Nonequilibrium potential for coexisting attractors

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(Received 23 August 1985)

The weak-noise limit of Fokker-Planck models with coexisting attractors is analyzed and the nonequilibrium potential defined by the weak-noise limit of the probability density in the steady state is constructed. The nonequilibrium potential is found to be continuous with discontinuities in its first derivatives along certain surfaces in the configuration space, which are created by the coexistence of attractors. The general rules determining the position of these surfaces in configuration space are given. Numerical and approximate analytical results for the nonequilibrium potential of the Brownian motion in a periodic potential subject to an external force are obtained. A comparison with earlier work is presented.

I. INTRODUCTION

An old problem of nonequilibrium thermodynamics is whether it is possible to define entropylike macroscopic potentials for dynamical systems in steady states which are not states of thermodynamic equilibrium, and what the properties of such potentials might be. In a recent series of papers[1–4] we have approached this problem by considering dynamical systems subject to weak Gaussian white noise of strength η and taking the limit of weak noise $\eta \to 0$. A nonequilibrium potential $\phi(q)$ with properties analogous to a course-grained free energy or negative entropy can be defined for such systems,[5–10] provided there exists a unique time-independent probability distribution $P(q,\eta)$ which is approached from any given initial distribution as the weakly stochastic dynamical system approaches its steady state:

$$\phi(q) = -\lim_{\eta\to 0} \eta \ln P(q,\eta) , \tag{1.1}$$

where q denotes the vector of the n state variables of the system, $q=(q^1,q^2,\dots,q^n)$.

By assumption the probability density $P(q,\eta)$ satisfies a time-independent Fokker-Planck equation:

$$-\frac{\partial}{\partial q^\nu}K^\nu(q)P + \frac{\eta}{2}\frac{\partial^2}{\partial q^\nu \partial q^\mu}Q^{\nu\mu}(q)P = 0 , \tag{1.2}$$

where $K^\nu(q)$ is a drift vector, which describes the deterministic dynamics of the system for $\eta=0$ via

$$\dot q^{\,\nu} = K^\nu(q) , \tag{1.3}$$

and $Q^{\nu\mu}(q)$ describes the properties of the noise. The summation convention is implied. It is assumed that the symmetric matrix $Q^{\nu\mu}$ does not vanish in the attractors, saddles, or repellors of the deterministic system (1.3).

The potential ϕ defined by Eq. (1.1) satisfies a Hamilton-Jacobi equation

$$\frac{1}{2}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^\nu}\frac{\partial\phi}{\partial q^\mu} + K^\nu(q)\frac{\partial\phi}{\partial q^\nu} = 0 \tag{1.4}$$

with energy zero which follows from Eq. (1.2). For positive definite $Q^{\nu\mu}$ (and often also under the weaker condition of non-negative $Q^{\nu\mu}$, cf. the example of Sec. II) Eq. (1.4) implies an H theorem which $\phi(q)$ satisfies under the deterministic dynamics (1.3):

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial q^\nu}K^\nu(q) = -\frac{1}{2}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^\nu}\frac{\partial\phi}{\partial q^\mu} \le 0 , \tag{1.5}$$

where equality holds only for $\partial\phi/\partial q^\nu = 0$. Thus $\phi(q)$ must then be minimal in the attractors of the deterministic system. A corresponding argument for the time-reversed dynamics following from Eq. (1.3) by the transformation $t \to -t$ shows that $\phi(q)$ must be maximal in the repellors of (1.3). Finally, one may apply the argument to trajectories starting or ending in the saddles of Eq. (1.3) with the conclusion that $\phi(q)$ must have saddles in the saddles of Eq. (1.3). The same conclusions may be drawn from the fact that $P(q,\eta)$ for $\eta\to 0$ has local maxima in the attractors, local minima in the repellors, and local saddles in the saddles of Eq. (1.3). Its extremum properties, the H theorem (1.5), and the relation of ϕ with the fluctuations of the system via Eq. (1.1) are entropylike properties of the nonequilibrium potential ϕ.

A further property with an analogy in thermodynamics is the representation by a minimum principle,[10,3] which reads, for the case where Eq. (1.3) has only one attractor,

$$\phi(q) = \min \int_{q(-\infty)\in\mathscr{A}}^{q(0)=q} dt\, L_0(q(t),\dot q(t)) + C(\mathscr{A}) . \tag{1.6}$$

Here $C(\mathscr{A})$ is the value of the potential on the attractor. $L_0(q,\dot q)$ denotes the Lagrangian associated to the Hamiltonian

$$H = \tfrac{1}{2}Q^{\nu\mu}(q)p_\nu p_\mu + K^\nu(q)p_\nu \tag{1.7}$$

of Eq. (1.4) by a Legendre transformation

$$L_0 = \tfrac{1}{2}Q_{\nu\mu}(q)[\dot q^{\,\nu} - K^\nu(q)][\dot q^{\,\mu} - K^\mu(q)] . \tag{1.8}$$

The minimum in Eq. (1.6) is taken over all paths starting in the attractor $\mathscr{A}$ for $t\to -\infty$ and ending in the point q at $t=0$. L_0 is analogous to a negative excess entropy production caused by fluctuations, which vanishes for the deterministic dynamics, and which is positive for positive definite $Q^{\nu\mu}$, for all fluctuations away from the deterministic dynamics caused by the weak noise. Equation

(1.6) with Eq. (1.8) implies that $\phi(q)$ for a single attractor in a simply connected configuration space can only increase along any path starting in the attractor $\mathscr{A}$, which is everywhere transverse to equipotential surfaces $\phi=\text{const}$.

Still a further property of ϕ which generalizes a well-known result of statistical thermodynamics is the Arrhenius-type relation for the average exit rate $R(\mathscr{A},\eta)$ of the dynamical system out of a finite domain of attraction of an attractor $\mathscr{A}$:

$$\lim_{\eta\to 0} \eta \ln R(\mathscr{A},\eta) = -[\phi(P_S) - C(\mathscr{A})] \;. \tag{1.9}$$

Here $\phi(P_S)$ is the lowest value of ϕ on the boundary surrounding the domain of the attractor $\mathscr{A}$ (cf. Refs. 11–13 for the extensive recent literature on this aspect).

After noting these close analogies between ϕ and a negative entropy, it is important to know whether ϕ also has properties which reflect the nonthermodynamic nature of the underlying dynamical system. In our preceding work[1–4] one such property was found to be the appearance of discontinuities in the first-order derivatives of the potential $\phi(q)$ in a region of configuration space sufficiently far away from—but still within the domain of—attractor $\mathscr{A}$.

The reason for the appearance of these discontinuities in the first derivative can be summarized as follows: The extremum properties of ϕ as defined by Eq. (1.1) imply that ϕ has to be determined as the solution of Eq. (1.4) which is stationary in the limit set points of Eq. (1.3). This means that the n-dimensional surface in (p,q) phase space

$$p_\nu = \frac{\partial \phi}{\partial q^\nu}, \quad \nu = 1,2,\ldots,n \tag{1.10}$$

is a separatrix of the Hamiltonian system defined by Eq. (1.4) with canonical momentum $p=(p_1,p_2,\ldots,p_n)$. This separatrix passes through the limit set points of Eq. (1.3) for $p=0$, and must be smooth near the attractors of Eq. (1.3), where a representation of ϕ of the form of Eq. (1.6) exists. However, smooth separatrices are structurally unstable in general Hamiltonian systems. Thus, for a nonthermodynamic system the separatrix (1.10) in general cannot be smooth everywhere and has wild oscillations as q approaches nonattracting limit set points of Eq. (1.3), which would make a continuously differentiable function ϕ defined by $\phi(q)=\int p_\nu dq^\nu$ a multibranched function of q. Equation (1.6) then serves to single out the minimum among these different branches which reduces $\phi(q)$ to a single-valued function. However, at all points where the minimum is transferred from one branch to another, $\phi(q)$ remains continuous, but its first derivatives, in general, have discontinuities. In thermodynamic systems such discontinuities do not appear since here the Hamiltonian-Jacobi equation (1.4) has a further special property—a detailed balance symmetry with respect to a time-reversal transformation independent of η^2—which guarantees the smoothness of the separatrix (1.10). Special nonthermodynamic systems with a continuously differentiable potential $\phi(q)$ are those for which the Hamiltonian system associated with Eq. (1.4) is integrable, at least on the hypersurface $H(q,p)=0$.

The discontinuities in the derivatives of ϕ appear already in systems with a single attractor, to which Eq. (1.5) applies. However, it is also interesting to enquire into the properties of ϕ in dynamical systems with several coexisting attractors, and to analyze how ϕ changes as one changes a parameter from a region with a single attractor through a coexistence region of two or more attractors into another region with a single attractor. This problem is studied in the present paper. For the sake of concreteness the problem is analyzed for a particular system, the damped driven nonlinear pendulum with thermal noise. The presence of the driving force makes this a nonthermodynamic system, but for vanishing driving force thermodynamics is recovered.

This particular stochastic system for finite noise intensity has, in fact, been studied extensively in the literature, either by perturbative methods[14] or numerically by the method of matrix continued fractions.[15] Here we are interested specifically in the limit of weak noise, which cannot be treated by these methods. Previous work on the limit of weak noise in this model has also been presented in the literature,[16] but the approximations employed there are controlled by a small parameter only in the limit of small dissipation and small external force, i.e., very close to the state of thermodynamic equilibrium (cf. also Sec. VI). Our goal here is to analyze the weak-noise limit for arbitrary departures from thermodynamic equilibrium. A comparison with the work of Ref. 16 will be given at the end of this paper. This special system is chosen because of its interesting physical applications[15,17,18] and because the increase of the driving force for fixed dissipation rate allows one to pass through several regimes: from a regime of a single attracting fixed point (if the configuration space is compact in the phase variable, otherwise one has infinitely many periodically equivalent fixed points) one goes to a coexistence regime of an attracting limit cycle with the attracting fixed point and then to a regime where only the limit cycle is stable. Another advantage of the example chosen here is the fact that it belongs to a class of Brownian motions for which the Hamiltonian (1.7) is integrable for $H(q,p)=0$.[4] Therefore, those discontinuities of the derivatives of the potential which are associated with the nonintegrability of the Hamiltonian system do not appear and it is possible to analyze the consequences of coexisting attractors without this further complication.

