypersurface. These findings can be summarized with the existing body of knowledge in a hierarchy of integrability properties of differing strength, which are interrelated by mutually forming necessary or sufficient conditions for each other. Denoting by $A \rightarrow B$ that A is a sufficient condition for B (and, therefore, B is necessary for A) we may write the hierarchy we find as follows: Positive result of the singular-point analysis $\leftarrow$ weak Painlevé or Painlevé property $\rightarrow$ complete integrability $\rightarrow$ integrability on a special energy hypersurface $\rightarrow$ smoothness of a special separatrix.

The application of the singular-point analysis turns out to provide a practical tool to detect special separable cases in parametrized classes of Fokker-Planck models. A concrete example is given in the Appendix. On the other hand, the same method does not provide an efficient tool for detecting models with a smooth potential, partly because nonpolynomial and explicitly time-dependent first integrals are not well handled by the available methods but are quite common in integrable Fokker-Planck Hamiltonians, and partly because the tested property of complete integrability in the entire phase space turns out to be unnecessarily restrictive. Integrability at $E=0$ is a more relevant condition, appropriate methods for testing it are, however, not yet available.

II. MECHANICAL ANALOGY IN THE WEAK-NOISE LIMIT OF STOCHASTIC PROCESSES

We consider systems characterized by a few macroscopic variables $\{q^\nu, \nu=1,2,\ldots,n\}$ the dynamics of which is governed by Langevin-type equations

$$\dot{q}^\nu = K^\nu(q) + \eta^{1/2} g_i{}^\nu(q)\xi^i(t) , \qquad (2.1)$$

where K^ν stands for a deterministic drift term, generally nonlinear in q, while the noise ξ^i describes the influence of the fast variables coupled to the slow ones through $g_i{}^\nu$. (Summation over repeated lower and upper indices is implied.) ξ^i is assumed to be a Gaussian white noise with $\langle \xi^i(t) \rangle = 0$, $\langle \xi^i(t)\xi^j(0) \rangle = \delta^{ij}\delta(t)$; the intensity of the noise is measured by the dimensionless number η. We shall be interested in the limit of weak noise, $\eta \rightarrow 0$, which is quite realistic for macroscopic systems. It is worth noting that the case $\eta=0$, the deterministic case, and the limit $\eta \rightarrow 0$ must be distinguished. The latter means, by definition, that in all quantities the leading contribution in η will be kept only. Such a calculation is analogous in spirit with the semiclassical approximation of quantum mechanics.[14,15]

When turning to the description of the probability density $P(q,\eta,t)$ associated with (2.1) there is no need to decide whether the Ito or the Stratonovich[4,6] calculus is used since the drift terms which appear in the equation of the probability distribution in the two different interpretations coincide in leading order in η. Thus, (2.1) in the weak-noise limit leads to the Fokker-Planck equation

$$\frac{\partial P(q,\eta,t)}{\partial t} = \left[-\frac{\partial}{\partial q^\nu} K^\nu(q) + \frac{\eta}{2} \frac{\partial^2}{\partial q^\nu \partial q^\mu} Q^{\nu\mu}(q) \right] \times P(q,\eta,t) \qquad (2.2)$$

with $Q^{\nu\mu}(q)=g_i{}^{\nu}(q)g_j{}^{\mu}(q)\delta^{ij}$ as the diffusion matrix, assumed to be positive semidefinite. In spite of the explicit η dependence, the second term cannot be ignored since $P(q,\eta,t)$ proves to be a singular function of η.

We focus our attention on the stationary probability density $P(q,\eta)$, the time-independent solution of (2.2). For small values of η it has the form

$$P(q,\eta)=N(\eta)\exp[-\phi(q)/\eta] , \tag{2.3}$$

where $N(\eta)$ is a normalization constant and $\phi(q)$ is a continuous (and η independent) function. This follows from the path integral solution of the Fokker-Planck equation.[11] We call ϕ the potential of the macroscopic system. If ϕ is in addition smooth, (2.3) is of the same type as the distributions governing fluctuations around thermodynamic equilibrium (where $1/\eta$ is proportional to Avogadro's number). There are well-known nonequilibrium systems (undergoing also instabilities) which possess a smooth potential.[16–20] This is, however, not the case in general.[11]

The equation determining ϕ follows immediately after inserting (2.3) into (2.2) and keeping the leading terms being of order $1/\eta$. It reads

$$\frac{1}{2}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^{\nu}}\frac{\partial\phi}{\partial q^{\mu}}+K^{\nu}(q)\frac{\partial\phi}{\partial q^{\nu}}=0 . \tag{2.4}$$

Next, we have to specify the boundary conditions under which (2.4) is to be solved. First, we note that the deterministic dynamical system $\dot q^{\nu}=K^{\nu}(q)$ has, in general, one or more attractors. Then, it is possible to speak formally about a stationary distribution also for $\eta=0$, which consists of δ functions concentrated on the attracting set. If a weak, nonvanishing noise is switched on, the distribution will be broadened with a width typically of order $\eta^{1/\alpha}$, $\alpha\geq 2$, but, in the same leading order, it will not be shifted. Thus, in the weak-noise limit $\phi(q)$ must have a minimum on the attractors of the deterministic system. Similarly, on the repellors the probability density must have a minimum, i.e., ϕ must be maximal. (On the saddles ϕ is extremal.) That ϕ can be considered as a potential is because of these properties and of the fact that the drift always contains a gradient term in ϕ. One may always write according to (2.4)

$$K^{\nu}(q)=-\tfrac{1}{2}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^{\mu}}+r^{\nu}(q) \tag{2.5}$$

with

$$r^{\nu}(q)\frac{\partial\phi}{\partial q^{\nu}}=0 . \tag{2.6}$$

The circulation r^{ν} defined by (2.5) is, therefore, that part of the drift which corresponds to a motion along equipotential surfaces. A given potential can characterize several systems differing in their circulation terms. A substantial difference between equilibrium and nonequilibrium systems is that in the former class r^{ν} can be determined simply since it transforms like $\dot q^{\nu}$ under the microscopically defined transformation of time reversal.[21] Thus, a smooth potential always exists which can be obtained by integrating $Q^{-1}_{\mu\nu}(K^{\nu}-r^{\nu})$. In the nonequilibrium case, however, neither r nor ϕ are known. One has to solve (2.4) before r can be specified.

For a continuously differentiable ϕ we may write the boundary condition discussed above in a somewhat weaker form:

$$\left.\frac{\partial\phi}{\partial q^{\nu}}\right|_{q\in\Gamma}=0 , \tag{2.7}$$

where Γ denotes the union of the limit sets (attractors, repellors, saddles , . . .) of the deterministic system. If there exists a smooth solution of (2.4) with (2.7) it is a global smooth potential for the macroscopic system. In general, however, such a solution does not exist.[10] Condition (2.7) must then be weakened by requiring the disappearance of the derivative at the attractors only since in the vicinity of them, contrary to that of repellors, ϕ is always smooth (this follows again from the path integral solution[11]). The potential obtained from (2.4) is then typically multivalued. Among the different branches the minimal one is to be kept because it gives the dominant contribution to $P(q,\eta)$ for small values of η.[11] Therefore, the potentials of macroscopic nonequilibrium systems are, in general, nondifferentiable functions.

