PHYSICAL REVIEW A VOLUME 32, NUMBER 1 JULY 1985

Multivariate stochastic processes with exponentially correlated broadband noise

Axel Schenzle and Tamás Tél*
Fachbereich Physik, Universität Essen Gesamthochschule, 4300 Essen, West Germany
(Received 4 December 1984)

Conditions are derived under which the steady-state probability density of a multivariate stochastic process, driven by exponentially correlated broadband noise, can be obtained in an analytical form, provided the steady-state density of the corresponding white-noise process is known explicitly. The general results are illustrated by three examples: optical bistability, a laser model with detuning, and a model of Brownian motion.

I. INTRODUCTION

The stochastic description of a macroscopic physical system is an attempt to formulate the dynamics of a many-body problem in terms of only a small number of variables, while the irregularities—due to the many degrees of freedom of the system itself as well as the noise imposed by a fluctuating environment—are represented by a fluctuating force. The separation into collective variables and random terms is based on the intuitive physical picture that these processes are characterized by vastly different time scales—a slow global evolution on a coarse-grained time scale with superimposed random excursions that are visible only on a much higher resolution. The time constant of the gross behavior, which is specified by the deterministic parameters of the model, has to be compared with the correlation time of the fluctuations which in most physical cases is much shorter. However, when a clear separation of time scales is not possible, this approach looses its intuitive meaning and a stochastic description is no longer appropriate.

A first attempt to describe dynamical systems under the influence of noise in chemical reaction dynamics or quantum optics is based on the assumption that the time scale of noise is so short that it is entirely irrelevant, not accessible experimentally and, therefore, can safely be set to zero. In this limit the problem can be formulated in terms of a continuous Markov process where the hierarchy of probability distributions satisfies a Fokker-Planck equation.[1–4]

There is also a good physical reason why Markov processes should play a dominant role for problems with fluctuations. As is well known, noise becomes especially visible on a macroscopic scale near points of instability, where the restoring forces of the system diminish, making it susceptible to any kind of internal or external perturbation. In such cases of continuous bifurcations the weakening of the deterministic forces causes the system to slow down dramatically when approaching the bifurcation point. Therefore, in the neighborhood of a critical point the time scale of the fluctuations can always be expected to be much smaller than the time scale of the gross evolution of the system.

This picture, however, changes entirely when we consider the case of a discontinuous instability where two deterministically stable states are connected by fluctuations. In this case the deterministic dynamics is not characterized by a diverging time scale in regions of parameter space where fluctuations become important. Therefore, it is quite possible that the time scales of the fluctuations—still short compared to the times of the deterministic evolution—do not become irrelevant and the assumption of zero correlation time may break down. This is just one example which shows that there exist interesting physical systems where it is necessary to study the effects of noise with a finite nonvanishing correlation time.

In this paper we want to demonstrate how one can analyze systems with fluctuations which are correlated over a short but finite time without leaving the framework of Markov processes. This is done by enlarging the phase space of the process and treating the fluctuating forces as dynamically evolving variables as well. It has been shown that in the physically relevant limit of weak noise this concept can be made into a practical tool for analyzing simple model systems.[5–7] This is done by eliminating the auxiliary degrees of freedom by a systematic adiabatic method.[5,8] The phase space is thereby reduced again to its original dimension.

As is well known, the properties of a one-dimensional Markov process in a steady state can be studied analytically since the steady-state distribution can always be found up to quadratures. When we now enlarge the phase space for inclusion of noise with a finite correlation time, integration of the Fokker-Planck equation in general is no longer possible. Therefore, the adiabatic reduction of the phase space mentioned above is essential in order to turn the problem back into a solvable one.

Multivariate Markov processes, however, cannot be solved analytically and in a systematic way for their stationary distribution. In those cases where the system has the property of detailed balance an explicit solution becomes possible.[9–11] When we now try to generalize these processes to also include non-white noise by doubling the phase space and then return to the original dimension again by adiabatic reduction, we will end up with a modified process in more than one dimension which in general does not satisfy the condition of detailed balance any more. It will turn out, however, that with short-range memory noise the stationary density can be constructed explicitly in a broad class of systems, independent of their detailed balance property, if the stationary distribution in the white-noise case is known.

It is the aim of the paper to formulate conditions under which the stationary distribution of the final model can be found by systematic methods. The paper is organized as follows. In Sec. II we formulate the problem in terms of a two-dimensional process with non-white noise. Section III presents the adiabatic reduction of the phase space back to the original two dimensions which yields a Fokker-Planck-type equation describing the long-time behavior of the reduced distribution in the case of short-range memory. In the following Sec. IV we derive the stationary distribution and the condition under which it can be expressed directly in terms of the stationary distribution of the corresponding white-noise limit. We then illustrate this formal concept by three explicit examples—optical bistability, a laser model with detuning, and a model of Brownian motion—in the next three sections. Finally, some concluding remarks are given.

II. THE MODEL

We investigate here a general stochastic process in two dimensions described by the variables x^1 and x^2 and characterized by the deterministic drifts $k^i(x^1,x^2)$, $i=1,2$, and colored external noise. The Langevin equation of this system is written as

$$\dot{x}^i=k^i(x^1,x^2)+\zeta^i(t),\quad i=1,2 \tag{2.1}$$

where ζ^i is assumed to be a Gaussian noise characterized by a finite bandwidth ϵ^{-2}. In order to stay within the framework of Markov processes, an exponentially decaying correlation function has been taken:

$$\langle \zeta^i(t)\zeta^j(t')\rangle=\frac{\eta}{2\epsilon^2}Q^{ij}\exp(-\,|\,t-t'\,|\,\epsilon^{-2})\ . \tag{2.2}$$

For simplicity, the symmetric correlation matrix Q^{ij} is assumed to be independent of the variables x. We have chosen the normalization in such a way that the limit $\epsilon\to 0$ corresponds to the white-noise case. For later convenience, the factor η, which measures the noise intensity, has been separated from the matrix Q^{ij}. For the purpose of adiabatic elimination it is useful to scale the non-white force as $\zeta^i=y^i/\epsilon$ where y^i denotes a stationary Ornstein-Uhlenbeck process:

$$\dot{y}^i=\epsilon\dot{\zeta}^i(t)=-y^i/\epsilon^2+\xi^i(t)/\epsilon,\quad i=1,2 \tag{2.3}$$

driven directly by Gaussian white noise ξ^i, the correlation matrix of which is just ηQ^{ij}.

The joint process of (2.1) and (2.3) represents a multivariate Markov process in four dimensions and thus traditional methods can be applied. The Fokker-Planck equation for the joint single-time probability distribution $P(x^1,x^2,y^1,y^2,t)$, abbreviated by $P(x,y,t)$, follows immediately:

$$\frac{\partial}{\partial t}P(x,y,t)=-\frac{\partial}{\partial x^i}\left[k^i+\frac{y^i}{\epsilon}\right]P+\frac{1}{\epsilon^2}\frac{\partial}{\partial y^i}y^iP+\frac{\eta Q^{ij}}{2\epsilon^2}\frac{\partial^2}{\partial y^i\partial y^j}P\ , \tag{2.4}$$

and the complete hierarchy of multitime distribution functions can be formulated in the usual way.[1–5] Here and in the following, summation over repeated indices is implied.