The most important general problem appearing for the case of several coexisting attractors is the necessity to generalize Eq. (1.6), since the minimum in Eq. (1.6) has then to be taken over all trajectories starting from any of the several attractors $\mathscr{A}_i$ of the system. However, before the minimum over the various attractors can be taken it is necessary to fix the relative size of the constants $C(\mathscr{A}_i)$ appearing in Eq. (1.6). These constants determine the relative weight of the attractors $\mathscr{A}_i$ in the steady-state distribution according to Eq. (1.1). A simple general rule for fixing these constants is derived in the present paper and applied to the example at hand.

As a general result we find that the coexistence of attractors in nonequilibrium systems leads to additional discontinuities in the first derivatives of ϕ beyond those which are already created by the nonintegrability of the Hamilton-Jacobi equation. These new discontinuities appear on surfaces in configurational space where the minimum principle generalizing Eq. (1.6) and including the minimum over the attractors changes its selection from one attractor to another one. Fortunately, however, these discontinuities appear only for values of the potential at or above the saddle-point value and are, therefore, not problematic in practical calculations. For the special example chosen here the potential ϕ and its surface of discontinuous first derivatives are determined qualitatively and are calculated either numerically or by analytical approximations. Thus, for the first time, we construct an entropylike potential for a nonequilibrium system with coexisting attractors of direct physical importance.

The paper is organized as follows. In Sec. II we present the dynamical system which we choose as our working example. In Sec. III we determine ϕ analytically in the two limiting cases of weak and strong damping. The discontinuities of the derivatives of ϕ appear here in a natural way and are not yet related to a minimum principle. In Sec. IV the generalization of Eq. (1.5) for several attractors is presented and a general method for finding the $C(\mathcal{A}_i)$ is given. In Sec. V and the Appendix we analyze the example introduced in Sec. II and present results of a numerical analysis. In Sec. VI we reconsider an earlier analysis of the special example which appeared in the literature[16] and which is here found to be restricted in its validity to the case of weak damping and weak driving force. Our main general conclusions have already been mentioned in this introduction.

II. BROWNIAN MOTION IN A PERIODIC POTENTIAL WITH A CONSTANT EXTERNAL FORCE IN THE WEAK-NOISE LIMIT

We investigate here the Brownian motion of a particle of unit mass along the x axis in a smooth periodic potential $\omega_0^2 g(x)$ in the presence of a constant external force $\bar{F}>0$. The x coordinate is measured in dimensionless units such that the potential is 2π periodic: $g(x+2\pi)=g(x)$. The function $g(x)$ is assumed to have a single maximum and minimum inside an interval of length 2π. The value of $g(x)$ at its maximum in $0\leq x<2\pi$ is normalized to be zero. The equation of motion has the form

$$\frac{d^2x}{d\bar{t}^2}=-\bar{\gamma}\frac{dx}{d\bar{t}}-\omega_0^2 f(x)+\bar{F}+\bar{\xi}(\bar{t}) , \qquad (2.1)$$

where $\omega_0^2 f(x)$ denotes the negative force associated with the periodic potential: $f(x)=g'(x)$ and $\bar{\gamma}$ stands for the damping constant. The random noise $\bar{\xi}$ is considered to be white and Gaussian with zero mean value and with the correlation function

$$\langle \bar{\xi}(\bar{t})\bar{\xi}(\bar{t}')\rangle=2\eta\bar{\gamma}\bar{\Theta}\delta(\bar{t}-\bar{t}') . \qquad (2.2)$$

Here the dimensionless number η is a measure of the noise intensity. In the case of thermal noise η can be considered as the inverse of Avogadro's number and, owing to Einstein's relation, $\bar{\Theta}=RT$ where R denotes the universal gas constant and T the temperature. In nonthermodynamic systems these relations are not valid. In many cases, however, the noise intensity turns out to be weak. As a particular choice we shall use the potential

$$g(x)=-\cos x-1 . \qquad (2.3)$$

Equations (2.1) and (2.2) appear in quite different contexts ranging from Josephson junctions[17] and charge-density-wave-transport[18] to communication problems. A nice review of applications has been given in Ref. 15. Here we consider this system as a prototype of bistable stochastic systems and illustrate how the weak-noise-limit description of monostable systems[3] can be extended to multistable ones. The results for the particular example may also be of practical relevance since the very powerful method of matrix continued fractions[15] breaks down just in the limit of weak noise.[15]

First, we make the equation of motion dimensionless by measuring the time in units of $1/\omega_0$. Equations (2.1) and (2.2) then become equivalent to

$$\begin{aligned} &\dot{x}=v ,\\ &\dot{v}=-\gamma v-f(x)+F+\xi(t) , \qquad (2.4)\\ &\langle \xi(t)\xi(t')\rangle=2\eta\gamma\Theta\delta(t-t') , \end{aligned}$$

where v is the dimensionless velocity, the dot means differentiation with respect to $t=\omega_0\bar{t}$, and

$$\gamma=\bar{\gamma}/\omega_0,\quad F=\bar{F}/\omega_0^2,\quad \Theta=\bar{\Theta}/\omega_0^2 \qquad (2.5)$$

are the new parameters.

In the weak-noise limit there exists a Hamiltonian associated with the stochastic process (2.4) [cf. (1.7)]

$$H=\gamma\Theta p_v^2+p_v[-\gamma v-f(x)+F]+p_x v , \qquad (2.6)$$

where p_v and p_x are the momenta conjugate to v and x, respectively. Any smooth solution of the corresponding Hamilton-Jacobi equation,[1–4]

$$\gamma\Theta\left[\frac{\partial\phi}{\partial v}\right]^2+\frac{\partial\phi}{\partial v}[-\gamma v-f(x)+F]+\frac{\partial\phi}{\partial x}v=0 \qquad (2.7)$$

with the boundary condition that ϕ is minimal at one of the attractors of the deterministic dissipative system [(2.4) with $\xi=0$], gives a local solution of the Fokker-Planck equation in the form

$$P(x,v,\eta)\sim\exp[-\phi(x,v)/\eta] \qquad (2.8)$$

around the given attractor. A direct solution of the Hamilton-Jacobi equation for some stochastic models has been given in Refs. 19–21, 1, and 2. Since the solutions of (2.7) scale with $1/\Theta$ we may set $\Theta=1$ without restriction of generality. Thus only two free parameters γ and F are left in the weak-noise limit $\eta\to 0$, just as in the dissipative system. The general rules for the construction of the global potential from local pieces follow from the representation of ϕ by Eq. (1.6) or a generalization thereof. This will be discussed in Sec. IV. In the present case, where noise occurs only in the second equation of (2.4), the expression for L appearing in Eq. (1.6) as given by Eq. (1.8) has to be modified by eliminating v via $v=\dot{x}$ and the integrand in Eq. (1.6) now reads

$$L_0=\frac{1}{4\gamma}[\ddot{x}+\gamma\dot{x}+f(x)-F]^2 \ . \tag{2.9}$$

Equation (1.6) then assumes the form

$$\phi(x)=\min \int_{(x(-\infty),\dot{x}(-\infty)\in\mathscr{A})}^{(x(0)=x,\dot{x}(0)=v)} d\tau L_0(x,\dot{x},\ddot{x})+C(\mathscr{A}) \ . \tag{2.10}$$

We shall be interested in a nonequilibrium steady state maintained by the presence of a stationary probability current, therefore we are looking for a global potential which is continuous and singled valued on the surface of the cylinder $0\leq x<2\pi$, $-\infty<v<+\infty$.

Since the local pieces of the potential are associated with the attractors of the deterministic system it is worth studying here briefly the motion in the potential $V(x)=g(x)-Fx$. The maximum of $f(x)=g'(x)$ is assumed to be unity in the following. For $F<1$ the equilibrium condition $f(x)=F$ has two solutions in the interval $[0,2\pi)$. The solutions x_S and x_0—belonging to the local maximum and to the the local minimum of $V(x)$—characterize a saddle point $S'=(x_S,v=0)$ and a stable fixed point $P_0=(x_0,v=0)$ of the dynamics, respectively. The flow associated with the dynamics is periodic in x owing to the periodicity of $f(x)$. It is therefore sufficient to consider an interval of length 2π in x, which for $F<1$ will be chosen to be bounded by two subsequent saddle points, $(x_S,x_S+2\pi]$. The configuration space $x_S<x\leq x_S+2\pi$, $-\infty<v<+\infty$ is then the surface of a cylinder.

For small values of F the flow diagram is a somewhat distorted version of the flow diagram of the free damped pendulum, the distortion being due to the presence of the $(v\rightarrow -v)$-symmetry breaking force F. There is only a single attractor, the fixed point P_0. The stable and unstable manifolds of the saddle $S=(x_S+2\pi,v=0)$ and its equivalent $S'=(x_S,v=0)$ are typical trajectories characterizing the flow [Fig. 1(a)]. When increasing the force F at fixed damping, motion in the positive x direction is supported by the force and there is a strong deformation of the trajectories in the upper half-plane, unless $\gamma v \gg F$. At a critical value $F=F_c(\gamma)$ the upper unstable manifold of the saddle point S returns to the saddle point after one trip around the cylinder and coincides with the upper stable manifold, forming a periodic homoclinic trajectory. (If S and S' are not considered as equivalent the trajectory would be heteroclinic.) For $F>F_c(\gamma)$ this periodic trajectory becomes separated from the manifolds of the saddle point and is an attractive running solution, i.e., a stable limit cycle closed around the cylinder. The upper unstable manifold of the saddle point approaches the limit cycle asymptotically and the stable fixed point attracts trajectories from a certain region of the configuration space only [Figs. 1(b) and 1(c)]. Two attractors are coexisting. Upon further increase of F this remains so until at $F=1$ the attracting fixed point and the saddle point coalesce, and for $F>1$ the limit cycle becomes the only attractor [Fig. 1(d)]. If $F<1$ is fixed and γ is increased instead, starting from the coexistence region, at a critical value of the damping $\gamma=\gamma_c(F)$ the limit cycle ceases to exist just as by lowering F at a fixed γ. For γ sufficiently large P_0