In stable systems the probability density $P(q,\eta)$ must be normalizable in q. This implies an extra condition for $\phi(q)$ which we shall investigate after (2.4) with (2.7) has been solved. According to our experience the normalizability is in many cases automatically satisfied.

When studying the question of solvability of (2.4) with (2.7) in more detail a mechanical analogy turns out to be helpful. It should be noticed that the nonlinear equation (2.4) has the form of a Hamilton-Jacobi equation

$$H\left[q,\frac{\partial S}{\partial q}\right]=E , \tag{2.8}$$

where H stands for a Hamiltonian of a mechanical system with generalized coordinates q^{ν}, E, and $S(q)$ denote the energy and the action, respectively, and $\partial S/\partial q^{\nu}=p_{\nu}$ defines the momenta. A comparison with (2.4) gives $E=0$ and

$$H(q,p)=\tfrac{1}{2}p_{\nu}p_{\mu}Q^{\nu\mu}(q)+p_{\nu}K^{\nu}(q) \tag{2.9}$$

as the Hamiltonian associated with the macroscopic system. Since H can be uniquely constructed from the coefficients of the Fokker-Planck equation we call (2.9) a Fokker-Planck Hamiltonian. It is useful to note that in (2.2) H appears as an operator, which we obtain from (2.9) if we put $p_{\nu}=-\eta\partial/\partial q^{\nu}$ and preserve the order of p and q defined by (2.9). The first term of (2.9) can be interpreted as a kinetic energy term with an anisotropic mass tensor, the second one, however, is different from the usual potential energy. Similar terms appear when describing the motion of charged particles in an external magnetic field; $K^{\nu}(q)$ is, thus, analogous with a vector potential. The part of $K^{\nu}(q)$ which is derivable from the macroscopic potential $\phi(q)$ [cf. (2.5)] is of pure gauge type. It can be transformed into an equivalent scalar potential by the gauge transformation $p_{\nu}\rightarrow\bar p_{\nu}$ with

$$p_\nu = \bar{p}_\nu + \tfrac{1}{2}\frac{\partial\phi}{\partial q^\nu} \tag{2.10}$$

which after taking into account (2.6) changes the Hamiltonian into

$$\bar{H}(q,\bar{p}) = \frac{1}{2}\bar{p}_\nu\bar{p}_\mu Q^{\nu\mu}(q) + \bar{p}_\nu r^\nu(q) - \frac{1}{8}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^\nu}\frac{\partial\phi}{\partial q^\mu} . \tag{2.11}$$

Owing to the special form of the Hamiltonian (2.9) or (2.11) the mechanical motion described by the canonical equations is typically unbounded in q even at energy zero. Another unusual feature of Fokker-Planck Hamiltonians follows from (2.11). At a given value of the energy a certain component p_γ of the momentum is given by solving a quadratic equation. This means that in a case with nonvanishing circulation there are, in general, two different values of p_γ associated with any accessible points leading to a double foliation of the energy hypersurface (Fig. 1). For vanishing circulation, when (2.11) is of the form describing the motion in a scalar potential, the two values of p_γ are equal in modulus and the foliation becomes degenerated.

It follows from (2.8) and (2.9) that the potential ϕ plays the role of an action in the mechanical system. In fact, we are only looking for the special action satisfying also (2.7). In the mechanical picture it means that the momentum associated with this action must vanish in those points q which belong to the limit sets of the deterministic systems. This statement becomes clearer by noting that the $2n$-dimensional Hamiltonian phase space contains an invariant n-dimensional hypersurface S_0, where the deterministic motion, described by $\dot{q}^\nu = K^\nu(q)$, takes place. One sees directly from the canonical equations that S_0 is defined by $p_\nu = 0$. Thus, the limit sets of the deterministic system appear on the plane $p_\nu = 0$, but being embedded into the $2n$-dimensional space, they become hyperbolic objects. They have, therefore, stable and unstable invariant manifolds also transverse to S_0 on which not all p_ν vanish. We are interested in the action ϕ just on these separatrices since the relation

$$p_\nu = \frac{\partial\phi}{\partial q^\nu}, \quad \nu = 1,2,\ldots,n \tag{2.12}$$

on them automatically ensures condition (2.7) if ϕ is defined globally. If it turns out that the nontrivial (transverse to S_0) separatrices are smooth and are given by unique functions $p_\nu = p_\nu(q)$, the potential is obtained by simply integrating (2.12). Thus, the existence of a smooth potential requires the existence of a single nontrivial smooth separatrix in the Hamiltonian phase space connecting all limit sets of the deterministic system.

It is well known that smooth separatrices are structurally unstable against weak general perturbations,[22,23] therefore, the existence of a smooth potential is exceptional among nonequilibrium systems.

Since integrability guarantees the smoothness of all trajectories, the complete integrability of the Hamiltonian system (2.9) is obviously a sufficient condition for finding a smooth separatrix. (By complete integrability is meant the existence of n independent globally defined smooth functions of q,p, and eventually of the time, whose time derivative vanishes.) Therefore, it is interesting to look for the conserved quantities of the Hamiltonian system and to discuss the relation between the potential and the constants of the motion in the associated mechanical system.

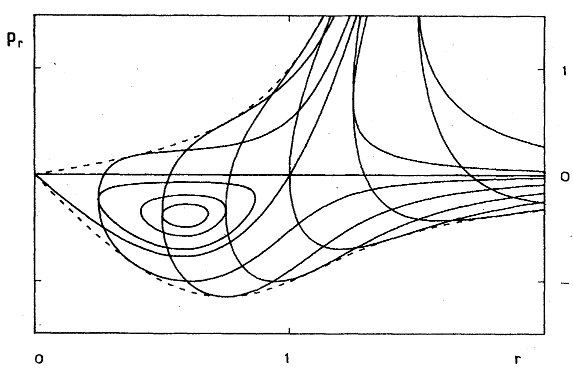

FIG. 1. Foliation of the plane $E=0$, $\varphi=$const in the Fokker-Planck Hamiltonian system defined by the unit diffusion matrix and (3.1) with $f(r^2)=1-r^2$, $g(r^2)=1$. The dashed lines give the boundary of the domain accessible on the $E=0$ hypersurface. Through all points in this region there pass two constant-L curves. We have plotted couples of these curves passing through the points $r_0 = 0.25, 0.5, \ldots, 1.5$, $p_{r0} = -r_0 f(r_0^2)$. Curves with negative L intersect the r axis, those with positive L do not. The potential is associated with the $L=0$ curve given by $p_r = -2rf(r^2)$. For $g \to 0$ the accessible domain shrinks to that between $p_r=0$ and $p_r=-2rf(r^2)$ and constant-L curves with opposite L values coalesce, as in the usual case of motion in a scalar potential.