Our aim is to investigate how the stochastic properties of the dynamical variables x^1,x^2 change by varying the memory of the noise. In a realistic physical situation one cannot follow the rapid variations of the noise in detail and $P(x,y,t)$, therefore, contains more information than there is experimentally accessible. Thus we are interested only in the reduced probability distributions obtained by integrating the joint densities over the y variables. The most elementary and most important of these reduced distributions is

$$W(x,t)=\int P(x,y,t)dy \tag{2.5}$$

specifying all transient moments and containing the stationary density $W_0(x)$ too. The reduced dynamics on the two-dimensional phase space x, however, is non-Markovian and, consequently, the entire hierarchy of the reduced multitime distributions cannot be constructed from the knowledge of $W(x,t)$ and $W_0(x)$ alone.[1–5]

III. ADIABATIC ELIMINATION EQUATION FOR THE SINGLE-TIME REDUCED DISTRIBUTION

In the case of short-range memory when the relaxation time of the noise is much shorter than any other characteristic times of the system, i.e., for small ϵ, the variables y^i relax rapidly and follow the evolution of x^i in an adiabatic way. In this case a generalization of the expansion method developed by Wilemsky[12] can be used successfully. In a variety of one-dimensional examples a resummation of this series was possible, yielding results for arbitrary values of ϵ.[5]

In order to find the reduced probability $W(x,t)$ we proceed as follows. By means of the definition

$$J^{n,m}(x^1,x^2,t)=\int (y^1)^n(y^2)^mP(x^1,x^2,y^1,y^2,t)dy^1dy^2 \tag{3.1}$$

we introduce time-dependent moments of the irrelevant variables y. In this notation it is obvious that we have to identify $W(x,t)=J^{0,0}(x,t)$. The equations of motion for $J^{n,m}$ are obtained by multiplying the evolution equation (2.4) by $(y^1)^n(y^2)^m$ and integrating over y^1,y^2. Using the Laplace transform $j^{n,m}(x,z)$ of $J^{n,m}(x,t)$

$$j^{n,m}(x,z)=\int_0^\infty J^{n,m}(x,t)\exp(-zt)dt \tag{3.2}$$

we find

$$\begin{aligned}\left[n+m+\epsilon^2\left[z+\frac{\partial}{\partial x^i}k^i\right]\right]j^{n,m}&\\=J^{n,m}(x,t=0)-\epsilon j_1^{n+1,m}-\epsilon j_2^{n,m+1}&\\+\eta Q^{11}n(n-1)j^{n-2,m}/2+\eta Q^{12}nmj^{n-1,m-1}&\\+\eta Q^{22}m(m-1)j^{n,m-2}/2\ .&\end{aligned} \tag{3.3}$$

Here and in the following a subscript i stands for a derivative with respect to x^i. Our aim is now to deduce a closed equation for $j^{0,0}(x,z)$ and its Laplace transform, the reduced single-time density, from the hierarchy (3.3). For arbitrary values of ϵ this is, in general, not possible analytically. It turns out, however, that for small ϵ the equations with $n+m\leq 4$ only play a role. In the following we will not care about the terms $J^{n,m}(x,t=0)$ in (3.3) as these terms are only remnants of the rapid initial transients and we will be interested here only in the asymptotic behavior of $W(x,t)$.[5,6,13,14]

The leading-order solutions of (3.3) are listed below. Terms of order unity (ϵ^0):

$$\begin{aligned} j^{2,0}&=\frac{\eta}{2}Q^{11}j^{0,0},\quad j^{1,1}=\frac{\eta}{2}Q^{12}j^{0,0}\,,\\ j^{4,0}&=\frac{3\eta^2}{4}(Q^{11})^2j^{0,0},\quad j^{3,1}=\frac{3\eta^2}{4}Q^{11}Q^{12}j^{0,0}\,,\\ j^{2,2}&=\frac{\eta^2}{4}[Q^{11}Q^{22}+2(Q^{12})^2]j^{0,0}\,. \end{aligned} \tag{3.4}$$

Terms of order ϵ^1:

$$\begin{aligned} j^{1,0}&=-\frac{\epsilon\eta}{2}(Q^{11}j_1^{0,0}+Q^{12}j_2^{0,0})\,,\\ j^{3,0}&=-\frac{3\epsilon\eta^2}{4}[(Q^{11})^2j_1^{0,0}+Q^{11}Q^{12}j_2^{0,0}]\,,\\ j^{2,1}&=-\frac{\epsilon\eta^2}{4}\{3Q^{11}Q^{12}j_1^{0,0}+[Q^{11}Q^{22}+2(Q^{12})^2]j_2^{0,0}\}\,. \end{aligned} \tag{3.5}$$

The elements $j^{m,n}$ are easily obtained from $j^{n,m}$ by interchanging the indices 1 and 2. Substituting (3.4) and (3.5) into the equation of $j^{0,0}$ we find in lowest order in ϵ^2

$$\left[z+\frac{\partial}{\partial x^i}k^i\right]j^{0,0}=\frac{\eta}{2}Q^{ij}j_{i,j}^{0,0} \tag{3.6}$$

which is just the Laplace transform of the Fokker-Planck equation in the white-noise case.

In order to keep all terms up to the first nontrivial correction, i.e., up to order ϵ^2, in the equation of $j^{0,0}$ we also need to know $j^{1,0}$ and $j^{0,1}$ up to order ϵ^3 as these terms appear with a prefactor ϵ^{-1}. Since the equations for $j^{1,0}$ and $j^{0,1}$ contain $j^{2,0}$, $j^{1,1}$, and $j^{0,2}$ one has also to calculate the latter elements up to order ϵ^2. For this reason, we formally divide the corresponding equations of (3.3) by the operator

$$\left[n+m+\epsilon^2\left[z+\frac{\partial}{\partial x^i}k^i\right]\right]$$

and expand the right-hand side in successive powers of ϵ^2. By using the formulas (3.4) and (3.5), terms like $zj^{0,0}$ will also appear which are to be expressed through (3.6) leading to expressions which become independent of z and, consequently, no further time derivatives are introduced after inversion of the Laplace transformation.