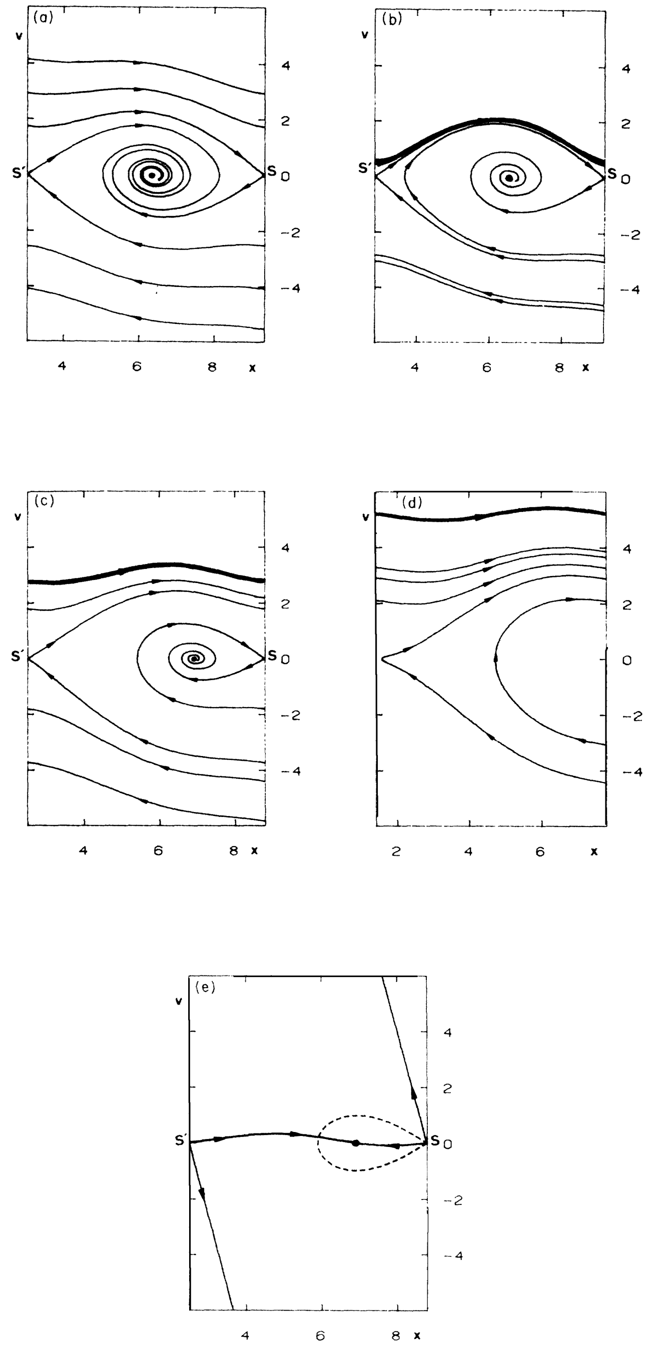

FIG. 1. Typical trajectories of the deterministic motion in the case of the potential $g(x)=-\cos x-1$. (a) Stable and unstable manifolds of the saddle point $(x_S=\pi-\arcsin F, v=0)$ on the cylinder $x_S<x\leq x_S+2\pi$, $-6<v<6$. The stable fixed point P_0 is denoted by a dot. $\gamma=0.2$, $F=0.1$. (b) Same as (a) for $\gamma=0.2$, $F=0.28$ just above the critical value $F=0.252$. The limit cycle is represented by a bold line. (c) Same but for $\gamma=0.2$, $F=0.6$. (d) Two typical trajectories on the cylinder $\pi/2<x\leq 5\pi/2$. No fixed points exist. $\gamma=0.2$, $F=1.05$. (e) Stable and unstable manifolds of the saddle point on the cylinder $x_S<x\leq x_S+2\pi$, $-6<v<6$ in the overdamped case with the fixed point P_0 as the only attractor. $\gamma=5$, $F=0.6$. Dashed line denotes a curve with $\phi=0$ as discussed in Sec. V.

turns from a focus into a node, i.e., into a fixed point with real negative eigenvalues [Fig. 1(e)]. In this regime of γ there is practically no value of F where coexisting attractors would be present. A quantitative description of the phase diagram in the (γ,F) plane for the cosine potential has been given in Refs. 15 and 16. Here, only the qualitative picture given in terms of the invariant manifolds of the saddle point will be important in the following.

III. TWO LIMITING CASES

A. Weak damping

Since the Hamilton-Jacobi equation (2.7) cannot be solved exactly under the required boundary conditions (ϕ being minimal at the attractors of the deterministic system and continuous on the cylinder $x_S < x \le x_S + 2\pi$; $-\infty < v < +\infty$) it is highly desirable to study limiting cases where analytic solutons become possible. We shall see that a peculiar qualitative feature of the potential, its nondifferentiality along certain curves, shows up already in limiting cases. First, the limit of low friction, $\gamma \ll 1$, will be considered. Note, that this assumption is made only after the weak-noise limit has been taken, i.e., the complete condition is $\eta \ll \gamma \ll 1$. A stationary solution in the presence of a weak damping can be reached only if the external force is weak, too. Therefore, it is worth scaling the latter with γ, i.e., to write

$$F=\gamma F_0 , \tag{3.1}$$

where F_0 is of order unity. Equation (2.4) describes then a weakly perturbed motion in the mechanical potential $g(x)$. Thus, one expects that the variable

$$E=\frac{v^2}{2}+g(x) \tag{3.2}$$

is well suited for characterizing the system for small γ. In particular, in leading order the potential ϕ will be a function of E only.

To show this, we first eliminate the variable v by writing

$$\phi^{\pm}(x,E)=\phi(x,\pm v(x,E)) , \tag{3.3}$$

where the function $v(x,E)$ is defined as

$$v(x,E)=\{2[E-g(x)]\}^{1/2} \tag{3.4}$$

with $g(x)<E$. We recall that the maximum value of g, taken at a certain $\bar{x}_S$, is chosen to be zero. After transforming the Hamilton-Jacobi equation into the variables x,E we find

$$\gamma v(x,E)\left[\left[\frac{\partial\phi^{\pm}}{\partial E}\right]^2-\frac{\partial\phi^{\pm}}{\partial E}\right]\pm\gamma F_0\frac{\partial\phi^{\pm}}{\partial E}\pm\frac{\partial\phi^{\pm}}{\partial x}=0 . \tag{3.5}$$

Next, we assume that ϕ can be expanded into a power series in γ:

$$\phi^{\pm}(x,E)=\sum_{n=0}^{\infty}\gamma^n\phi_n^{\pm}(x,E) . \tag{3.6}$$

In the low-friction case Eq. (3.6) with the first few contributions may then provide an approximation for the potential. The accuracy of such approximations, in general, will depend on E and x. Inserting (3.6) into (3.5) and considering terms of different order in γ separately, we find that $\phi_0^{\pm}$ does not depend on x. In order γ

$$\left[\frac{\partial\phi_0^{\pm}}{\partial E}\right]\left[v(x,E)\left[\frac{\partial\phi_0^{\pm}}{\partial E}-1\right]\pm F_0\right]=\mp\frac{\partial\phi_1^{\pm}}{\partial x} \tag{3.7}$$

follows. Assuming $E\ge 0$ and requiring the periodicity of $\phi_1^{\pm}$ in x we integrate (3.7) over x along a path encircling the cylinder between subsequent saddle-point coordinates $\bar{x}_S$ and $\bar{x}_S+2\pi$ and find

$$\phi_0^{\pm}(E)=E\mp\int_0^E\frac{F_0}{\bar{v}(E)}dE , \tag{3.8}$$

where

$$\bar{v}(E)=\frac{1}{2\pi}\int_{\bar{x}_S}^{\bar{x}_S+2\pi}v(x,E)dx \tag{3.9}$$

denotes the average velocity along a running trajectory of energy E in the conservative system. After substituting (3.8) in (3.7) we obtain

$$\phi_1^{\pm}(x,E)=F_1^{\pm}(x,E)+h_1^{\pm}(E) \tag{3.10}$$

with

$$F_1^{\pm}(x,E)=-F_0\left[1\mp\frac{F_0}{\bar{v}(E)}\right]\times\left[x-\bar{x}_S-\frac{1}{\bar{v}(E)}\int_{\bar{x}_S}^{x}v(x,E)dx\right]$$

illustrating the explicit x dependence appearing already in the first correction to $\phi_0^{\pm}$. The function $h_1^{\pm}$ is determined from the equation arising in second order in γ and may be represented as

$$h_1^{\pm}(E)=\int_0^E k_1^{\pm}(\tilde{E})d\tilde{E} ,$$
$$k_1^{\pm}(E)=\frac{1}{2\pi[F_0\mp\bar{v}(E)]}\times\int_0^{2\pi}\frac{\partial F_1^{\pm}(x,E)}{\partial E}\left[F_0\left[1-\frac{2v(x,E)}{\bar{v}(E)}\right]\pm v(x,E)\right]dx . \tag{3.11}$$

As the leading contribution to the potential depends only on the energy, the equipotential surfaces are given by the condition $E=$const.

Another solution of (3.7) is

$$\phi_0^{\pm}(E)=E\quad(E<0) \tag{3.12}$$

with the correction

$$\phi_1^{\pm}(x,E)=-F_0(x-\bar{x}_S-2\pi) . \tag{3.13}$$

The latter correction is not periodic in x but this does not cause any problem, since the solution will only be needed in a subinterval of $\bar{x}_S<x\le\bar{x}_S+2\pi$, where the solution

(3.8) does not hold. In a higher-order calculation no further contribution arises since

$$\phi_0(x,v)=\frac{v^2}{2}+g(x)-F(x-\bar{x}_S-2\pi) \tag{3.14}$$

is an exact local solution of the Hamilton-Jacobi equation (2.7). Note that the potential given by (3.14) is minimal at the attracting fixed point of the deterministic system obtained from (2.4) without noise.

In summary, the global potential ϕ in leading order is given by (3.8) for $E>0$ and by (3.12) for $E<0$. The two pieces fit continuously together along the line defined by $E=0$, however, with a discontinuity in the first derivative. The result up to this order agrees with that obtained by Risken and Vollmer[22] who considered the case of low friction without taking the weak-noise limit (i.e., $\gamma \ll \eta$). In the next order of γ the line where the local pieces fit together deviates a little from that given by $E=0$. It can be calculated from (3.8)–(3.12) as follows.