III. FOKKER-PLANCK HAMILTONIANS AND CONSERVED QUANTITIES

Statements on integrability of the Fokker-Planck Hamiltonian (2.9) can only be made for special examples and subclasses. The most important points can be illustrated by considering relatively simple examples. In what follows we restrict ourselves to stochastic systems with two variables (four-dimensional Hamiltonian phase space x,y,p_x,p_y) and with a constant diagonal diffusion matrix. To prove integrability we have to find then a conserved quantity independent of the Hamiltonian.

As a first example we consider a macroscopic system specified by the unit $\underline{Q}$ matrix and the drift terms

$$K^x = xf(r^2) - yg(r^2) ,$$
$$K^y = yf(r^2) + xg(r^2) \tag{3.1}$$

with f and g arbitrary smooth functions and $r^2 = x^2 + y^2$. The system possesses a rotational symmetry. In polar coordinates r,φ the equations of motion for the deterministic system become

$$\dot{r} = rf(r^2), \quad \dot{\varphi} = g(r^2) . \tag{3.2}$$

Consequently, the limit sets are circles of radius r such that $f(r^2)=0$, and in addition a singular point at the origin. A circle of radius r_0 is an attracting limit cycle if $f'(r_0^2)<0$; the origin is a stable focus for $f(0)<0$. (Prime denotes derivative with respect to the argument.) A direct substitution shows that

$$\phi(x,y)=-\int^{r^2} f(z)dz \tag{3.3}$$

is a smooth solution of (2.4) satisfying (2.7). Moreover, ϕ is minimal at the attractors of (3.2) and maximal at the repellors. Thus, (3.3) represents a smooth potential. The circulation terms [cf. (2.5)] are identified as $r_x=-yg(r^2)$, $r_y=xg(r^2)$. From (3.2) one can see that they generate a deterministic motion along the equipotential lines which are concentric circles. This explains why the potential is independent of g.

The Fokker-Planck Hamiltonian (2.9) for this system is invariant under rotation. Using polar coordinates one finds that the angle φ is a cyclic variable and, therefore, the conjugated momentum

$$L=p_\varphi=xp_y-yp_x \tag{3.4}$$

is a conserved quantity. Consequently, this Hamiltonian system is integrable at all values of the energy. The value of L specifying the smooth separatrix associated with the potential is $L=0$ since p vanishes at the attractors. Figure 1 shows some constant-L curves on the $E=0$ hypersurface in the special case of $f(r^2)=1-r^2$, $g(r^2)\equiv 1$ illustrating the double foliation mentioned in Sec. II.

This example demonstrates how the potential and the conserved quantity are, in general, related. The knowledge of ϕ is not sufficient to reconstruct the conserved quantity, not even at $H=0$, since ϕ is not the general action but rather a special one restricted to the asymptotic trajectory connecting the limit sets of the deterministic system transverse to the hypersurface S_0. Inversely, the conditions $H=0$, $L=L(q\in\Gamma,\ p=0)$ uniquely specify ϕ. In the present case $H=0$, $L=0$ immediately yield $p_x=-2xf(r^2)$, $p_y=-2yf(r^2)$ which are just the equations for the smooth separatrix emanating from the limit sets.

It is worth mentioning that one can specify integrable Fokker-Planck Hamiltonian systems by means of the relations (2.10) and (2.11) in the following way. If among the integrable systems associated with motion in a scalar potential $V(q)$ and with a non-negative mass tensor Q a case is found where a smooth function $\phi(q)$ exists so that

$$V(q)=-\frac{1}{8}Q^{\nu\mu}(q)\frac{\partial\phi(q)}{\partial q^\nu}\frac{\partial\phi(q)}{\partial q^\mu},$$

the application of the inverse of (2.10) leads to an integrable Fokker-Planck Hamiltonian system with vanishing circulation on the potential $\phi(q)$. For example, the conserved quantity of the Fokker-Planck Hamiltonian associated with a two-dimensional system with unit Q matrix and rotationally invariant scalar potential $V(x^2+y^2)$ is obtained from the angular momentum as

$$L=y\left[p_x-\frac{1}{2}\frac{\partial\phi}{\partial x}\right]-x\left[p_y-\frac{1}{2}\frac{\partial\phi}{\partial y}\right]. \tag{3.5}$$

It follows from (3.5) that the deterministic motion restricted to the hypersurface $p=0$ possesses also a conserved quantity, consequently, the limit set points of the deterministic dynamics can be reached only along the curves specified by $p=0$ and the surface $y\partial\phi/\partial x=x\partial\phi/\partial y$. Therefore, the condition $H=0$, $L=0$ is fulfilled only on the intersection of the nontrivial separatrix and this surface.

Another obviously integrable case is realized if the Euler-Lagrange equations are decoupled. This is always found for

$$K^x=f(x),\quad K^y=g(y) \tag{3.6}$$

and

$$\underline{Q}=\begin{bmatrix} Q_x & 0\\ 0 & Q_y\end{bmatrix},$$

where f and g are arbitrary smooth functions vanishing in the limit sets and Q_x,Q_y are constants. This corresponds to a situation where the stochastic equations (2.1) for x and y are independent. The potential is then

$$\phi(x,y)=-2\int^x f(z)dz/Q_x-2\int^y g(z)dz/Q_y\ . \tag{3.7}$$

A possible choice for the conserved quantity is given by

$$L=Q_xp_x^2/2+p_xf(x)\ . \tag{3.8}$$

It is now quadratic in p_x and again $L=0$ specifies the smooth separatrix at $E=0$.

The examples shown above all have the property that for a polynomial choice of f and g the conserved quantities are polynomials, too. The majority of the explicitly known systems among the Hamiltonians describing motion in scalar potentials also supports this rule. We turn now to Fokker-Planck Hamiltonians where this is not the case.

We start with a linear system specified by the drift

$$\underline{K}=\underline{B}\underline{q}\ , \tag{3.9}$$

where $\underline{B}$ is a constant matrix, and by a constant (symmetric) diffusion matrix $\underline{Q}\cdot\underline{q}$ denotes the column vector $(x,y)^T$. (The results will be valid for arbitrary dimensions.) The limit set of the deterministic system is the origin. It is attracting if $\underline{B}$ is negative definite. Equation (2.1) then describes the stochastic approach of a stable steady state.

The potential has a quadratic form

$$\phi=\tfrac{1}{2}\underline{q}^T\underline{C}^{-1}\underline{q}\ . \tag{3.10}$$

$\underline{C}$ stands here for the solution of the matrix equation

$$\underline{B}\underline{C}+\underline{C}\underline{B}^T+\underline{Q}=\underline{0} \tag{3.11}$$

as it can be checked by a direct substitution into (2.4). According to (3.10) the elements of the symmetric matrix $\eta\underline{C}$ yield the quadratic mean values in the steady state. The circulation is given by $r=\frac{1}{2}(\underline{B}-\underline{C}\underline{B}^T\underline{C}^{-1})\underline{q}$.