Collecting all the contributions, we obtain

$$\begin{aligned} j^{1,1}&=\frac{\eta Q^{12}}{2}j^{0,0}+\frac{\epsilon^2\eta^2}{4}(Q^{12}Q^{im}j_{i,m}^{0,0}+\det Q j_{1,2}^{0,0})\,,\\ j^{2,0}&=\frac{\eta Q^{11}}{2}j^{0,0}+\frac{\epsilon^2\eta^2}{4}(Q^{11}Q^{im}j_{i,m}^{0,0}-\det Q j_{2,2}^{0,0})\,, \end{aligned} \tag{3.7}$$

and similarly for $j^{0,2}$. The subscripts i,j denote partial derivatives with respect to x^i and x^j. By inserting (3.7) into the equation of $j^{1,0}$ one finds after some minor rearrangements that the terms containing quadratic powers of Q^{ij} and partial derivatives higher than second order cancel, resulting in

$$j^{1,0}=-\frac{\epsilon\eta}{2}Q^{1m}j_m^{0,0}-\frac{\epsilon^3\eta}{2}Q^{1m}(k_m^i j^{0,0})_i\,. \tag{3.8}$$

The expression of $j^{0,1}$ is obtained by replacing the superscript 1 by 2 on the right-hand side. As $j^{1,0}$ and $j^{0,1}$ do not contain any explicit z dependence, one may easily perform the inverse Laplace transformation of $j^{0,0}(x,z)$ and find the equation of the single-time reduced distribution $W(x,t)$, in the form

$$\frac{\partial}{\partial t}W(x,t)=-\frac{\partial}{\partial x^i}k^iW+\frac{\eta}{2}\frac{\partial^2}{\partial x^i\partial x^j}K^{ij}(x)W \tag{3.9}$$

with

$$K^{ij}=Q^{ij}+\tfrac{1}{2}\epsilon^2(Q^{im}k_m^j+Q^{jm}k_m^i)\,. \tag{3.10}$$

In the present approximation, the initial conditions which would contribute through terms like $J^{n,m}(t=0)$ in Eq. (3.3) would generate an inhomogeneity in Eq. (3.9) of the structure $I(x,\partial/\partial t)\delta(t)$. However, these terms would not influence the asymptotic behavior of the solution of Eq. (3.9) but only renormalize the initial values.

The resulting equation, (3.9), for the reduced single-time probability density valid for $t/\epsilon^2\gg 1$ has the formal appearance of a Fokker-Planck equation of a process with multiplicative noise owing to the fact that up to order ϵ^2, higher than second-order derivatives cancel in (3.8). It has to be emphasized, however, that the reduction of the process onto the variables x^1,x^2 leaves us with a process which is no longer Markovian, thus, $W(x,t)$ alone only provides a partial description of the system. The solution of Eq. (3.9) subject to a δ initial condition yields the conditional two-time probability density $W_{1|1}(x,t\,|\,x',t')$ which together with $W(x,t)$ specifies the joint distribution $W_2(x,t;x',t')$; multitime distributions like $W_3(x,t;x',t';x'',t'')$, however, cannot be expressed through W and $W_{1|1}$. Their calculation is also possible along similar lines by starting from the equation for multitime distributions like, e.g., $P(x,y,t;x',y',t';x'',y'',t'')$ of the four-variable Markov process and performing the adiabatic elimination in those schemes.[5]

IV. THE REDUCED STATIONARY DISTRIBUTION

The stationary distribution is one of the most important characteristics of a stochastic system since it describes the fluctuations in the asymptotic state reached for large times. This distribution is universal in the sense that it does not contain any information about the initial state. The negative logarithm of this density $W_0(x)$ may be considered as a generalized thermodynamic potential of the nonequilibrium system under consideration. More precisely, we will write $W_0=\exp(-\Phi/\eta)$ and if Φ *does not* depend on the noise intensity η it will be called the potential of the stochastic system which plays the role of a Lyapunov function for the corresponding deterministic dynamics.[15]

The stationary distribution $W_0(x)$ of the process with broad-bandwidth noise is the time-independent solution of the evolution equation (3.9). It is the principle aim of this paper to find out whether it is possible to construct the distribution $W_0(x)$ from the stationary density of the underlying white-noise case without explicitly solving the partial differential equation (3.9).

We first assume that the drift terms k^i possess a potential ϕ with respect to the diffusion matrix Q^{ij}, i.e., one may write

$$k^i = -\tfrac{1}{2}Q^{ij}\phi_j \; . \tag{4.1}$$

This means that the system in the limit $\epsilon \to 0$ satisfies the potential condition in the sense of Stratonovich[1] and, thus, the stationary distribution is given by $\exp(-\phi/\eta)$.

We now ask whether the stationary solution of (3.9) and (3.10),

$$W_0(x) = \exp[-\Phi(x,\eta)/\eta] \; , \tag{4.2}$$

can be expressed in terms of ϕ. By inserting the ansatz (4.2) into the stationary evolution equation we find

$$\left[\Phi_i - \eta\frac{\partial}{\partial x^i}\right](2k^i + K^{ij}\Phi_j - \eta K^{ij}_j) = 0 \; , \tag{4.3}$$

which is the well-known condition for the stationary solution of the Fokker-Planck equation.[3,4] A possible solution of Eq. (4.3) is

$$\Phi_j = (K^{-1})^{ji}(-2k^i + \eta K^{im}_m) \; , \tag{4.4}$$

where K^{-1} stands for the inverse of the matrix K of (3.10). Note that under general conditions the stationary solution is unique.[16] By substituting (4.1) and (3.9) one finds after some straightforward algebra that the right-hand side of (4.4) is in fact a gradient, and a solution is possible in the form

$$\Phi = \phi + \frac{\epsilon^2}{4}Q^{ij}\phi_i\phi_j - \eta\frac{\epsilon^2}{2}Q^{ij}\phi_{i,j} \; . \tag{4.5}$$

This implies that the stationary density of the non-Markovian process in the limit of short-range memory can easily be constructed if the stationary distribution of the corresponding white-noise system is known.

Next, we turn to the more general case when the drift k^i not only contains gradient terms of a function ϕ but can be written as

$$k^i = -\tfrac{1}{2}Q^{ij}\phi_j + r^i \tag{4.6}$$

with

$$\eta r^i_i - r^i\phi_i = 0 \; . \tag{4.7}$$

The stationary distribution is then given by $\exp(-\phi/\eta)$. The solution of Eqs. (4.6) and (4.7) is equivalent to that of the time-independent Fokker-Planck equation in the white-noise case. Note that in general ϕ and $\mathbf{r}$ may depend on the noise intensity η. If ϕ is independent of η and $\mathbf{r}$ is divergence-free, $\mathbf{r}$ is called the circulation as it describes the motion on equipotential surfaces of ϕ. In general, the vector $\mathbf{r}$ represents a contribution to the drift term which does not appear explicitly in the density itself but which causes a persistent probability current given by $\mathbf{r}\exp(-\phi/\eta)$ even in the steady state.

If there is a nonzero $\mathbf{r}$ in the white-noise case, it must be present in the system with short-range memory as well. In fact, we may search for a general solution of Eq. (4.3) in the form

$$\Phi_j = (K^{-1})^{ji}(-2k^i + \eta K^{im}_m + 2R^i) \; , \tag{4.8}$$

where

$$\eta R^i_i - R^i\Phi_i = 0 \; . \tag{4.9}$$

The stationary probability current in the present case is given by $\mathbf{R}W_0$.