First, we note that E will remain close to zero and it is sufficient to consider $\phi_0^{\pm}(x,E)+\gamma\phi_1^{\pm}(x,E)$ for small values of E only $[E=0(\gamma)]$ where we obtain from Eqs. (3.8) and (3.10)

$$\phi_0^{\pm}(E)+\gamma\phi_1^{\pm}(x,0) \approx \left[1\mp\frac{F_0}{\bar{v}(0)}\right] \times\left[E-\gamma F_0\left[x-\bar{x}_S-\frac{1}{\bar{v}(0)}\int_{\bar{x}_S}^{x} v(\tilde{x},0)d\tilde{x}\right]\right]. \tag{3.15}$$

Next, we determine the saddle points of the potentials $\phi_0^{\pm}(x,v)+\gamma\phi_1^{\pm}(x,v)$ and $\phi_0(x,v)$ to first order in γ and notice that the saddle points of both potentials in the interval $\bar{x}_S<x\le\bar{x}_S+2\pi$ coincide in

$$S=\left[\bar{x}_S+\frac{\gamma F_0}{g''(\bar{x}_S)}+2\pi,0\right]. \tag{3.16}$$

Furthermore, both potentials are already normalized in such a way that they both agree to order γ at the saddle points. As will be shown in Sec. IV this normalization is necessary in order to determine the curve where both potentials are joined continuously (but with discontinuous derivatives)

$$\phi_0^{\pm}(E)+\gamma\phi_1^{\pm}(x,0)=\phi_0(x,E) \tag{3.17}$$

for $\bar{x}_S<x\le\bar{x}_S+2\pi$. Explicitly we find the two asymmetric and nonperiodic curves

$$E=E_c^{\pm}(x)=\gamma\left[F_0(x-\bar{x}_S)\mp 2\pi\bar{v}(0) - \left[\frac{F_0}{\bar{v}(0)}\mp 1\right]\int_{\bar{x}_S}^{x} v(\tilde{x},0)d\tilde{x}\right] \tag{3.18}$$

and hence

$$v=\pm\left\{-2g(x)+2\gamma\left[F_0(x-\bar{x}_S)\mp 2\pi\bar{v}(0) - \left[\frac{F_0}{\bar{v}(0)}\mp 1\right]\int_{\bar{x}_S}^{x} v(\tilde{x},0)d\tilde{x}\right]\right\}^{1/2} \tag{3.19}$$

which correspond to two continuous nonclosing curves on the cylinder $\bar{x}_S<x\le\bar{x}_S+2\pi$, $-\infty<v<+\infty$ for $v\ge 0$ and $v\le 0$, respectively, which are connected at the point $x=\bar{x}_S+2\pi$, $v=0$. Of course, the present analysis is only valid as long as the second term under the square root in (3.19) is small compared to the first term.

Another set of curves, which are of special interest, are the equipotential lines $\phi=0$ passing through the common saddle point S. They are defined by $\phi_0(x,E)=0$, $\phi_0^{\pm}(x,E)+\gamma\phi_1^{\pm}(x,E)=0$. We obtain from ϕ_0

$$E=E_0(x)\equiv\gamma F_0(x-\bar{x}_S-2\pi) \tag{3.20}$$

and, using again the fact that E is close to zero along these equipotential lines as long as γ is small, from $\phi_0^{\pm}+\gamma\phi_1^{\pm}$

$$E=E_0^{\pm}(x)\equiv\gamma F_0\left[x-\bar{x}_S-\frac{1}{\bar{v}(0)}\int_{\bar{x}_S}^{x} v(\tilde{x},0)d\tilde{x}\right]. \tag{3.21}$$

In the interval $x_S<x\le x_S+2\pi$ the curve (3.20) is restricted to $E\le 0$ and aperiodic in x. The curves (3.21) are defined for $E \gtrless 0$ and are periodic in x. They correspond to closed curves winding around the cylinder $x_S\le x\le x_S+2\pi$, $-\infty<v<+\infty$ for $v>0$ and $v<0$. From Eqs. (3.18) and (3.21) it follows that for $x_S<x\le x_S+2\pi$ the inequalities $E_c^{+}(x)\le E_0^{+}(x)$ and $E_c^{-}(x)\ge E_0^{-}(x)$ hold. Hence the equipotential line $E=E_0^{-}(x)$ for $v<0$ is irrelevant, since it lies in the domain where the corresponding potential $\phi_0^{-}+\gamma\phi_1^{-}$ does not apply. On the other hand, the equipotential line $E=E_0^{+}(x)$ for $v>0$ is relevant, as are the lines $E=E_0(x)$ for $v\gtrless 0$.

Numerical results for equipotential lines in cases where the present perturbation theory does not apply are presented in Sec. V.

Finally, let us remark that the global potential ϕ everywhere is realized by the smallest of the three local solutions $\phi_0^{\pm}+\gamma\phi_1^{\pm}$ and ϕ_0. Also it may be worth pointing out that in the region of parameters $F_0>\bar{v}(0)$ where a limit cycle of the deterministic dissipative system exists the minimum of $\phi_0^{+}+\gamma\phi_1^{+}$ given by (3.8)–(3.10) coincides with the curve of the limit cycle evaluated also up to first order in γ. Thus, in the case of weak damping we have found the solution of the Hamilton-Jacobi equation which fulfills all the boundary conditions. The potential obtained has a discontinuous first derivative along a generally nonclosing line winding around the cylinder.

B. Strong damping

We consider the Brownian motion in the potential $V(x)=g(x)-Fx$ in the limiting case of extremely strong damping so that at first the noise intensity is kept finite. The weak-noise limit will be taken afterwards ($\gamma^{-1} \ll \eta \ll 1$). Since the friction is very strong the velocity of the particle is a fast variable and can be adiabatically eliminated. Thus, we arrive at a Smoluchowski-type equation for the position

$$\dot{x}=[F-f(x)]/\gamma+\xi(t)/\gamma \; , \tag{3.22}$$

where the properties of the noise $\xi(t)$ have been given in (2.4). The corresponding stationary Fokker-Planck equation reads

$$\frac{\partial}{\partial x}\left[\frac{F-f(x)}{\gamma}P(x,\eta)-\frac{\eta}{\gamma}\frac{\partial}{\partial x}P(x,\eta)\right]=0 \; . \tag{3.23}$$

The 2π-periodic stationary probability density of this one-variable process obtained in the presence of a constant probability current ηJ can be written as[14,15]

$$P(x,\eta)=N\exp\left[-\frac{V(x)}{\eta}\right]\int_x^{x+2\pi}\exp\left[\frac{V(\tilde{x})}{\eta}\right]d\tilde{x} \; , \tag{3.24}$$

where

$$N=\frac{\gamma J}{1-\exp(-2\pi F/\eta)} \; . \tag{3.25}$$

In the limit of weak noise we expect a solution in the form

$$P(x,\eta)\sim\exp[-\phi(x)/\eta] \; . \tag{3.26}$$

In order to find $\phi(x)$ we perform the limit $\eta\to 0$ in (3.24). The integral can then be evaluated by means of the saddle-point method. We consider the x interval $(x_S,x_S+2\pi]$. Remember that x_S is the coordinate of the saddle point in $[0,2\pi)$ and, thus, $V(x)$ has a local maximum there. The interval is divided into two parts by the value $\bar{x}_1$ defined via $V(\bar{x}_1)=V(x_S+2\pi)$ (see Fig. 2). Since the saddle-point method picks up the maximum value of the exponent in the integrand, quite different results are obtained in the two subintervals. For $x\in(x_S,\bar{x}_1]$ the maximum point is x itself, while for $x\in(\bar{x}_1,x_S+2\pi]$ the maximum is taken at $x_S+2\pi$. Therefore, from (3.24) and (3.26)

$$\phi(x)=\begin{cases}0, & x_S<x\le\bar{x}_1 \\ V(x)-V(x_S+2\pi), & \bar{x}_1\le x\le x_S+2\pi\end{cases} \tag{3.27}$$

follows which is again not continuously differentiable (Fig. 2). With increasing F the point x_1 approaches $x_S+2\pi$.

Beyond $F=1$ the fixed points disappear and the potential associated with the limit cycle becomes a constant. (In the strongly damped case no coexistence of attractors is possible as mentioned earlier.) We shall see in the next sections that the same potential $\phi(x)$ arises also in the reversed limit $\eta\ll\gamma^{-1}\ll 1$ after the variable v is integrated out.

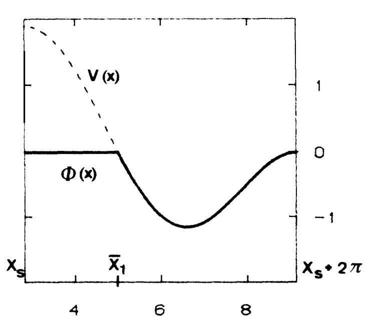

FIG. 2. The potential $\phi(x)$ (bold line) in the overdamped case in the interval $x_S<x<x_S+2\pi$. Dashed line belongs to the mechanical potential $V(x)=\cos x-\cos x_S-F(x-x_S-2\pi)$.

IV. GLOBAL POTENTIAL FOR COEXISTING ATTRACTORS

In this section we wish to formulate the general principles and rules which determine the global potential in terms of the different local potentials associated with different attractors. Our basic tool is the minimum principle satisfied by the potential, which we derived in a preceding paper[3] from the path integral of the Fokker-Planck equation in the weak-noise limit. It can be formulated as

$$\phi(q)=\min_{(i)}\phi_i(q) \tag{4.1}$$

with

$$\phi_i(q)=\min\int_{q(-\infty)\in\mathscr{A}_i}^{q(0)=q}L_0(q(\tau),\dot{q}(\tau))d\tau+C(\mathscr{A}_i) \tag{4.2}$$

generalizing Eq. (1.6). Here the index i is used to label the various different attractors consisting of the point sets $\mathscr{A}_i$. $\phi_i(q)$ is the local potential associated with the attractor $\mathscr{A}_i$. The absolute minimum in Eq. (4.2) is taken over all attractors $\mathscr{A}_i$ whose potential $\phi_i(q)$ can be defined for the point q according to Eq. (4.2). The Lagrangian L_0 [Eq. (1.8)] is associated with the Hamiltonian (1.7) of the Hamilton-Jacobi equation (1.4) describing the weak-noise limit. The absolute minimum in Eq. (4.2) is taken over all trajectories $q(\tau)$ starting on $\mathscr{A}_i$ at $\tau=-\infty$ and ending in q at $\tau=0$. The constants $C(\mathscr{A}_i)$ represent a normalization of the potential $\phi(q)$. Clearly, the constants $C(\mathscr{A}_i)$ for all different i must be fixed relative to each other [apart from a common additive constant C representing the overall normalization of the global potential $\phi(q)$] before the minimum in Eq. (4.1) acquires an absolute well-defined meaning. The solution of this task was left undiscussed in Ref. 3, where the competition of different attractors was not considered explicitly, and we therefore wish to address this problem here. A general abstract alogorithm for determining the constants $C(\mathscr{A}_i)$ was developed by Freidlin and Ventsell.[11] However, their algorithm was designed for the most general case and is rather complicated to handle even in very simple special cases. Therefore, we find it worthwhile to present a more intuitively applicable procedure, which becomes simple in simple examples.