After rearranging the canonical equations one finds a vector of time-dependent first integrals

$$L=\exp(-Bt)[q(t)-Cp(t)]\ . \tag{3.12}$$

Along the nontrivial separatrix of the origin L is identically zero.

Next, we consider a model introduced by Hietarinta (case A in Ref. 24). In the original version $\underline{Q}$ is the unit matrix but one of the drift terms is rather singular. Therefore, we perform a canonical transformation leading to

$$K^x=0,\quad K^y=xy^3\ . \tag{3.13}$$

Owing to this change the diffusion matrix becomes $\begin{pmatrix}1&0\\0&y^4\end{pmatrix}$. The integrability of this system at any values of the energy was established by finding a conserved quantity.[24] In the variables we use, a possible choice is

$$L=\exp(p_x)(1-p_x-xy^3p_y)\ . \tag{3.14}$$

The line $y=0$ is a limit set of the deterministic system (which for negative values of x is attracting). Therefore, on the nontrivial separatrix of the limit set $L=1$. Now, by means of the conserved quantity we construct the potential. From $H=0$ and $L=1$ one obtains an implicit set of equations for the separatrix in the form

$$\begin{aligned} x^2y^2&=\frac{1}{2}\frac{[1-p_x-\exp(-p_x)]^2}{\exp(-p_x)-1+p_x-p_x^2/2}\ ,\\ xy^3&=\frac{1-p_x-\exp(-p_x)}{p_y}\ , \end{aligned} \tag{3.15}$$

where $p_x=\partial\phi/\partial x$, $p_y=\partial\phi/\partial y$. This suggests an ansatz for the potential

$$\phi(x,y)=xh(xy) \tag{3.16}$$

with an even function h. By inserting it into the condition $H=0$ we find a quadratic equation for the derivative of h and choose the branch of its solution which is consistent with (3.15) for $x,y\to 0$. Thus, $h(z)$ is specified by

$$\frac{dh}{dz}=-\frac{h+z^2+z(2h-h^2+z^2)^{1/2}}{z(1+z^2)}\ , \tag{3.17}$$

and the free parameter in h is fixed by (3.15). The asymptotic form for h is obtained as

$$h(z)=-\tfrac{4}{9}z^2+\tfrac{4}{27}z^4-\tfrac{92}{1215}z^6+\cdots \tag{3.18}$$

for $z\to 0$, and

$$h(z)=-2\ln(z)+2-\ln 2-z^{-2}\ln^2(z)+\cdots \tag{3.19}$$

for $z\to\infty$. According to the numerical solution of (3.17) the asymptotic regions are smoothly connected (see Fig. 2). The conserved quantity (3.14) on the separatrix with (3.16), (3.17) has been found to remain equal to one showing the consistency of the ansatz in the whole regime. The potential is thus minimal along the negative x axis and maximal along the positive one. The probability distribution is not normalizable not even for $x<0$ owing to the extended structure of the attractor.

Finally, we turn to the example of the Brownian motion of a particle in an external potential. The drift terms are then given by

$$K^x=y,\quad K^y=-\gamma y+f(x)\ , \tag{3.20}$$

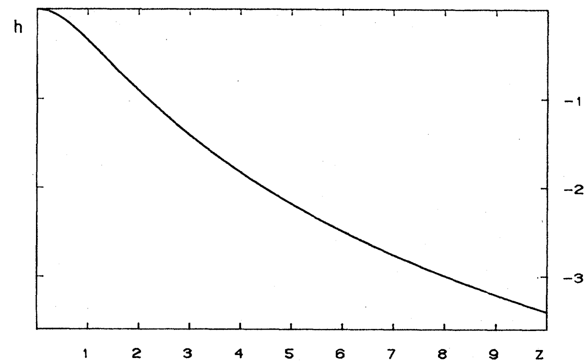

FIG. 2. Plot of the function $h(z)$ specifying the potential ϕ through (3.16) obtained by numerically integrating (3.17) with $h(0)=0$.

where y denotes the velocity, γ stands for the damping constant, and f represents the external force depending smoothly on x. Since a random force appears in the equation for the acceleration only, one can write

$$\underline{Q}=\begin{bmatrix}0&0\\0&2\gamma'\end{bmatrix}$$

with a constant γ'. If k_BT is identified with η and $\gamma=\gamma'$, the fluctuation-dissipation theorem is fulfilled and a thermodynamic equilibrium state (at temperature T) will be reached. In general, one obtains for the potential

$$\phi(x,y)=\frac{\gamma}{\gamma'}\left[\tfrac{1}{2}y^2-\int^x f(z)\,dz\right] \tag{3.21}$$

implying a circulation $r^x=y$, $r^y=f(x)$.

We have found a first integral at $H=0$ but have not succeeded in finding any at nonzero energies. The conserved quantity at $H=0$ is again explicitly time dependent

$$L=\exp(\gamma t)\left[\frac{y(t)}{p_y(t)}-\frac{\gamma'}{\gamma}\right]. \tag{3.22}$$

Making use of $H=0$ and the conservation of L, the canonical equations are reduced to

$$\dot y=-\gamma\frac{L\exp(-\gamma t)-\gamma'/\gamma}{L\exp(-\gamma t)+\gamma'/\gamma}y+f(x),\quad \dot x=y\ . \tag{3.23}$$

It is to be noted that (3.23) describes the deterministic motion of a particle in the potential $-\int f(x)\,dx$ with a time-dependent damping coefficient. As for $t\to\infty$, the damping goes to a constant this motion cannot be chaotic. The smooth separatrix of the potential is specified again by $L=0$.

The examples considered in this section show that it may occur quite often that a polynomial Fokker-Planck Hamiltonian possesses a nonpolynomial or time-dependent conserved quantity. Unfortunately, there are no standard methods available for finding such first integrals in Hamiltonian systems. On the other hand, the last example suggests that integrability may occur at energy zero only which also makes the search for integrability in this class difficult.

IV. PAINLEVÉ ANALYSIS

In order to understand better the relation between the integrability of the Fokker-Planck Hamiltonian, its consequences for the time-dependent solution of the Fokker-Planck equation, and for the existence of a smooth nonequilibrium potential ϕ, we apply in this section a singular point analysis of the Euler-Lagrange equations of (2.9). Such an analysis provides a strong necessary condition for the system having the Painlevé or the weak Painlevé property.

It is said that a system of ordinary differential equations has the Painlevé property if the only movable singularities of all its solutions in the complex time plane are poles. In the case of the weak Painlevé property also certain algebraic branch points are allowed. Investigations of these properties have been done in both dissipative and Hamiltonian systems.[13,25–37] The practical relevance of these studies lies in the Painlevé conjecture[13] and its extension[29] according to which a sufficient condition for the complete integrability of the system in question is the Painlevé and the weak Painlevé property, respectively. Therefore, the singular point analysis allows us to identify systems which are candidates for integrability, but the final confirmation is only provided by the explicit construction of the necessary number of conserved quantities. A related result proved recently[32] is that systems obeying a certain similarity property and having singularities characterized by irrational exponents cannot have rational invariant phase-space functions.