So far it is not clear *a priori* whether the presence of a nonvanishing vector $\mathbf{r}$ may modify the expression (4.5) which was obtained in the absence of such terms. Therefore, we may ask under which conditions the solution Eq. (4.5) still remains a solution in the general case in spite of the fact that the drifts k^i are now expressed by (4.6) and (4.7). From Eqs. (4.8) and (4.5) one may explicitly calculate R^i which reads

$$R^i = r^i + \frac{\epsilon^2}{4}[Q^{im}r^j_m\phi_j + Q^{jm}r^i_m\phi_j - \eta(Q^{im}r^j_{m,j} + Q^{jm}r^i_{m,j})] \; . \tag{4.10}$$

The quantities R^i, however, must fulfill the relation (4.9) as well. Inserting (4.10) and remembering that r^i and ϕ_i are not independent owing to Eq. (4.7), many terms cancel and Eq. (4.9) reduces to the condition

$$\eta Q^{ij}r^m_j\phi_{i,m} = 0 \tag{4.11}$$

which must also be true in order to have the distribution (4.5) unchanged. In other words, the terms r^i in (4.6) do not modify the form of the stationary density as long as the additional condition (4.11) is fulfilled simultaneously. The vector $\mathbf{R}$ of the non-Markovian system is then given by (4.10). In cases where the stationary solution of the white-noise system is characterized by such a ϕ and $\mathbf{r}$ that do not satisfy (4.11) Φ is no longer given by (4.5). It will have terms more complicated than such simple expressions containing derivatives of ϕ and can be obtained only by a direct solution of Eq. (3.9). In a whole class of systems specified by (4.11), however, Φ is given by (4.5) independently of the presence of an additional $\mathbf{r}$ in the Markovian limit.

It may be worthwhile to briefly discuss the changes introduced by finite-bandwidth noise in the shape of the probability density from a general point of view. This can most easily be done by restricting our considerations to weak noise intensities which is a realistic description for most macroscopic systems. The third term in (4.5) can then be neglected and one finds that in the weak-noise limit a potential Φ_0 exists in the form

$$\Phi_0 = \phi_0 + \frac{\epsilon^2}{4}Q^{ij}\phi_{0i}\phi_{0j} \; , \tag{4.12}$$

where ϕ_0 stands for the potential of the white-noise system in the weak-noise limit. In order for the expansion to be valid we have to assume that ϕ_0 is smoothly differentiable which in nonequilibrium systems will not always be the case.[17,18] First, we note that the extrema of ϕ_0 and those of Φ_0 coincide, since the first derivatives of Φ_0 vanish in those points where ϕ_{0i} vanishes as well. On the other hand, it has been pointed out[17–19] that in the weak-noise limit of stochastic processes with white noise the extrema of the potential coincide with the limit sets of the deterministic dynamical system defined by $\dot{x}^i = k^i(x)$. This implies, e.g., that the potential has a minimum or maximum in those points x where the deterministic dynamics possesses an attractor or repellor. We conclude, therefore, that this coincidence remains valid also in the presence of broad-bandwidth colored noise. Away from the extrema the difference between Φ_0 and ϕ_0, however, can be quite significant. As the diffusion matrix in general is positive definite, or at least positive semidefinite, $Q^{ij}\phi_{0i}\phi_{0j}$ cannot be negative and, consequently, Φ_0 is never smaller than ϕ_0. We can, therefore, say that the minima of Φ_0 are situated on the attractors of the deterministic dynamical system and that the corresponding peaks of the probability density, in general, become narrower with increasing noise correlation time ϵ^2. The narrowing of the peaks in the stationary distribution is the most important qualitative change when replacing white noise by colored noise. In the case of a finite noise intensity the extrema of Φ may be shifted from those of Φ_0 according to the more general expression (4.8) and the change of the width of the distribution depends on the details of the model.

V. OPTICAL BISTABILITY WITH FINITE-BANDWIDTH NOISE

As a first example illustrating the general ideas discussed above we consider a system with vanishing probability current in the stationary state. The phenomenon of optical bistability is an example of a discontinuous instability which exhibits a hysteresis-type behavior analogous to that of first-order phase transitions.[20–24] Besides the practical interest in such optical devices they also provide some elementary, but nevertheless realistic, physical models for studying theoretically the influence of colored noise on the steady state of nonequilibrium systems. Contrary to continuous bifurcations, where due to the phenomenon of critical slowing down, the white-noise approximation seems to be always realistic, at least in the vicinity of the transition point, a finite bandwidth of the noise may lead to qualitative changes in multistable systems. These systems are characterized by several time constants which do not diverge simultaneously.

We consider the case of absorptive optical bistability where the driving field is in resonance with the Fabry-Perot étalon on the one hand and with an atomic resonance of the nonlinear medium on the other hand.[21,22] In the adiabatic limit, where the time scale of the atomic relaxation is short compared to the damping of the cavity mode, the macroscopic behavior of the system is described by the complex amplitude $E(t)$ of the cavity mode. This field changes in time due to the excitation by a driving field of constant amplitude E_0, due to the linear loss of the cavity, due to the nonlinear loss and saturation in the medium characterized phenomenologically by a function $f(|E^2|)$—a function of the field intensity—and finally, due to random perturbations. Measuring the time scale in units of the cavity damping time the equation of motion for E is thus given by

$$\dot{E}(t) = -E[1+f(|E^2|)] + E_0 + F(t) . \tag{5.1}$$

$F(t)$ represents all fluctuating influences coming from spontaneous emission, thermal motion, and fast changes in the pump field E_0, among which the last may well dominate all other influences. We assume that $F(t)$ has a characteristic time scale ϵ^2, and is exponentially correlated in time:

$$\langle F(t)\rangle = 0, \quad \langle F(t)F(t')\rangle = 0 ,$$
$$\langle F^*(t)F(t')\rangle = \frac{\eta}{\epsilon^2}\exp(-|t-t'|/\epsilon^2) . \tag{5.2}$$

The function $f(|E^2|)$ describing the nonlinear response of the medium can be derived from a microscopic model. For large intensities $f(|E^2|)$ goes to zero owing to the nonlinearity of the material response. In the case of a homogeneously broadened two-level system as a model for the medium one finds,[24,25] for example,

$$f(|E^2|) = \Gamma^2/(1+|E^2|) , \tag{5.3}$$

where Γ^2 is a material constant.

By decomposing the field E and the random force in their real and imaginary parts by

$$E = x^1 + ix^2, \quad F = (y^1 + iy^2)/\epsilon , \tag{5.4}$$

and assuming, without the loss of generality, that E_0 is a real parameter, $E_0 = x_0$, one realizes that the stochastic process of (5.1) and (5.2) is a special case of the general process (2.1)—(2.3) with

$$k^i(x^1,x^2) = -x^i h((x^1)^2 + (x^2)^2) + x_0\delta^{i1} ,$$
$$h(z) = 1 + f(z), \quad Q^{ij} = \delta^{ij} . \tag{5.5}$$

Therefore, the results of Sec. IV can directly be applied to the present problem. It is easy to find the potential ϕ associated with the drift through (4.1),[22] and the function Φ [Eq. (4.5)] in the presence of colored noise is then obtained in the form

$$\begin{aligned}\Phi(x^1,x^2,\eta) = & \int^{r^2} h(z)dz + \epsilon^2 r^2 h^2(r^2) \\ & - 2rx_0\cos\varphi[1+\epsilon^2 h(r^2)] \\ & + x_0^2(1+\epsilon^2) - 2\epsilon^2\eta[h(r^2) + r^2 h'(r^2)] ,\end{aligned} \tag{5.6}$$

where the prime denotes the derivative with respect to the argument. For the sake of simplicity, polar coordinates $x^1 + ix^2 = r\exp(i\varphi)$ have been used.