The basic principle fixing the constants $C(\mathscr{A}_i)$ relative to each other is the balance between the probability currents into and out of the domain of attraction of all attractors $\mathscr{A}_i$ in the steady state. As already mentioned in Ref. 3, the exit rate $R(\mathscr{A}_i,\eta)$ out of the domain of attraction $\mathscr{A}_i$ with the boundary $B(\mathscr{A}_i)$ satisfies in the weak-noise limit (cf. Refs. 11–13)

$$\min_{q\in B(\mathscr{A}_i)} \phi_i(q) = -\lim_{\eta\to 0} \eta \ln R(\mathscr{A}_i,\eta) \ . \tag{4.3}$$

Note that the validity of this relation depends on the proper normalization of $\phi_i(q)$, i.e., on the proper choice of $C(\mathscr{A}_i)$ in Eq. (4.2). The exit rate $R(\mathscr{A}_i,\eta)$ appears again as the entrance rate into the domain of attraction of another attractor which borders on the domain of attraction of $\mathscr{A}_i$ at the point $q\in B(\mathscr{A}_i)$ which is selected in Eq. (4.3) by the minimum principle. And, finally, the exit rate $R(\mathscr{A}_i,\eta)$ must be balanced by an equal rate of entrance into the domain of attraction of $\mathscr{A}_i$, which need not occur through the same point $q\in B(\mathscr{A}_i)$ which is selected by the minimum in (4.3) for the exit. These considerations are taken into account by the following procedure for fixing the constants $C(\mathscr{A}_i)$.

(i) For all attractors $\mathscr{A}_i$ the local pieces $\phi_i(q)$ are defined by Eq. (4.2). Then for each $\phi_i(q)$ the absolute minimum

$$M_i = \min_{q\in B(\mathscr{A}_i)} \phi_i(q) \tag{4.4}$$

is determined, which $\phi_i(q)$ achieves on the separtrices surrounding the basin of attraction of $\mathscr{A}_i$. In general, these minima are located in singular points or singular sets of points on the separatrices. For simplicity we assume that the point q selected by the minimum in Eq. (4.4) is unique for each attractor. Let us call M_i the separatrix minimum of $\mathscr{A}_i$ in the following.

(ii) The set of all attractors $\{\mathscr{A}_i\}$ can now be decomposed into different disjoint closed sets S_ν each of which consists of all attractors which can be connected with all other members of the same set by exits and entrances through (not necessarily coinciding) separatrix minima, either directly or indirectly. Attractors $\mathscr{A}_i$ whose domain cannot be reached from any other attractors $\mathscr{A}_i'$ through the separatrix minima of $\mathscr{A}_i$ each form a set S_ν containing only $\mathscr{A}_i$. Attractors in different sets S_ν can then not be reached from each other by exits and entrances through separatrix minima and may therefore be considered as being decoupled from each other for sufficiently small noise. In simple cases there will be only one or a few such sets S_ν.

(iii) From our preceding remarks about the balance of exit and entrance rates in the steady state it now follows that within each of the sets S_ν

$$M_i = -\lim_{\eta\to 0} \eta \ln R(\mathscr{A}_i,\eta) = k_\nu \tag{4.5}$$

for all $\mathscr{A}_i \in S_\nu$, where the k_ν are constants independent of i. The equality of all M_i for $\mathscr{A}_i\in S_\nu$ can be achieved by choosing the constants $C(\mathscr{A}_i)$ in Eq. (4.2) appropriately. This fixes the $C(\mathscr{A}_i)$ up to an additive constant C_ν. The procedure can now be repeated with the different sets S_ν replacing the attractors $\mathscr{A}_i$ of the first step. Thus it is again necessary to determine the separatix minima

$$M_\nu = \min_{q\in B(S_\nu)} \phi_\nu(q) \ , \tag{4.6}$$

where

$$\phi_\nu(q) = \min_{(\mathscr{A}_i \in S_\nu)} \phi_i(q) \tag{4.7}$$

and $B(S_\nu)$ is the boundary of the union of the domains of attraction of all $\mathscr{A}_i\in S_\nu$. M_ν defined by Eq. (4.6) is necessarily larger than M_i defined by Eq. (4.4), otherwise the set S_ν would not have been closed. Again the location of the exit and entrance points selected by the minimum in Eq. (4.3) serves to decompose the set $\{S_\nu\}$ of all S_ν into disjoint closed sets whose members can be connected with all other members by exits and entrances through separatrix minima M_ν. The equality of all M_ν within each of the sets then fixes the additive constants C_ν within each set up to a common additive constant. The procedure must be repeated up to the level where there remains only one closed set, and the constants $C(\mathscr{A}_i)$ are then all fixed relative to each other up to a common additive constant, which may be used to normalize $\min_{(q)}\phi(q)=0$. In typical practical cases this means that the procedure has to be carried through only once or twice.

As two simple examples we consider the two limiting cases analyzed in Secs. III A and III B. In the case of Sec. III A two attractors, a fixed point and a limit cycle, coexist provided that $F_0 > \overline{v}(0)$. The potential ϕ_0 given by Eq. (3.14) is generated via Eq. (4.2) by trajectories starting from the fixed point. The potential $\phi_0^+ + \gamma\phi_1^+$ given by Eqs. (3.8) and (3.10) is correspondingly generated by trajectories starting from the limit cycle. Both potentials have a saddle point at the same point $(x_S+2\pi,0)$, which, therefore, must be the position of their separatrix minima. By the general rule given above, both potentials must coincide in this common point. This rule was already used in Sec. III A in order to find the line $E=E_c^+(x)$, where ϕ_0 and $\phi_0^+ + \gamma\phi_1^+$ are continuously joined.

The examples of Sec. III B we may consider as a one-dimensional periodic system with infinitely many attracting fixed points. The local potential generated via Eq. (4.2) by trajectories starting at the fixed point $\mathscr{A}_j$ at $x=x_j=x_0+2\pi j$ may be written as

$$\phi_j(x) = g(x) - Fx + C_j \ , \tag{4.8}$$

for

$$x_{Sj} \le x \le x_{Sj} + 2\pi \ ,$$

where $x_{Sj} = x_S + 2\pi j$. We recall that $g(x)$ was assumed to be 2π periodic and to have one maximum and one minimum in the periodicity interval 2π and we assume presently that $F<1$ so that $\phi_j(x)$ still has one maximum and one minimum in a 2π interval of x. Now, it is clear that the separatrix minimum $M_j=\phi_j(x_{Sj}+2\pi)$ of the attractor at x_j is achieved at the local maximum of ϕ_j at $x_{Sj}+2\pi > x_j$. As all the attractors $\mathscr{A}_j$ are connected by exits and entrances through separatrix minima it follows that they are all in one set S within which the separatrix minima are the same. Hence, the constant C_j in Eq. (4.8) must be adjusted in such a way that $M_j = M$ independent of j. Choosing $M = 0$ we obtain

$$C_j = -g(x_{Sj}) + F(x_{Sj} + 2\pi) \ . \tag{4.9}$$

Before the minimum principle of Eq. (4.1) can be applied it is still necessary to extend the calculation of $\phi_j(x)$ outside the interval $(x_{Sj}, x_{Sj} + 2\pi]$ by using Eq. (4.2). This is easy since a trajectory of the deterministic system exists which connects the point $x = x_{Sj} + 2\pi$ with all points in $x_{Sj} + 2\pi \le x \le x_j + 2\pi$. Along this deterministic trajectory L_0 of Eq. (4.2) vanishes, hence

$$\phi_j(x) = g(x_{Sj} + 2\pi) - F(x_{Sj} + 2\pi) + C_j = 0 \tag{4.10}$$

for $x_{Sj} + 2\pi \le x \le x_j + 2\pi$. Similarly, a deterministic trajectory connects the point $x = x_{Sj}$ with all points $x_{Sj} \ge x \ge x_j - 2\pi$, hence, by the same token

$$\phi_j(x) = g(x_{Sj}) - Fx_{Sj} + C_j = 2\pi F \tag{4.11}$$

for $x_{Sj} \ge x \ge x_j - 2\pi$. Extending x beyond $x_j - 2\pi$ and $x_j + 2\pi$ no deterministic trajectories are available any more and ϕ_j must increase in both directions. It now follows that for $x_{Sj} \le x \le x_{Sj} + 2\pi$

$$\phi(x) = \min_{(i)} \phi_i(x) = \begin{cases} 0, & x_{Sj} \le x \le \bar{x}_j \\ g(x) - F(x - x_{Sj} - 2\pi) - g(x_{Sj}), & \bar{x}_j \le x \le x_{Sj} + 2\pi \ . \end{cases} \tag{4.12}$$

Here $\bar{x}_j$ is a point in the interval where the first derivative of $\phi(x)$ has a discontinuity, which is given by

$$g(\bar{x}_j) - F\bar{x}_j = g(x_{Sj}) - F(x_{Sj} + 2\pi) \ . \tag{4.13}$$

We, therefore, recover our result (3.27) in an entirely different way.

V. NONEQUILIBRIUM POTENTIAL FOR THE BROWNIAN MOTION (2.4)

A. First integral for the Hamiltonian dynamics

First, we investigate the dynamics defined by the Hamiltonian (2.6). The canonical equations ($\Theta = 1$) read

$$\begin{aligned} \dot{x} &= v \ , \\ \dot{v} &= -\gamma v - f(x) + F + 2\gamma p_v \ , \\ \dot{p}_x &= p_v f'(x) \ , \\ \dot{p}_v &= \gamma p_v - p_x \ . \end{aligned} \tag{5.1}$$

It can be directly seen that the deterministic motion takes place on the plane $p_x = p_v = 0$ of the four-dimensional phase space. The value of the energy specifying the Hamiltonian dynamics associated with a Fokker-Planck process is zero [see Eq. (1.4)]. From the condition $H = 0$ and (5.1) then follows that a time-dependent first integral exists on the energy-zero hypersurface which is given by

$$A = \left[\frac{v(t)}{p_v(t)} - 1 \right] \exp(\gamma t) \tag{5.2}$$

as one can check directly. Since $H = 0$ and $A = \text{const}$, a Hamiltonian trajectory is uniquely defined by the equations

$$\dot{x} = v, \quad \dot{v} = -\gamma(t) v - f(x) + F \tag{5.3}$$

with

$$\gamma(t) = \frac{A \exp(-\gamma t) - 1}{A \exp(-\gamma t) + 1} \gamma \ . \tag{5.4}$$

The motion along Hamiltonian trajectories is, therefore, a damped motion in the mechanical potential $V(x) = g(x) - Fx$, however, with a time-dependent friction coefficient (5.4). The value of A is specified by the initial data v_0, p_{v0}. Since for $t \to \infty$ the damping goes to a constant $(-\gamma)$ the motion is nonchaotic. Thus, we can exclude the possibility, investigated in earlier papers,[2,3] that a nondifferentiable potential would arise because of the nonintegrability of the Hamiltonian system associated with the stochastic process.