Next, we outline the main steps of the singular point analysis. A system of n variables q^α, $\alpha=1,\dots,n$, satisfying n second-order differential equations of polynomial form is considered. We try to find solutions around a movable singularity situated at t_0 in the complex time plane in the form

$$q^\alpha(t)=\sum_{j=0}^{\infty} a_j^\alpha \tau^{-\mu_\alpha+j/d}, \tag{4.1}$$

where $\tau=t-t_0$, and the μ_α's are positive rational numbers with a common integer denominator $d>0$. With an integer μ_α and $d=1$ (4.1) describes a pole singularity. According to Ref. 29 the system has the weak Painlevé property if all solutions around movable singularities are of the form (4.1) with $d\neq 1$.

As the first step of the analysis, only the most singular terms ($j=0$) of (4.1) are kept, and inserted into the equations of motion. For certain values of μ_α, $\alpha=1,\dots,n$ it may turn out, in general, that some terms of the equations balance, while others can be ignored. The former ones are called the dominant terms of the equations. This step provides the equations determining the amplitudes a_0^α, $\alpha=1,\dots,n$. All possibilities of dominant behavior specified by the sets $\{(\mu_\alpha,a_0^\alpha;\ \alpha=1,2,\dots,n\}$ must then be investigated.

Considering the complete form (4.1), a direct substitution into the equations of motion and a comparison of the coefficients in the power series leads to a set of recursion relations of the type $\{a_i=(a_i^1,a_i^2,\dots,a_i^n)\}$

$$M_j(d,a_0)a_j=R_j(d,a_0,\dots,a_{j-1}),\quad j>0 \tag{4.2}$$

where M_j and R_j are, in general, nonlinear functions.

In the second step of the analysis one decides whether the general solution of the equations of motion containing $2n$ free parameters and special solutions with less free parameters can be of the form (4.1). Since for all j for which the matrix $\underline{M}_j$ is nonsingular a_j can be expressed uniquely in terms of the amplitudes a_i, $i<j$, the possibility of a free parameter appears for those values of j, $j=r$, for which the determinant of $\underline{M}_j$ vanishes. These special values r are called resonances[13] and can be calculated from the $2n$th-order algebraic equation obtained from $\det\underline{M}_r=0$. One of the resonances turns out to be always $r=-d$ and represents the arbitrariness of the location t_0 of the movable singularity. (As for the special case of a double resonance at $r=-d$ see Appendix; if there is a free parameter among the amplitudes a_0^α, a resonance at $r=0$ also appears.) Thus, in general, one may write the set of resonances at a couple (μ_α,a_0^α) as $r=-d$, $r_1,r_2,\dots,r_{2n-1}$. These resonances fall into two groups according to the sign of their real part.

(1) Resonances with negative real parts ($\neq d$) are not consistent with the assumption that the leading terms are $a_0^\alpha\tau^{-\mu_\alpha}$, therefore, these values are to be excluded. If such resonances are present the number of free parameters in (4.1) is less than $2n-1$ and the corresponding solution of type (4.1) can represent only a special solution.

(2) The group of resonances with non-negative real parts may have noninteger elements. If so, this indicates the existence of a solution not of the form (4.1). The system cannot have then the Painlevé or the weak Painlevé property. (i) If all elements of the group are different integers (4.1) may represent a solution of the equation. (ii) If among the non-negative integer resonances some are multiple ones, the rank of the matrix $\underline{M}_r$ is to be considered in order to specify the number of free parameters appearing in (4.1). If, e.g., at a double resonance $r_1=r_2$ the rank of $\underline{M}_{r_1}$ is $n-2$, there are two free parameters among the amplitudes a_{r_1}.

One has to calculate the resonances of all couples $\{(\mu_\alpha,a_0^\alpha),\ \alpha=1,2,\dots,n\}$. Owing to the presence of resonances with negative real parts or of multiple integer resonances it may happen that to none of the couples belong $2n-1$ free parameters. Then, the general solution cannot be of the type (4.1) and must exhibit logarithmic or even stronger singularities. The system, therefore, does not exhibit the Painlevé or the weak Painlevé property. If this is not the case and no noninteger resonances with a positive real part have been found, one can proceed to the third step.

It is of great practical importance that the matrix $\underline{M}_r$ and the resonances r can be calculated also without explicitly constructing the hierarchy (4.2).[13] This is done by inserting

$$q^\alpha(t)=a_0^\alpha\tau^{-\mu_\alpha}+c^\alpha\tau^{-\mu_\alpha+r/d} \tag{4.3}$$

into the *dominant* terms of the equations of motion and keeping terms being *linear* in c only. When comparing the coefficients in the power series terms of type R_j of (4.2) do not appear and one obtains the condition $\underline{M}_r\underline{c}=0$.

The third step of the singular-point analysis is then to calculate R_j with $j \le r_k$, where r_k is the largest resonance, from the *complete* equations of motion and to check whether the relation (4.2) up to $j = r_k$ has indeed a solution belonging to different couples $\{(\mu_\alpha, a_0^\alpha),\ \alpha = \alpha_1$ and α_2, with $\alpha_1 \neq \alpha_2\}$. If no inconsistency has been found the system is a good candidate to exhibit the Painlevé ($d = 1$) or the weak Painlevé ($d > 1$) property. However, the singular point analysis outlined here does not signal essential singularities and this is why it does not provide also a sufficient condition for these properties to be satisfied.[38] In order to detect the complete integrability a direct search for the conserved quantities is finally necessary which can often efficiently be made directly after the second step of the analysis.

Now we discuss shortly the statistical physical consequences following in the special cases for which the whole program outlined above can be carried through. We restrict our discussion to the case of two variables. The method then leads to the identification of completely integrable Fokker-Planck Hamiltonians $H(q,p)$ and a second smooth conserved phase-space function $L(q,p)$. The knowledge of the latter can be very valuable for the purpose of solving the original time-dependent Fokker-Planck equation, if the operator versions of H and L obtained by the replacement $p_\nu = -\eta \partial / \partial q^\nu$ commute.[39] The eigenvalue problem of the Fokker-Planck equation

$$H\left[q, -\eta \frac{\partial}{\partial q}\right] P_i(q) = -\eta \lambda_i P_i(q) \tag{4.4}$$

obtained from Eq. (2.2) by writing

$$P(q,t) = \sum_i \exp(-\lambda_i t) P_i(q) \tag{4.5}$$

can then be solved simultaneously with the eigenvalue problem

$$L\left[q, -\eta \frac{\partial}{\partial q}\right] P_i(q) = l_i P_i(q) \; . \tag{4.6}$$

The (generally non-Hermitian) operators H and L form a complete set, and the (generally complex) eigenvalues λ_i and l_i uniquely label the eigenfunction $P_i(q)$. The $P_i(q)$ and the eigenfunctions $\bar{P}_i(q)$ of the adjoint equation

$$H^+ \bar{P}_i(q) = -\eta \lambda_i^* \bar{P}_i(q) \tag{4.7}$$

form a biorthogonal set. The $\bar{P}_i(q)$ can be chosen as simultaneous eigenfunctions of L^+, the adjoint of L, since $[H,L]=0$ implies $[H^+,L^+]=0$.