The changes introduced by altering the correlation time ϵ^2 of the fluctuations are most easily seen in a plot of the probability density. As a particular choice we have taken the case of the homogeneously broadened two-level system characterized by (5.3) with $\Gamma^2 = 25$. It is well-known from the deterministic theory[21] that bistable behavior is then observed in the range $9.73 < x_0 < 13.54$. For a field amplitude $x_0=10.75$ the probability distribution $\exp(-\Phi/\eta)$ is shown in Fig. 1(a) in the limit of white noise ($\epsilon=0$). It possesses two peaks: a narrower one concentrated around a small value of the field (absorptive branch) and a broader one representing the transmitting state of the bistable device. With increasing ϵ both peaks

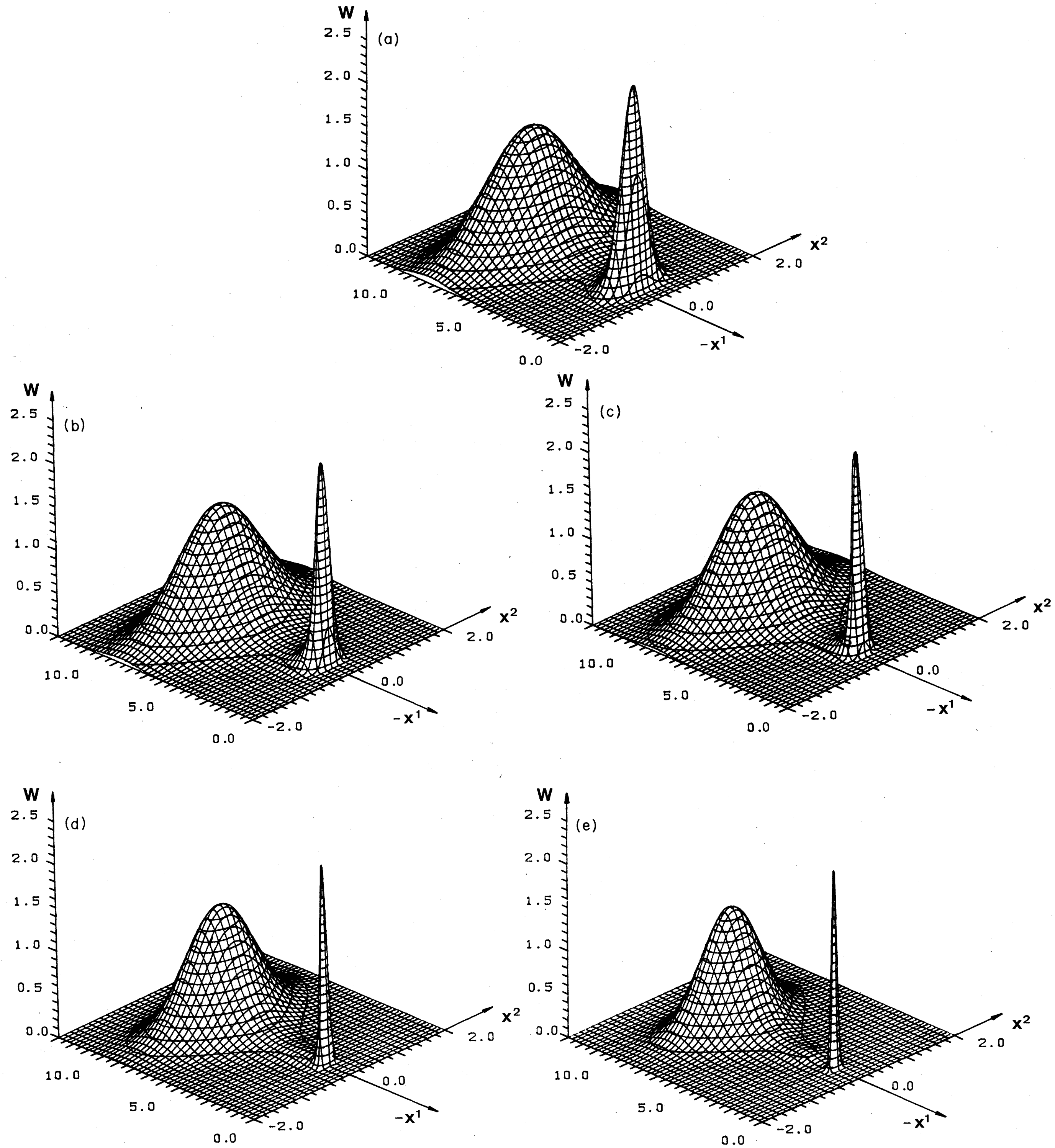

FIG. 1. This series of three-dimensional plots shows the steady-state probability density, $W_0(x^1,x^2)$ for optical bistability for $\Gamma^2=25$, $x_0=10.75$, and $\eta=2$. White-noise result, (a), is compared with the finite-bandwidth solutions in (b)–(e). Correlation time of noise increases from (b) to (e): (b) $\epsilon^2=0.1$, (c) $\epsilon^2=0.2$, (d) $\epsilon^2=0.5$ (e) $\epsilon^2=1.0$.

become narrower but the width of the absorptive peak shrinks much more drastically than that of the transmitting one [Figs. 1(b)—1(e)]. A direct comparison of these changes is shown in Fig. 2 where we have cut through the probability density along the x^1 axis. This shows that the absorptive state is much more sensitive to the decrease of the noise bandwidth than the transmitting state. This tendency becomes more pronounced when the distribution is plotted by cutting through the most probable values orthogonal to the phase of the external field (i.e., at $x^1=\text{const}$). Here an observable amount of shrinking in the absorptive branch is already obtained for values of ϵ^2 as small as 0.03 [Fig. 3(a)], while no modifications are yet to be seen for the peak corresponding to the upper branch [Fig. 3(b)]. Thus the increase of the correlation time of noise results in a lowering of the statistical weight of the lower branch, causing the device to be less likely to be observed in the lower absorbing state. Note that in order to visualize clearly the shrinking of the peaks in the figures we have chosen a rather large value for η, $\eta=2$, but still plotted the distribution as valid in the weak-noise limit. Otherwise, the peaks would be very narrow already in the white-noise case.