Let us now apply our results to the construction of the potential ϕ for our working example.

B. Nonequilibrium potential for an attracting fixed point on a cylinder

We consider the different regimes of the model separately and start with the case where the only attractor is the fixed point P_0. Thus, we may apply Eq. (4.2) in its form (2.10) directly. The minimizing trajectories connecting P_0 with a given point (x,v) must satisfy Hamilton's equation, and thus also the reduced equation (5.3). The constant A in Eq. (5.4) is constant along any Hamiltonian trajectory and is only a function of the end point (x,v) at $t = 0$. Inserting Eq. (5.3) in Eqs. (2.9) and (2.10) we obtain

$$\phi(x,v) = \gamma \int_{(x(-\infty)=x_0, \dot{x}(-\infty)=0)}^{(x(0)=x, \dot{x}(0)=v)} \frac{\dot{x}^2}{(1 + Ae^{-\gamma\tau})^2} d\tau + \phi(x_0, 0) \ , \tag{5.5}$$

where the integral can be taken along a solution of Eq. (5.3). Note that Eq. (5.3) implies the identity

$$\gamma \int_{(x(-\infty)=x_0, \dot{x}(-\infty)=0)}^{(x(0)=x, \dot{x}(0)=v)} \frac{1 - Ae^{-\gamma\tau}}{1 + Ae^{-\gamma\tau}} \dot{x}^2(\tau) d\tau = \tfrac{1}{2} v^2 + g(x) - Fx - g(x_0) + Fx_0 \ . \tag{5.6}$$

Let us first determine those end points (x,v) to which the value

$$A(x,v) = 0 \tag{5.7}$$

is associated. For $A=0$ the integral in Eq. (5.5) can be evaluated with the help of the identity (5.6) and we obtain

$$\phi=\phi_0(x,v)=v^2/2+g(x)-Fx-g(x_S)+F(x_S+2\pi) \ . \quad (5.8)$$

Here we normalized ϕ_0 so that it vanishes at $x=x_S+2\pi$, $v=0$. Equation (5.3) now reduces to

$$\dot{x}=v \ ,$$
$$\dot{v}=\gamma v+F-f(x) \ . \quad (5.9)$$

We notice that these equations are equivalent to the deterministic equations of our system after the transformation

$$t\rightarrow -t, \quad v\rightarrow -v \ . \quad (5.10)$$

Thus P_0 is a repellor of Eq. (5.9). In order to find all points with $A(x,v)=0$, i.e., points (x,v) which can be reached at $t=0$ by solutions of Eq. (5.9) starting in its repellor P_0 at $t\rightarrow -\infty$, it is merely necessary to determine the domain of attraction of P_0 in the deterministic system, which is enclosed by the stable manifold of the saddle points S,S' and to apply the transformation $v\rightarrow -v$ to this domain. The domain with $A(x,v)=0$ and its borders are thus obtained by a reflection of the domain of the attractor P_0 and its surrounding separatrices on the x axis. In Fig. 3 we give a qualitative plot of the result. The dashed lines in this plot are the reflected separatrices of the deterministic system [cf. Figs. 1(a) and 1(e)], which form the borders of the region where $A(x,v)=0$. In this region the potential $\phi_0(x,v)$ given by Eq. (5.8) can be defined. $\phi=\phi_0(x,v)$ is therefore the desired potential at least in a neighborhood of P_0. Notice that in Fig. 3 any point on either side of the reflected separatrices can be reached from P_0 without crossing a reflected separatrix, by traveling n times around the cylinder. The number $n\gtrless 0$ determines where $\phi_0(x+2n\pi,v)$ is defined. In this way some value of n and $\phi_0(x+2n\pi,v)$ is assigned to any point on the cylinder. Notice, however, that an acceptable global potential ϕ outside the domain enclosed by the curve $\phi_0(x,v)=0$ is not yet defined in this way. Indeed along the reflected separatrices $\phi_0(x+2\pi n,v)$ and $\phi_0(x+2\pi(n+1),v)$ do not coincide, violating the continuity of ϕ. A further indication that something is yet missing is the fact, which we now prove, that the correct global potential ϕ must be constant along a part of the deterministic unstable manifold of S'. This follows from the observation that in the saddle S we have $\phi=\phi_0(x_S,0)=0$, on the equipotential line through S, $\phi_0(x,v)=\phi_0(x_S,0)$ (closed curve in Fig. 3), we also have $\phi=\phi_0(x_S,0)=0$; but on the other hand ϕ cannot increase along the deterministic unstable manifold of S'. We therefore have to conclude that $\phi=0$ on that part of the unstable manifold of S', which, in Fig. 3, connects S' and the point of intersection $\bar{P}$ with the equipotential line $\phi_0(x,v)=0$. This conclusion must be true as long as the point of intersection $\bar{P}$ exists. The integral expression (5.5) leads to the same conclusion, because the integral from P_0 to an end point on the unstable manifolds of S' can be taken along a curve from P_0 to S, where $\phi=0$, and from S' (which is equivalent to S) to the desired end point on the manifold, which does not increase the integral, because the integrand vanishes identically on the manifold. Hence, $\phi(x,v)=0$ in the end point. For end points on the unstable manifold of S' beyond the point of intersection $\bar{P}$, for which $\phi_0(x,v)<0$, the minimum principle selects $\phi(x,v)=\phi_0(x,v)$. Hence, $\phi(x,v)$ is continuous at $\bar{P}$, but has a discontinuous first derivative there.

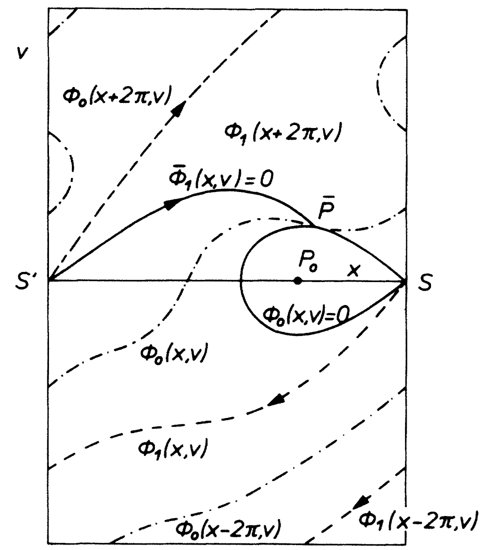

FIG. 3. Qualitative plot of the potential in a region where the fixed point P_0 is the only attractor. Equipotential lines $\phi(x,v)=0$ are shown as solid lines; reflected separatrices along which $\phi_0(x,v)=\phi_1(x,v)$ with continuous derivatives are shown as dashed lines; curves where $\phi_1(x+2\pi,v)=\phi_0(x,v)$ with discontinuous derivatives are shown as dashed-dotted lines. The local representation of $\phi(x,v)$ by $\phi_0(x+2\pi n,v)$ and $\phi_1(x+2\pi n,v)$ is indicated.

The preceding analysis showed that an important part of the potential ϕ is obtained from Eq. (5.5) for trajectories connecting P_0 with the endpoint, which pass arbitrarily close to the saddle point S. Let us call this part of the potential $\phi_1(x,v)$. It can be represented as

$$\phi_1(x,v)=\min \int_{(x(-\infty)=x_s,\dot{x}(-\infty)=0)}^{(x(0)=x,\dot{x}(0)=v)} \frac{1}{4\gamma}[\ddot{x}+\gamma\dot{x}+f(x) -F]^2 d\tau \ . \quad (5.11)$$

There again we make the convention that for each jump of the path from the right border of Fig. 3 to the left border the argument x of ϕ_1 increases by 2π. Jumps in the opposite direction count as -2π. Unfortunately, ϕ_1 cannot be calculated exactly. However, a few exact properties can be easily derived. From the representation (5.11) it follows that

$$\phi_1(x,v)\geq 0 \ , \quad (5.12)$$

where the equality sign can only hold for points which can be reached from S (or S') via a deterministic trajectory. These are exclusively the points on the unstable manifold of S' discussed before. For end points on the reflected separatrices of Fig. 3 the expression (5.11) for ϕ_1 can be evaluated in the same way as we formely evaluated ϕ_0

[again trajectories of Eq. (5.3) with $A=0$ must be used in the integral (5.5)]. The result is that along the reflected separatrices.

$$\phi_1(x,v)\equiv\phi_0(x,v) . \tag{5.13}$$

It follows, as we now show, that the equipotential lines $v=v(x)$ of $\phi_1(x,v)$ and $\phi_0(x,v)$ meet on the reflected separatrices continuously, and even with continuous slope, i.e.,

$$\left.\frac{\partial v}{\partial x}\right|_{\phi_0}=\left.\frac{\partial v}{\partial x}\right|_{\phi_1} \tag{5.14}$$

on the reflected separatrices. Equation (5.14) is proven by noting the identity

$$\left.\frac{\partial v}{\partial x}\right|_{\phi}=-\frac{p_x}{p_v} \tag{5.15}$$

and expressing p_x in terms of x, v, and p_v by using $H(x,v,p_x,p_v)=0$. The result is

$$\left.\frac{\partial v}{\partial x}\right|_{\phi}=\frac{1}{v}(\gamma p_v-\gamma v+F-f) . \tag{5.16}$$

Recalling now that for ϕ_0 and ϕ_1 the condition $A=0$ on the reflected separatrices must be used, i.e., $p_v=v$ in both cases, it follows that

$$\left.\frac{\partial v}{\partial x}\right|_{\phi_0}=\frac{1}{v}[F-f(x)]=\left.\frac{\partial v}{\partial x}\right|_{\phi_1} . \tag{5.17}$$

Thus, $\phi_0(x,v)$ and $\phi_1(x,v)$ are connected along the reflected separatrices of Fig. 3 continuously with continuous first derivatives.