In general, therefore, the knowledge of the conserved operator L provides valuable information about the Fokker-Planck operator which may render possible the solution of the associated eigenvalue problem and its adjoint. Also, as we have seen the smooth nonequilibrium potential ϕ can be calculated.

In the next part of this section we summarize our observations obtained by performing the singular point analysis of several Fokker-Planck Hamiltonian systems. The Appendix contains an example with a relatively detailed computation.

First, we consider the systems mentioned in Sec. III. It is easy to check that the system with rotational symmetry specified by a unit diffusion matrix and the drift (3.1) with f and g polynomial passes the singular point analysis. The operator angular momentum commutes with the Fokker-Planck operator which separates in polar coordinates. On the contrary, it has been shown in Ref. 24 that the solution for system (3.13) is not of the form (4.1) owing to the presence of logarithmic terms such as $\tau \ln \tau$, although it is also completely integrable. The nonpolynomial form of the conserved L with respect to p makes it difficult, in this case, to use the operator L. In the case of the Brownian particle under the influence of an external force (3.20) the canonical equations can be transformed into a fourth-order differential equation for x

$$x^{(\mathrm{IV})} = (2f + \gamma^2)\ddot{x} + f'' \dot{x}^2 - f'f \; .$$

For a polynomial f of degree k (odd and different from 1) one identifies the dominant behavior in the form of $x = a_0 \tau^{-2/(k-1)}$ with two families for the amplitudes: (a) $a_0^{k-1} = 2(k+1)/(k-1)^2$, (b) $a_0^{k-1} = 2[(3k-1)/(k-1)^2]$. Using the ansatz (4.3) we find the resonances in case (a) as $r = -(k-1)$, $(k-1)$, $4k$, $2(k+1)$, and in case (b) as $r = -(k-1)$, $4k$, $[(3k+1) \mp (33k^2 - 26k + 9)^{1/2}]/2$. Owing to the presence of irrational resonances the system does not possess the weak Painlevé property. This can be a consequence of the fact that integrability is maintained on the hypersurface $E = 0$ only.

We have performed the singular point analysis for several Fokker-Planck models with quadratic or cubic drift terms like, e.g., those discussed in Ref. 40. In systems where the potential is not known in a whole region of parameters we have found only special values where the (weak) Painlevé property can be satisfied. These values were always associated with simple known integrable systems and potentials. Also models for which a potential ϕ exists in a whole region of the parameter space have passed the singular point analysis only for excep- tional values of the parameters (see Appendix). Infinitesimal changes in the drift orthogonal or parallel to equipotential surfaces of ϕ may destroy the (weak) Painlevé property. In the example discussed in the Appendix the singular point analysis led to the identification of three integrable special cases for which the conserved $L(q,p)$ could be found explicitly. In all three cases the eigenvalue problem of H is found to be separable.

These observations illustrate the value of the singular point analysis for detecting cases where operators L exist commuting with the Fokker-Planck operator. Among general stochastic processes such cases are extremely rare.

Our findings also illustrate that the existence of a smooth potential does not imply the (weak) Painlevé property, and, vice versa, that the singular point analysis can hardly yield predictions about the potential. The reason for this failure is the fact that the (weak) Painlevé property turns out as too strong a requirement for this purpose. It is a sufficient condition for and, therefore, even stronger than the integrability at arbitrary energy. On the other hand, the existence of a smooth potential is only connected with integrability at energy $E = 0$, in the sense that all structural perturbations preserving integrability at $E = 0$ but destroying it for $E \neq 0$ still preserve the smoothness of the potential. In the following section we present numerical examples of cases where even integrability in the entire hypersurface $E=0$ is a stronger property than the existence of a smooth separatrix. In other words, there are found to exist special structural perturbations which destroy integrability on the hypersurface $E=0$, while preserving the smoothness of the separatrix $E=0$, $L(q,p)=L_0$. Even excluding such special cases it seems clear that methods testing for integrability on certain hypersurfaces in phase space, like $E=0$, would be better suited for the problem at hand, than tests for integrability in the entire phase space, but methods of this type remain to be developed. (For a first effort in this direction see Ref. 41.)

V. NUMERICAL RESULTS

We have seen in Sec. II that the mechanical motion described by the canonical equations of a Fokker-Planck Hamiltonian is typically unbounded. This makes a numerical investigation, in general, extremely difficult. The situation is better in periodic systems where the boundedness of at least one variable is ensured by periodicity. Integrability can then be studied by investigating the motion on an appropriate Poincaré section. Note, that a singular point analysis can in such cases not be applied as the equations are nonpolynomial.

First, we recall a model which has been investigated earlier in more detail.[10] It is defined by the drift terms

$$K^x=\epsilon(x-x^3)(1+a\cos y),\quad K^y=1 \tag{5.1}$$

and the diffusion matrix

$$\underline{\underline{Q}}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

and describes the Brownian motion of an overdamped anharmonic oscillator under the influence of a periodic forcing term with an x-dependent amplitude. For $a=0$ the Fokker-Planck Hamiltonian is trivially integrable. It represents a special case of the separable systems described by (3.6). The conserved quantity can be chosen to be the momentum p_y, and the potential is given by

$$\phi=-\epsilon(x^2-x^4/2)\ . \tag{5.2}$$

According to numerical investigations the system loses its integrability in the presence of the forcing term, i.e., for $a\neq 0$. This has been demonstrated by plotting the intersection points of trajectories with the Poincaré plane $y=0$, $E=0$ (Fig. 3). The attractors (limit cycles) of the deterministic system also for $a\neq 0$ are at $x=\pm 1$. The nontrivial separatrix of these attractors undergoes rapid oscillations around the origin (a repellor) illustrating that the nonequilibrium potential cannot be smooth.