This behavior can be understood qualitatively by noticing that the bistable device possesses two different time scales.[23] Below and above the multistable regime there exists only a single stationary solution and, consequently, a single characteristic relaxation time. For small external field amplitudes this is approximately $\tau_1\approx 1/(1+\Gamma^2)$, while for large amplitudes one finds $\tau_2\approx 1 \gg \tau_1$. In the bistable regime both time scales are relevant simultaneously and as the bandwidth of noise is narrowed the time scale of the lower branch (τ_1) of the bistable hysteresis cycle is approached first and, therefore, only the width of the narrow peak is affected while the other peak first remains roughly unchanged. Thus we may conclude that in a multistable system, in general, branches with the smallest characteristic time will more strongly feel the finiteness of the correlation time of the applied colored noise and will shrink more drastically compared to the corresponding white-noise case. This behavior has been demonstrated qualitatively in Figs. 1—3, where the bandwidth of noise is varied over a rather wide range. However, quantitatively correct distributions are only guaranteed

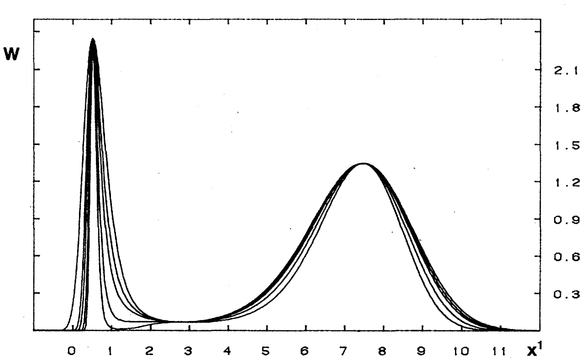

FIG. 2. $W_0(x^1,x^2=0)$—a cut through the probability distribution shown in Fig. 1 along the x^1 axis for the same parameters. Widest distribution corresponds to the white-noise case.

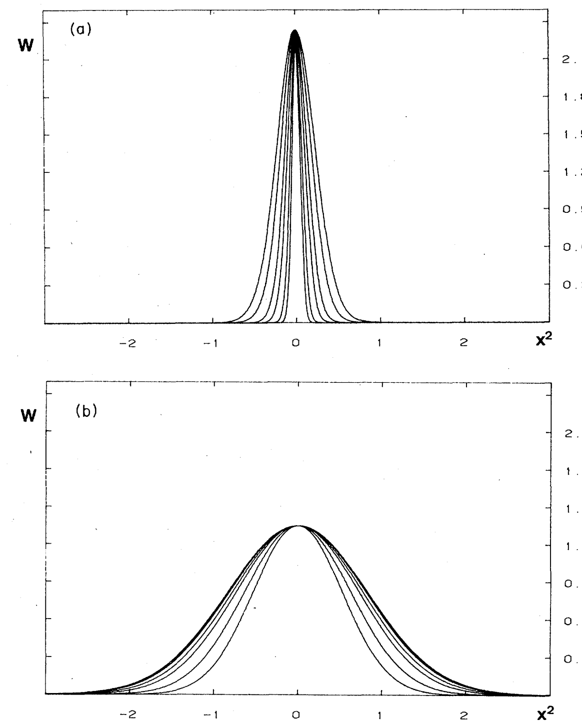

FIG. 3. Steady-state distribution cut parallel to the x^2 axis through the peaks of the probability density for $\epsilon^2=0$, 0.03, 0.1, 0.2, 0.5, 1.0, otherwise at the same parameters as Fig. 1. (a) displays the absorptive branch, (b) the transmitting one. Outermost curve corresponds to the white-noise case and can hardly be distinguished from that belonging to $\epsilon^2=0.03$ in (b).

as long as the correlation time ϵ^2 of the fluctuations remains shorter than τ_1.

VI. SINGLE-MODE LASER WITH FINITE-BANDWIDTH NOISE

In our second example we investigate a system with a nonvanishing probability current in the stationary state. The drift terms are supposed to be of the form

$$k^1(x^1,x^2)=-x^1h(r^2)-x^2g(r^2)\ ,$$
$$k^2(x^1,x^2)=-x^2h(r^2)+x^1g(r^2)\ , \tag{6.1}$$

where $r^2=(x^1)^2+(x^2)^2$, and the diffusion matrix Q is chosen to be the unit matrix. The process has a potential in the white-noise case:

$$\phi(x^1,x^2)=\int^{r^2}h(z)dz\ . \tag{6.2}$$

Consequently, the components of the circulation vector [cf. (4.6) and (4.7)] are given as

$$r^1=-x^2g(r^2),\quad r^2=x^1g(r^2)\ . \tag{6.3}$$

It is worth mentioning that the single-mode laser with detuning belongs to the class characterized by (6.1). The equation for the field amplitude $E(t)$ obtained after adiabatically eliminating the rapidly varying degrees of freedom is of the form[26,27]

$$\dot{E}(t) = -E[1+f(|E|^2)] + i\Delta E f(|E|^2) + F(t) . \quad (6.4)$$

The second term on the right-hand side describes the precession owing to the detuning Δ. In the case of a homogeneously broadened two-level system f is given by (5.3). Assuming that the noise $F(t)$ is Gaussian, has a characteristic time ϵ^2, and its correlation functions are given by (5.2), one finds that the real and imaginary parts of the field (x^1 and x^2) are described by a process of the form of (2.1)–(2.3) with (6.1) as the drift, where $h=1+f$ and $g=\Delta f$.

By applying the results of Sec. IV, we first notice that the condition (4.11) is satisfied by the potential ϕ and the circulation $\mathbf{r}$ of the white-noise case. Therefore, the stationary solution of the generalized model with colored noise is determined by Eq. (4.5) and, in particular, we find

$$\Phi(x^1,x^2,\eta) = \int^{r^2} h(z)dz + \epsilon^2 r^2 h^2(r^2) - 2\epsilon^2\eta[h(r^2)+r^2 h'(r^2)] . \quad (6.5)$$

The new vector $\mathbf{R}$ is obtained from Eq. (4.10) in the form

$$R^1 = -x^2 G(r^2), \quad R^2 = x^1 G(r^2) \quad (6.6)$$

with

$$G(r^2) = g(r^2) + \epsilon^2 r^2 g'(r^2) h(r^2) - \epsilon^2\eta[2g'(r^2) + r^2 g''(r^2)] . \quad (6.7)$$

Note that the symmetry of the stationary state has not been altered by the generalization to finite-bandwidth noise: Φ depends only on r^2 and $\mathbf{R}$ is orthogonal to the gradient of Φ. Again, the steady-state distribution narrows with decreasing bandwidth of the noise, thereby stabilizing the amplitude fluctuations of the laser. The presence of a nonvanishing vector $\mathbf{R}$ does not modify Φ, and, therefore, Eq. (6.5) is formally a special case of the problem (5.6) for vanishing external field E_0. The vector $\mathbf{R}$, however, depends on the potential ϕ owing to the presence of the second term of (6.7).