In a similar way one can see that $\phi_1(x,v)$ and $\phi_1(x+2\pi,v)$ can be joined together continuously with continuous derivatives along some curve in configuration space. The reason is that both $\phi_1(x,v)$ and $\phi_1(x+2\pi,v)$ approach the non-negative function $\phi_0(x,v)-\phi_0(x_0,0)$ in the same part of configuration space; $\phi_1(x,v)$ because of the existence of minimizing paths starting in S and passing from there on a deterministic trajectory arbitrarily close by P_0 and going from there on to the final point; $\phi_1(x+2\pi,v)$ because of the existence of other minimizing paths jumping from S to S', moving from there near to P_0 on a deterministic trajectory, and going on from there to the final point. As a consequence, by selecting everywhere the smaller of the two functions, $\phi_1(x,v)$ and $\phi_1(x+2\pi,v)$ can be combined to form a single periodic solution $\bar{\phi}_1(x,v)$ of the Hamilton-Jacobi equation,

$$\bar{\phi}_1(x,v)=\bar{\phi}_1(x+2\pi,v) , \tag{5.18}$$

which is continuous and continuously differentiable. The position of the curve where $\phi_1(x,v)$ and $\phi_1(x+2\pi,v)$ can be differentiably joined can be found as the line where the equipotential lines of $\phi_1(x,v)$, $\phi_1(x+2\pi,v)$, and $\phi_0(x,v)$ have the same slope, i.e., are tangential to each other.

The minimum principle finally determines which of the functions $\phi_0,\bar{\phi}_1$ represents $\phi(x,v)$ locally at a given point. The qualitative result is sketched in Fig. 3 where we distinguish the separate pieces $\phi_1(x,v),\phi_1(x+2\pi,v)$ of $\bar{\phi}_1$ for clarity. As a consequence of the minimum principle there appear discontinuities in the first derivatives of the potential ϕ along curves shown in Fig. 3 as dashed-dotted lines where

$$\phi_1(x+2\pi(n+1),v)=\phi_0(x+2\pi n,v) . \tag{5.19}$$

For $n=0$ this curve contains the point $\bar{P}$.

For sufficiently large γ we can actually do better: we can then calculate $\bar{\phi}_1(x,v)$ explicitly in an expansion in $1/\gamma$ as periodic solution of the Hamilton-Jacobi equation which is constant and minimal along the separatrix connecting S' with P_0. This expansion in γ^{-1} is given in the Appendix. The corresponding equipotential lines of $\phi(x)$ are shown in Fig. 4. The equipotential line $\phi=\phi_0(x,v)=\phi_0(x,0)=0$ is the closed curve on the right-hand side. It is intersected by the separatrix with $\phi=\bar{\phi}_1(x,v)=\phi_0(x_S,v)=0$ in the point $\bar{P}$. The dashed lines are the reflected separatrices where $\phi=\phi_1(x,v)=\phi_0(x,v)$ with continuous first derivative. The dashed-dotted line through $\bar{P}$ is again a line where $\phi=\phi_1(x,v)=\phi_0(x,v)$ but where the first derivative of ϕ is discontinuous.

Let us see how the potential $\phi(x)$ obtained in a special case in Sec. III B reemerges from the present more general result in a simple way: Integrating $P(x,v)$ $\sim\exp[-\phi(x,v)/\eta]$ over v in saddle-point approximation we obtain

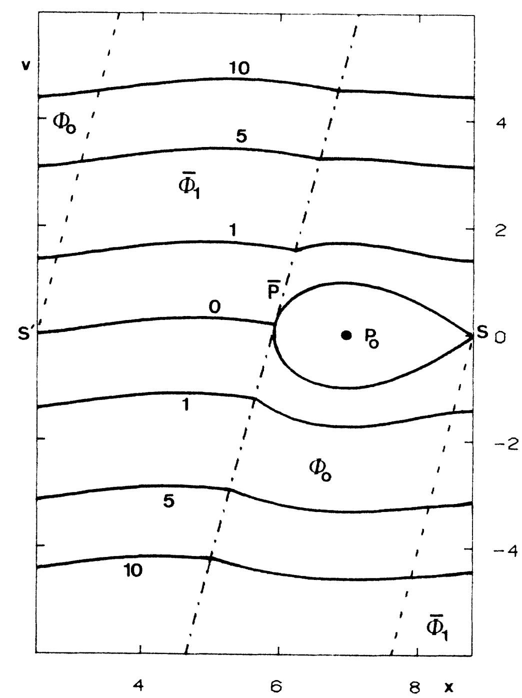

FIG. 4. The potential ϕ for $\gamma=5$, $F=0.6$. The fixed point P_0 is the only attractor. The local solution $\bar{\phi}_1$ around the unstable separatrix of the saddle point is evaluated by means of the $1/\gamma$ expansion (see Appendix). ϕ_0 is the local solution around the fixed point. ϕ_0 and $\bar{\phi}_1$ join continuously but with a discontinuous first derivative along the dashed-dotted line, and continuously with continuous first derivatives along the dashed lines. Numbers represent the value ϕ along equipotential lines. $\phi=-0.49$ at P_0.

$$P(x) \sim \begin{cases} \exp\left[-\dfrac{\phi_0(x_S,0)}{\eta}\right], & x_S < x \le \bar{x}_1 \\ \exp\left[-\dfrac{\phi_0(x,0)}{\eta}\right], & \bar{x}_1 \le x \le x_S + 2\pi \ . \end{cases} \tag{5.20}$$

Hence

$$\phi(x) = \begin{cases} 0, & x_S \le x \le \bar{x}_1 \\ \phi_0(x,0), & \bar{x}_1 \le x \le x_S + 2\pi \end{cases} \tag{5.21}$$

which is the result obtained in Sec. III B.

C. Nonequilibrium potential for coexisting attractors

Let us now see qualitatively how the potential ϕ changes as we increase F for fixed dissipation rate γ or decrease γ at fixed F. The point $\bar{P}$ of Fig. 3 will slowly move to the right until, for a critical value of the parameter, it coincides with the point S. At the same time the reflected separatrix of Fig. 3, along which $\phi_0(x,v)=\phi_1(x,v)$, moves up on the left-hand side until it passes through S'. It has then become just the mirror image of the unstable manifold of S'. The situation for this critical case is sketched qualitatively in Fig. 5. The dashed-dotted line where $\bar{\phi}_1(x,v)=\phi_0(x,v)$ holds with discontinuous derivative, and the reflected separatrix where $\phi_0(x,v)=\phi_1(x,v)$ with continuous derivative now intersect each other in the point Q. The point Q is determined by the condition that the reflected separatrix there is tangential to an equipotential line of $\phi_0(x,v)$. To see this, imagine that the reflected separatrix at Q would intersect transversally the equipotential lines of $\phi_1(x+2\pi,v)$, $\phi_1(x,v)$, and $\phi_0(x,v)$ which all coincide in a neighborhood of Q. Then, both $\phi_1(x+2\pi,v)$ and $\phi_1(x,v)$ would have to increase or decrease simultaneously at Q along the reflected separatrix. However, this is forbidden by their representation as a time integral over a positive function along paths starting in S' and S, respectively.

Beyond Q the reflected separatrix has not been drawn in Fig. 5, since there it has no longer any meaning for the potential ϕ, as in this region the minimum principle selects $\phi_1(x+2\pi,v)$ instead of $\phi_1(x,v)$. We have already shown that these two functions are joined by the minimum principle continuously with continuous derivative. Actually, the curve where they are joined contains the point Q. Note that it is impossible that $\phi=\phi_1(x,v)$ extends along the reflected separatrix all the way to S' as $\phi_1(x,v)\ge 0$ must increase along the reflected separatrix but $\phi=0$ at S'.

As we increase F further the periodic trajectory originally formed by the unstable manifolds of S' and S becomes disconnected from S,S' and develops into an attracting limit cycle. $\bar{\phi}_1$ correspondingly develops into a periodic local potential $\phi_p(x,v)$ which is now associated with the limit cycle. Figure 6 shows the situation for this case. The separatrix minimum on the separatrix between the fixed point P_0 and the limit cycle is taken at the saddle point S, since the separatrix is the stable manifold of S, and since the potential must decrease along deterministic trajectories according to (1.5). Following the general method of Sec. 4 the additive constants in the local pieces of the potential are to be determined so that the value of ϕ_p and ϕ_0 coincide at S. There are now discontinuities in the first derivatives of ϕ at the dashed-dotted line of Fig. 6,

$$\phi_p(x,v)=\phi_0(x,v) \ . \tag{5.22}$$

This line ends on the reflected separatrix at the point where the reflected separatrix becomes tangential to an equipotential line of $\phi_0(x,v)$. Along the reflected separatrix $\phi_p(x,v)$ and $\phi_0(x,v)$ are joined in a continuously differentiable way. The proof of these statements is analo-

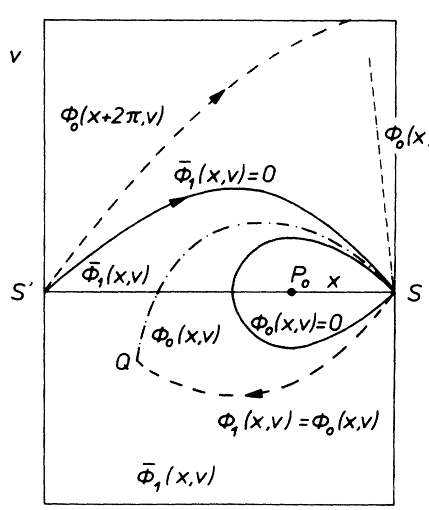

FIG. 5. Qualitative plot of the potential at a critical point where the periodic attractor first appears. Same conventions apply as in Fig. 3.

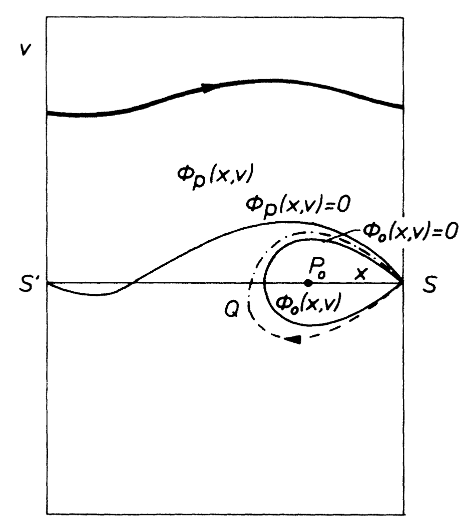

FIG. 6. Qualitative plot of the potential in the coexistence region of a fixed point and a limit cycle (bold line). Same conventions apply as in Fig. 3.

gous to the ones given above. Also shown in Fig. 6 are the limit cycle and P_0, and, qualitatively, the equipotential lines $\phi_p(x,v)=0$ and $\phi_0(x,v)=0$. We note that the discontinuous derivatives of ϕ all lie in a region of configuration space where $\phi \geq 0$. Hence, they are of little importance in all practical problems concerning the statistical distribution of the system in the weak-noise limit.