Let us now modify the model by taking

$$K^y=r(x,y)\ , \tag{5.3}$$

where r stands for a smooth function, 2π periodic in y. We concentrate on the case $a=0$. The potential is obviously (5.2) at any choice of r since r represents a circulation on equipotential surfaces of ϕ [cf. (2.6)]. As long as r depends only on x or y the system is also integrable with $L=p_y$ or $L=p_y r$ as a constant of the motion, respectively. On the separatrix, L takes the special value $L_0=0$. For a bivariate function $r(x,y)$ the functions L are no longer conserved, in general. Indeed, the numerical simulation of the Fokker-Planck dynamics suggests that general integrability on the hypersurface $E=0$ has been lost while the separatrix with $E=0$, $L=0$ remains smooth. For comparison we have plotted the points on the same

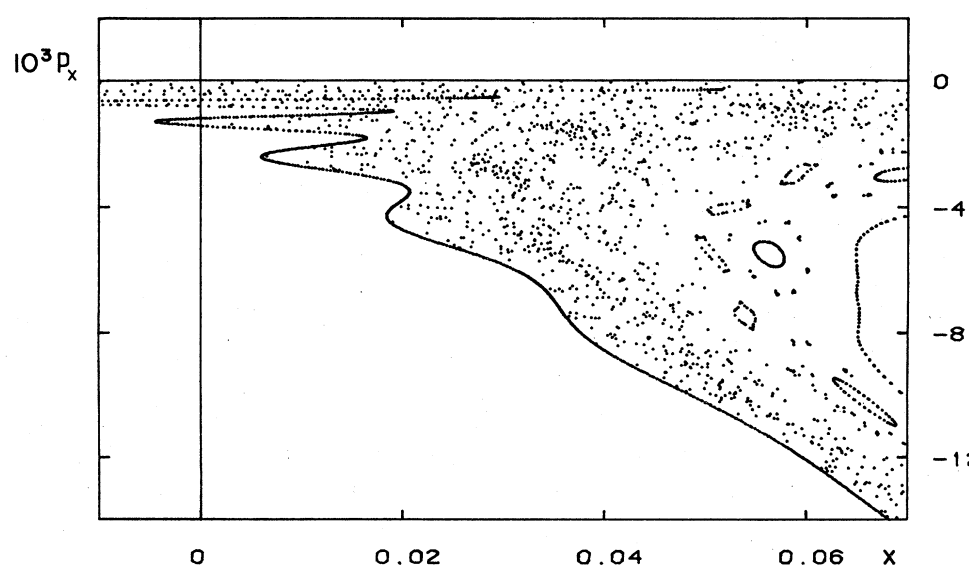

FIG. 3. The Poincaré surface [(x,p_x) plane at $y=0$] for system (5.1) with $a=1$, $\epsilon=0.1$ as obtained by numerical integration of the canonical equations of the Fokker-Planck Hamiltonian (2.9). The dots belong to the trajectories with initial condition $x_0=0.05, 0.1, \ldots, 0.65$, $p_{x0}=-\epsilon x_0$. The border line of this region is the wild separatrix emanating from the attractor point (1,0).

Poincaré plane as in the case of $r\equiv 1$, $a\neq 0$ (Fig. 3) with the same initial conditions. Chaotic trajectories appear inside the domain bounded by the nontrivial separatrix of the attractor, which remains smooth and passes through the origin (Fig. 4). This peculiar behavior can be understood by observing that the perturbation of the Fokker-Planck Hamiltonian relative to the case $r\equiv 1$, $a=0$ vanishes identically on the nontrivial separatrix of the unperturbed system, which, therefore, remains unchanged.

Nonintegrable mechanical systems with some integrable smooth separatrices have been found earlier in rather artificial examples only.[42-44] For Fokker-Planck Hamiltonians such cases can occur more commonly, although remain exceptional. An integrable system with conserved quantities $L_1, L_2, \ldots, L_{n-1}$, different from H and with a smooth potential ϕ associated with the separatrix $H=0$, $L_i=L_{i0}$, $i=1,2,\ldots,n-1$, can be perturbed by the Hamiltonian of the special form

$$\delta H = p_\nu \delta r^\nu(q) \quad (5.4)$$

with

$$\delta r^\nu(q)\frac{\partial\phi}{\partial q^\nu}=0 \quad (5.5)$$

which vanishes identically on the curve $p_\nu=\partial\phi/\partial q^\nu$. It follows from (2.6) that δr^ν is a shift of the circulation vector r^ν and, therefore, leaves ϕ unchanged. The separatrix $H=0$, $L_i=L_{i0}$ is thus preserved, but otherwise the integrability of the Fokker-Planck Hamiltonian at $E=0$ may be lost.

We conclude that a general method predicting the existence of a smooth potential in nonequilibrium systems is not provided by the usual techniques searching for complete integrability. In those special cases, however, where integrability is maintained at any values of the energy, such methods can be usefully applied. As a practical method to detect cases with a smooth potential there remains the possibility to compute perturbative solutions of the Hamilton-Jacobi equation for the potential with the boundary condition (2.7), expanding in some parameters for which the potential is known in zeroth order. If after a few steps the potential remains smooth, there is a good chance that the system possesses a smooth potential. A proof of this conjecture can then be sought by looking for an exact solution of the Hamilton-Jacobi equation. According to our experience the oscillating behavior of the nontrivial separatrix appears, perhaps on small scales only, immediately when making such a calculation.[10,11] Imposing on the parameters of a dissipative dynamical system the condition that oscillatory terms of the nontrivial separatrix vanish in the first few steps of such a perturbative analysis is, up to now, the most efficient practical method to detect those special cases where a smooth potential exists.

Also apart from this, the mechanical picture established in Sec. II remains useful in studying nonequilibrium potentials: on the one hand, it makes possible the investigation of general questions and, on the other hand, in periodic systems it provides a practical method for calculating numerically the separatrices of the attractors and from this the nonequilibrium potential itself.[11,45]

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APPENDIX

As an illustrative example we perform here the singular point analysis of the Fokker-Planck Hamiltonian system with a unit diffusion matrix and the drift terms

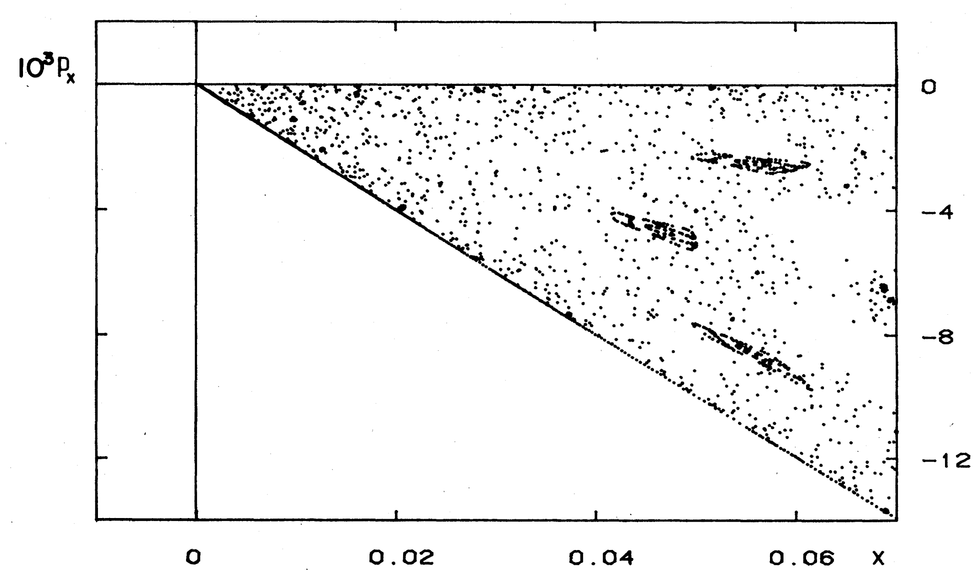

FIG. 4. The same as Fig. 3 with $a=0$, $K^y=1+x^3\cos y$.

$$K^x(x,y)=-x^3+\lambda xy^2,\ \ K^y(x,y)=-y^3+\lambda x^2 y\ . \tag{A1}$$

For $\lambda<1$ the origin is an attractor of the deterministic motion. For $\lambda>1$ it is a saddle point. For any value of λ a smooth potential

$$\phi(x,y)=x^4/2+y^4/2-\lambda x^2y^2 \tag{A2}$$

exists yielding a normalizable probability distribution for $\lambda<1$.