VII. "BROWNIAN MOTION" WITH A FINITE-BANDWIDTH NOISE

Finally, we consider the one-dimensional Brownian motion of a particle of unit mass in a mechanical potential $V(x^1)$ in the nonequilibrium situation characterized by an instantaneous damping and colored external noise with bandwidth ϵ^{-2}, acting on the momentum x^2. The system is, thus, defined by (2.1) and (2.2) with

$$k^1 = x^2, \quad k^2 = -\gamma x^2 - V'(x^1), \quad Q^{ij} = 2\gamma\delta^{i2} , \quad (7.1)$$

where γ denotes the damping constant. In the Markovian limit the stationary state is described by the potential

$$\phi(x^1,x^2) = (x^2)^2/2 + V(x^1) , \quad (7.2)$$

which leads to the well-known Boltzmann distribution with $\eta = k_B T$. This process carries a nonvanishing probability current even in steady state $\mathbf{j} = \mathbf{r}\exp(-\phi/\eta)$ with the circulation

$$r^1 = x^2, \quad r^2 = -V'(x^1) . \quad (7.3)$$

In the case of short-range memory a simple substitution in (3.10) shows that the diffusion matrix of the Fokker-Planck equation (3.9) remains constant, though no longer diagonal:

$$K^{11} = 0, \quad K^{12} = \epsilon^2\gamma, \quad K^{22} = 2\gamma(1-\epsilon^2\gamma) . \quad (7.4)$$

Then either a direct solution of the time-independent Fokker-Planck equation or an application of the general result (4.5) yields an η-independent Φ, i.e., a nonequilibrium potential

$$\Phi(x^1,x^2) = (1+\epsilon^2\gamma)(x^2)^2/2 + V(x^1) . \quad (7.5)$$

[The latter method is justified as (4.11) is found to be valid.] Note that the x^1 dependence has not been modified owing to the fact that only Q^{22} differs from zero. The "kinetic energy" part has been renormalized so that the distribution becomes narrower in the x^2 direction in accordance with the general observation of Sec. IV. The circulation vector has also been changed into

$$R^1 = (1+\epsilon^2\gamma/2)x^2, \quad R^2 = -(1-\epsilon^2\gamma/2)V'(x^1) \quad (7.6)$$

as it follows directly from (4.10).

It is worth briefly discussing the overdamped case where x^2 becomes a fast variable as well which can be eliminated adiabatically. Thus one obtains the following stochastic process for $\gamma \gg \epsilon^{-2}$:

$$\dot{x}^1 = -V'(x^1)/\gamma + y^1 ,$$
$$\dot{y}^1 = -y^1\epsilon^{-2} + \xi(2\eta\gamma^{-1}\epsilon^{-4})^{1/2} \quad (7.7)$$

with Gaussian white noise ξ of unit intensity. This, however, is a special case of (2.1)–(2.3) without the auxiliary variable of superscript 2. In the case of short-range memory, therefore, (4.5) can be applied yielding

$$\Phi(x^1,\eta) = V(x^1) + \epsilon^2[V'(x^1)]^2/(2\gamma) - \epsilon^2\eta V''(x^1)/\gamma . \quad (7.8)$$

For $\epsilon=0$ $\exp[-\Phi(x^1,\eta)/\eta]$ is identical to the solution of the Smoluchowski equation. For $\epsilon\neq 0$, however, $\Phi(x^1,\eta)$ cannot be obtained by merely integrating $\exp[-\Phi(x^1,x^2)/\eta]$ over x^2, where Φ is given by (7.5). The reason is that the limit of short-range memory of the nonoverdamped case is defined by $\epsilon^2\gamma \ll 1$, therefore, (7.5) is no longer valid in the limit $\gamma\to\infty$, with ϵ fixed.

The interplay between the limits $\gamma\to\infty$ and $\epsilon\to 0$ can most easily be understood by studying exactly solvable models. We have, therefore, investigated the motion in a harmonic potential $V(x^1) = \omega^2(x^1)^2/2$, described by the equations

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = -\gamma x^2 - \omega^2 x^1 + \epsilon^{-1}x^3 ,$$
$$\dot{x}^3 = -\epsilon^{-2}x^3 + \xi(2\eta\gamma\epsilon^{-2})^{1/2} . \quad (7.9)$$

The Fokker-Planck equation of this three-variable Markov process can easily be solved since the system is linear. The stationary distribution is of Gaussian form, which after integration over x^2 yields the following potential:

$$\Phi(x^1,x^2)=\tfrac{1}{2}(1+\epsilon^2\gamma+\epsilon^4\omega^2)\left[\frac{\omega^2}{1+\epsilon^2\gamma}(x^1)^2+(x^2)^2\right]. \tag{7.10}$$

This expression is valid without any restrictions on the parameters ϵ,γ. One immediately sees how the different asymptotic forms are obtained for $\epsilon\to 0$ with γ,ω fixed, and for $\gamma\to\infty$ with ϵ,ω fixed. In fact, the two results (7.5) and (7.8) are found to be consistent with (7.10) in the first and second limits, respectively. The potential (7.10) also nicely illustrates that, in general, the probability distribution becomes narrower with increasing ϵ^2 in both variables, and that equipartition is no longer valid in nonequilibrium situations.

It is also of importance to study a special case of (7.9), namely that of the free Brownian motion, $\omega=0$. No normalizable stationary solution will then be reached, but the temporal asymptotic behavior can be specified. Investigating the probability distribution at sufficiently large times so that initial conditions do not play a role any longer, one may find a solution with the following ansatz:

$$P(x^1,x^2,x^3,t)\sim t^{-1/2}\exp[-xA(t)x], \tag{7.11}$$

where the matrix A is assumed to contain constant terms as well as terms proportional to t^{-1}. After integration over x^3 one obtains

$$W(x^1,x^2,t)=\left[\frac{\gamma(1+\epsilon^2\gamma)}{4\pi\eta^2 t}\right]^{1/2} \times\exp\left[-\frac{1+\epsilon^2\gamma}{2\eta}\left[1+\frac{1+\epsilon^2\gamma}{2\gamma t}\right](x^2)^2 +\frac{1+\epsilon^2\gamma}{2\eta t}x^1x^2-\frac{\gamma}{4\eta t}(x^1)^2\right]. \tag{7.12}$$

The parameters ϵ and γ here are also arbitrary. Note that although this is not a stationary density and, therefore, the arguments of Sec. IV are not valid, the distribution in x^2 is again found to be narrower than in the white-noise case. On the other hand, there is no change in the x^1 direction and, consequently, after averaging over the momentum we obtain a pure diffusion with diffusion constant $D=2\eta/\gamma$ independent of the noise bandwidth.

Finally, a comment is in order on Brownian motion with colored noise in thermal equilibrium. According to the fluctuation-dissipation theorem[4,28,29] in this case the damping must then be noninstantaneous[30,31] and of the form $-\int_0^t X(t-t')x^2(t')dt'$, where $k_BTX(t-t')$ is the correlation function of the noise. With exponentially correlated Gaussian noise,

$$X(t)=(\gamma\epsilon^{-2})\exp(-|t|\epsilon^{-2}), \tag{7.13}$$

therefore, the system is specified by

$$\dot{x}^1(t)=x^2(t),$$
$$\dot{x}^2(t)=-(\gamma\epsilon^{-2})\int_0^t\exp(-|t-t'|\epsilon^{-2})x^2(t')dt' -V'(x^1(t))+x^3(t)\epsilon^{-1}, \tag{7.14}$$
$$\dot{x}^3(t)=-x^3(t)\epsilon^{-2}+\xi(2\eta\gamma\epsilon^{-2})^{1/2},$$

with $\eta=k_BT$. By differentiating the second equation with respect to time one can eliminate the variables x^2,x^3 and finds a third-order stochastic equation for x^1:

$$\dddot{x}^1=-\ddot{x}^1\epsilon^{-2}-\dot{x}^1\gamma\epsilon^{-2}-V''(x^1)\dot{x}^1-V'(x^1)\epsilon^{-2} +\xi(2\eta\gamma\epsilon^{-4})^{1/2}. \tag{7.15}$$

The corresponding third-order equation for the case defined by (7.1) turns out to be exactly of the same structure as (7.15), the only difference is found in the coefficient of the term $\dot{x}^1$ which there is $(\gamma+\epsilon^{-2})$ instead of ϵ^{-2}. This observation makes it plausible that, replacing $1+\epsilon^2\gamma$ by 1 in (7.5), (7.10), and (7.12), expressions characterizing thermodynamic equilibrium are recovered.

VIII. DISCUSSION

In this paper we have shown that the stationary probability density in a broad class of two-dimensional stochastic systems with a short-range memory noise can be explicitly constructed from the knowledge of the steady-state distribution of the same system with white noise.

It is well known that the time-independent Fokker-Planck equation of multivariate stochastic processes can only be solved systematically in the case of detailed balance. The possibility of finding a solution for the steady-state density of the colored-noise case, however, does not require the existence of the detailed balance property of the corresponding white-noise case. It only depends on the availability of an explicit solution for the latter. Nevertheless, we can ask whether detailed balance is valid for the Fokker-Planck equation (3.9), describing now a non-Markovian system, supposing it was satisfied in the original problem. Of course, since this equation governs only the asymptotic behavior of the system, even a positive answer does not involve detailed balance for the system itself in a strict sense.

As is usual,[9–11] we classify the variables according to their transformation properties with respect to time reversal and write for the transformed variables $\tilde{x}^i=\epsilon^i x^i$ where $\epsilon^i=1$ or -1 for even or odd variables. The drift term k^i is then split into an irreversible part which transforms in the same way as x^i itself and into a reversible part which transforms like the time derivative of x^i. The condition of detailed balance of a Fokker-Planck equation is shown to be satisfied if certain relations among the reversible and irreversible drift terms and the stationary distribution are fulfilled[9–11] and if the diffusion matrix $Q^{ij}(x)$ transforms according to

$$Q^{ij}(x)=\epsilon^i\epsilon^j Q^{ij}(\epsilon x). \tag{8.1}$$

Now we apply these conditions to the bivariate processes investigated above. Let us assume that the two variables have opposite transformation rules. Then, if detailed balance is satisfied in the white-noise case, the vector **r** [cf. (4.6) and (4.7)] must be identical with the reversible drift. It follows now from (3.10) that the diffusion matrix for the non-Markov process, in general, does not satisfy the condition (8.1) for $\epsilon=0$, i.e., detailed balance cannot be valid.

When the two variables follow the same transformation rule the complete drift must be irreversible in order to have detailed balance in the white-noise case. As one can see from Eqs. (3.10), (4.4), and (4.5), condition (8.1) is then valid, showing that detailed balance property of Eq. (3.9) can only be expected for processes with vanishing steady-state probability current.

Finally, we briefly discuss a consequence of the general result of Sec. IV when applied to the limit of weak noise intensity which is realized in most macroscopic systems. In leading order in the noise intensity η, the ansatz (4.2) with a smooth function Φ_0 [Eq. (4.12)] satisfies the stationary Fokker-Planck equation (3.9) identically—the additional constraints (4.11) must only be satisfied if arbitrary noise intensities are considered. It has been pointed out recently[17–19] that contrary to thermodynamic equilibrium, nonequilibrium systems under the influence of white noise do not necessarily have a smooth potential. The reason is that the equation specifying the potential turns out to be of the form of a Hamilton-Jacobi equation and the corresponding Hamiltonian system is typically nonintegrable. The stationary probability distribution at very small but finite noise intensity is found to have certain regions where its first derivatives change very rapidly.[19] This occurs where a given separatrix of the associated Hamiltonian system shows wild oscillations. Since Eq. (3.9) is of Fokker-Planck type and in the weak-noise limit a Hamilton-Jacobi equation can be associated with it, similar phenomena are expected in the case of colored noise, too.

ACKNOWLEDGMENT

We would like to thank Professor R. Graham for valuable discussions.

*Permanent address: Institute for Theoretical Physics, Eötvös University, H-1088 Budapest, Hungary.

[1]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[2]N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[3]C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, New York, 1982).
[4]H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1983).
[5]A. Schenzle, Habilitation Thesis, Universität Essen, 1984.
[6]A. Schenzle and R. Graham, Phys. Lett. **98A**, 314 (1983).
[7]A. Schenzle and T. Tél, in *Optical Bistability 2,* edited by C. M. Bowden, H. M. Gibbs, and S. L. McCall (Plenum, New York, 1984).
[8]J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).
[9]R. Graham and H. Haken, Z. Phys. **243**, 289 (1971); **245**, 141 (1971).
[10]H. Risken, Z. Phys. **251**, 231 (1972).
[11]P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).
[12]G. Wilemski, J. Stat. Phys. **14**, 153 (1976).
[13]F. Haake, Z. Phys. B **48**, 31 (1982).
[14]U. Titulaer, Z. Phys. B **50**, 71 (1982).
[15]R. Graham, in *Coherence and Quantum Optics,* edited by L. Mandel and E. Wolf (Plenum, New York, 1973).
[16]J. L. Lebowitz and P. G. Bergmann, Ann. Phys. (Leipzig) **1**, 1 (1957).
[17]R. Graham and T. Tél, Phys. Rev. Lett. **52**, 9 (1984).
[18]R. Graham and T. Tél, J. Stat. Phys. **35**, 729 (1984).
[19]R. Graham and T. Tél, Phys. Rev. A **31**, 1109 (1985).
[20]H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. **36**, 1135 (1976).
[21]R. Bonifacio and Lugiato, Opt. Commun. **19**, 172 (1976).
[22]A. Schenzle and H. Brand, Opt. Commun. **27**, 485 (1978).
[23]A. Schenzle and H. Brand, Opt. Commun. **31**, 401 (1979).
[24]R. Graham and A. Schenzle, Phys. Rev. A **23**, 1302 (1981).
[25]A. Schenzle, in *Laser '81,* edited by C. B. Collins (STS, McLean, Virginia, 1981).
[26]H. Haken, *Laser Theory,* Vol. XXV/2c of *Encyclopedia of Physics* (Springer, New York, 1970).
[27]H. Risken, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, 1970), Vol. 8.
[28]H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[29]R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).
[30]E. H. Hauge and A. Martin-Löf, J. Stat. Phys. **7**, 259 (1973).
[31]S. Grossmann, Z. Phys. **47**, 251 (1982).