By means of the singular point analysis we hope to identify the values of λ for which the Fokker-Planck Hamiltonian is completely integrable. It is worth noticing that the gauge transformation $p\rightarrow\bar{p}$ with

$$\bar{p}_x=p_x+2(\lambda-1)xy^2,\ \ \bar{p}_y=p_y+2(\lambda-1)x^2y \tag{A3}$$

maps the Hamiltonian with λ onto that with $2-\lambda$. Therefore, it is sufficient to restrict our investigations to the region $\lambda<1$. Furthermore, we mention that the scalar potential to which the drift terms (A1) are related via the gauge transformation (2.10) is a homogeneous sixth-order polynomial. The integrability of the motion in such a scalar potential was not yet studied.

The Euler-Lagrange equations are

$$\ddot{x}=3x^5+2\kappa x^3y^2+\kappa xy^4,\ \ \ddot{y}=3y^5+2\kappa x^2y^3+\kappa x^4y \tag{A4}$$

with

$$\kappa=\lambda(\lambda-2)\ . \tag{A5}$$

Proposing a singular behavior in the form

$$x=a_0\tau^{-\mu},\ \ y=b_0\tau^{-\nu}\ , \tag{A6}$$

$\tau=t-t_0$, we find that the singularities in the different terms of (A4) can balance for $\mu=\nu=\frac{1}{2}$, that all terms are dominant, and that a_0 and b_0 satisfy

$$\begin{aligned}&3a_0^4+2\kappa a_0^2b_0^2+\kappa b_0^4-3/4=0\ ,\\ &\kappa a_0^4+2\kappa a_0^2b_0^2+3b_0^4-3/4=0\ .\end{aligned} \tag{A7}$$

For $\kappa\neq 3$ this is equivalent to

$$a_0^4=\frac{3}{4}\frac{1}{3+(1+2\sigma)\kappa},\ \ b_0^2=\sigma a_0^2,\ \ \sigma=\pm 1\ , \tag{A8}$$

while for $\kappa=3$ there is only one relation

$$(a_0^2+b_0^2)^2=1/4\ . \tag{A9}$$

To find the resonances we substitute

$$\begin{aligned}&x=a_0\tau^{-1/2}+c\tau^{-1/2+r/2}\ ,\\ &y=b_0\tau^{-1/2}+d\tau^{-1/2+r/2}\end{aligned} \tag{A10}$$

and collect the terms linear in c and d. This gives us a linear system of the form

$$\underline{M}_r(2,a_0,b_0)\begin{bmatrix}c\\ d\end{bmatrix}=0\ , \tag{A11}$$

where the elements of the matrix are given by

$$\begin{aligned}&(\underline{M}_r)_{11}=-s+15a_0^4+6\kappa a_0^2b_0^2+\kappa b_0^4\ ,\\ &(\underline{M}_r)_{12}=(\underline{M}_r)_{21}=4\kappa(a_0^3b_0+a_0b_0^3)\ ,\\ &(\underline{M}_r)_{22}=-s+\kappa a_0^4+6\kappa a_0^2b_0^2+15b_0^4\ ,\end{aligned} \tag{A12}$$

and

$$s=(r-1)(r-3)/4\ . \tag{A13}$$

A necessary condition for the weak Painlevé property to be satisfied is that the resonances (the values of r for which $\det\underline{M}_r=0$) with positive real part are integers for all (a_0,b_0) solutions of (A7). Using the simplifying fact that $r=-2(s=15/4)$ must be a resonance owing to the arbitrariness of τ in (A6), the equation for the resonances is obtained as

$$(s-15/4)[s+15/4-(15+\kappa)(a_0^4+b_0^4)-12\kappa a_0^2b_0^2]=0\ . \tag{A14}$$

The second resonance at $s=15/4$ is $r=6$. The disappearance of the second factor in (A14) specifies resonances depending on λ and the branch σ considered. Altogether we find for $\lambda\neq -1$:

$$r=-2,6,2\pm 4/(1-\lambda)\ \ \text{for}\ \sigma=+1\ ,$$

$$r=-2,-2,6,6\ \ \text{for}\ \sigma=-1\ ,$$

and for $\lambda=-1$:

$$r=-2,0,4,6\ .$$

Only the branch of the solution for which the resonances are -2 and three non-negative integers can represent the general solution of (A4) in the neighborhood of the singularity.

Such a complete set of resonances is found for $\lambda=-3$, namely, $r=-2,1,3,6$ and also for $\lambda=-1$. A remark is in order about the double resonance at $r=-d=-2$. For $\lambda=0$ both branches $\sigma=\pm 1$ have this double root indicating that (A4) is separable. Then, two free parameters appear since x and y can have a singularity at different times t_0 and t_0'. In all other cases, however, no solution exists with $t_0\neq t_0'$. Therefore, a branch with a double resonance at $-d$ cannot be complete unless the Euler-Lagrange equations are separable in the original variables.

For $\kappa<3$, $\kappa\neq 0$ we also find singularities in the neighborhood of which x or y behaves as $\tau^{-1/2}$ and the other coordinate has an exponent $\bar{\mu}$ satisfying $\bar{\mu}(\bar{\mu}+1)=\kappa/4$. At those points, however, where the branches (A8) have integer resonances these additional branches are not present and, therefore, no additional restriction follows.

In summary, we find that the system with drift terms (A1) passes the second step of the singular point analysis for $\lambda=-3$, -1, and 0. In two cases the integrability can be proved immediately by explicitly giving the second constant of the motion. Thus, for $\lambda=0$ the system belongs to the class (3.6) with conserved quantity (3.8), the

Fokker-Planck operator is separable in the original coordinates. For $\lambda=-1$ the Hamiltonian possesses rotational symmetry and the angular momentum (3.4) is conserved, the Fokker-Planck operator is separable in polar coordinates. At $\lambda=-3$ the system passed also the third step of the singular point analysis and a direct search for the conserved quantity in the form of a polynomial second order in the momentum led to the result

$$L=p_x p_y - p_x(y^3+3x^2y) - p_y(x^3+3xy^2)\ . \qquad \text{(A15)}$$

In this case the Fokker-Planck operator is separable in the coordinates $x'=x+y$, $y'=-x+y$. Taking into account the gauge symmetry (A3) we have thus found altogether six values of λ where the system turns out to be completely integrable, the smooth potential, however, exists at any value of the parameter λ.

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