A more quantitative picture of the potential is given in Fig. 7 where the equipotential lines of ϕ_p have been determined numerically up to the level $\phi_p=0$. The equipotential line $\phi_0=0$ is represented by the closed curve. Since the minimum of ϕ_0 at the fixed point P_0 is very shallow no other equipotential lines of ϕ_0 appear in the figure. By means of the complete potential specified up to $\phi=0$ the description of jumping processes between the attractors is already possible in the weak-noise limit.

D. Nonequilibrium potential for a single periodic attractor

Finally, let us follow qualitatively what happens when increasing the external force F. As we have seen, the fixed point P_0 and the saddle point S approach one another, and at $F=1$ coincide. Beyond this value of F there are no fixed points in the system, the limit cycle becomes the only attractor [Fig. 1(d)]. Consequently, the global periodic potential ϕ is determined by Hamiltonian trajectories starting at $t=-\infty$ on the limit cycle. Figure 8 shows the equipotential lines of the potential obtained numerically at a relatively small value of the friction coefficient γ. In the overdamped case the evaluation of ϕ by means of the $1/\gamma$ expansion (see the Appendix) yields a rather accurate approximation for the potential.

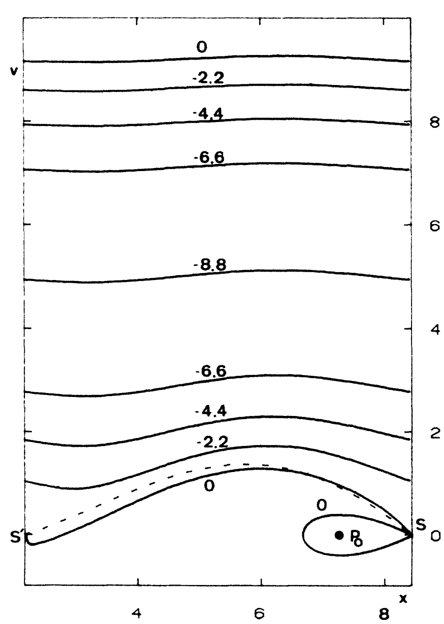

FIG. 7. Equipotential curves obtained numerically in the region of coexistence up to the level $\phi=0$. Dot denotes the fixed point P_0. Dashed line represents the curve $\phi=0$ obtained by the approximate method of Ref. 16. $\gamma=0.13$, $F=0.83$. Numbers give the value of the potential. At the limit cycle $\phi=-8.8$, at P_0, $\phi=-0.13$.

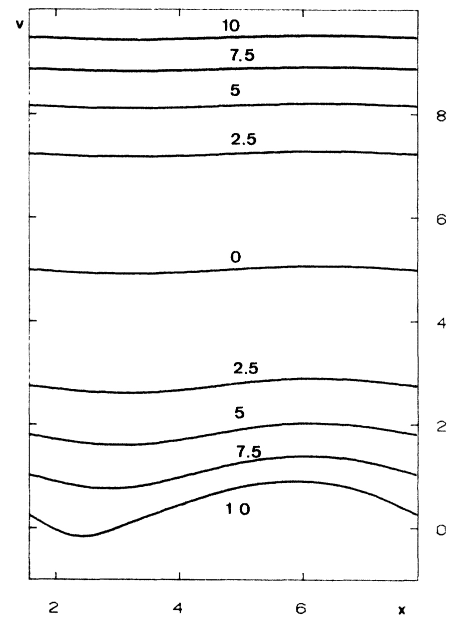

FIG. 8. Equipotential curves obtained numerically in a case where the limit cycle is the only attractor. $\pi/2<x<5\pi/2$. $\gamma=0.16$, $F=1.25$. Numbers give the value of the potential. On the limit cycle ϕ has been chosen to be zero.

E. Numerical method for evaluating the potential

In order to determine the potential generated by Hamiltonian trajectories starting from the limit cycle at $t\to-\infty$ we used a numerical method which can be applied in all cases where a well-defined dynamics on a Poincaré cross section can be found. This illustrates at the same time that the Hamiltonian analogy found in the weak-noise limit of stochastic processes can provide a practical method for calculating the stationary density in this limit. The Poincaré surface we considered was defined by $H=0$ and by a fixed value of x in $x_S<x\leq x_S+2\pi$. The coordinates on this plane are chosen to be v and p_v. The limit cycle appears as a fixed point $(v=\bar{v}(x),p_v=0)$ on the Poincaré surface. Since the potential must be minimal at the attractor, the relation $p_v=\partial\phi/\partial v$ defines a curve $p_v=p_v(v)$ passing through this fixed point of the (v,p_v) plane (Fig. 9) forming its unstable manifold. The problem, therefore, is to determine the unstable manifold of the fixed point $(\bar{v}(x),0)$ (its stable manifold is the v axis). A point of the unstable manifold may be found by fixing v and changing p_v until the preimages of the point reach a sufficiently small vicinity of the fixed point. After specifying the manifold $p_v=p_v(v)$, the integral

$$\phi(x,v)=\int_{\bar{v}(x)}^{v}p_v(\tilde{v})d\tilde{v}+C \tag{5.23}$$

yields the potential as a function of v at the x value where the Poincaré surface has been taken. C denotes the value of the potential along the limit cycle. The x dependence

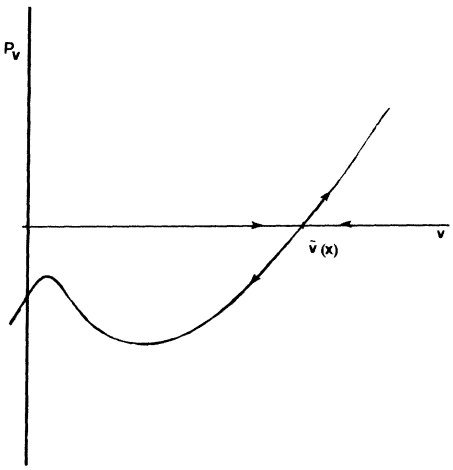

FIG. 9. Qualitative plot of the unstable manifold of the fixed point representing the limit cycle on the Poincaré surface v, p_v.

can then be followed by considering different Poincaré cross sections.

A practical problem is caused by a peculiar singular feature of the Hamiltonian dynamics. It can be seen by rewriting (5.2) as

$$p_v = \frac{v}{1+A\exp(-\gamma t)} . \tag{5.24}$$

Since the momentum cannot be infinite after a finite time, nominator and denominator must vanish simultaneously for all $A<-1$. It follows from the Hamiltonian (2.6) that at $v=0$ the momentum p_v equals either 0 or $[F-f(x)]/\gamma$, independent of p_x. This means that in the Poincaré surface there are two singular points: the origin and $(0,[F-f(x)]/\gamma)$ passed by an infinite number of trajectories. Partly, this problem can be overcome by using a local analytical solution around the point where the denominator of (5.24) vanishes, selecting the correct trajectory. Nevertheless, in practical calculations it was found that even then the accuracy of the numerical solution of Eqs. (5.3) and (5.4) rapidly deteriorates once v has passed well into the regime of negative velocities.

The results presented in Figs. 7 and 8 have been obtained by the method sketched in this section. Because of the difficulty mentioned above equipotential lines well inside the negative v half-plane have not been determined.

VI. COMPARISON WITH OTHER WORKS

Ben-Jacob *et al.*[16] also studied the weak-noise limit of the Brownian motion in a periodic potential with external force. We investigate here how that analysis fits into the framework of the present work. The idea of Ref. 16 was to find a general relationship between equipotential curves of the stochastic system and periodic trajectories of the deterministic system. We recall first that the drift $K^\nu(q)$, $\nu=1,2,\ldots,n$ of a stochastic system possessing a smooth potential $\phi(q)$ can always be written in the weak-noise limit as

$$K^\nu(q) = -\frac{1}{2}Q^{\nu\mu}\frac{\partial\phi(q)}{\partial q^\mu} + r^\nu(q) , \tag{6.1}$$

where the circulation r^ν is orthogonal to the gradient of the potential:

$$r^\nu(q)\frac{\partial\phi(q)}{\partial q^\nu} = 0 . \tag{6.2}$$

This follows directly from the Fokker-Planck equation with diffusion matrix[2] $Q^{\mu\nu}$. The dynamics defined by

$$\dot q^\nu = r^\nu(q) \tag{6.3}$$

describes a motion on equipotential surfaces since $(d/dt)\phi(q(t))=0$ according to (6.2) and (6.3). In the case of the Brownian motion (2.3) the drift matrix is $Q^{\nu\mu}=2\gamma\delta^{\nu 2}\delta^{\mu 2}$ and the dynamics (6.3) has the form

$$\dot x = v , \tag{6.4}$$

$$\dot v = -\gamma v - f(x) + F + \gamma\frac{\partial\phi}{\partial v} . \tag{6.5}$$

As a consequence of the orthogonality relation (6.2) there must exist a function $D(x,v)$ so that Eqs. (6.4) and (6.5) assume the form

$$\dot x = v = \frac{\partial\phi}{\partial v}D(x,y) , \tag{6.6}$$

$$\dot v = -\frac{\partial\phi}{\partial x}D(x,y) . \tag{6.7}$$

By means of Eq. (6.6) the dynamics on equipotential surfaces $\phi=\text{const}$ can then be written as

$$\begin{aligned} &\dot x = v , \\ &\dot v = -\Gamma(x,y)v - f(x) + F , \\ &\Gamma(x,y) = \gamma\left[1-\frac{1}{D(x,y)}\right] . \end{aligned} \tag{6.8}$$

This equation must have a periodic solution if ϕ is periodic.

If the function D happens to depend on the variables only through ϕ, which is constant under the dynamics (6.8), i.e., $D(x,v)=\tilde D(\phi(x,v))$, a considerable simplification occurs since the motion on equipotential surfaces is then of the same type as the deterministic motion. In particular, the equipotential curve $\phi=\text{const}$ is the curve of the limit cycle in the deterministic system with a friction coefficient

$$\Gamma = \gamma\left[1-\frac{1}{\tilde D(\phi)}\right] . \tag{6.9}$$

Assuming that this special case is realized, at least approximately, the next step is to determine the relation between Γ and ϕ. Equation (6.9) also means that the potential ϕ is constant along a curve $\Gamma=\Gamma(x,v)$, i.e., $\phi(x,v)=\tilde\phi(\Gamma(x,v))$. Since

$$\frac{\partial\phi}{\partial v} = \frac{d\tilde\phi}{d\Gamma}\frac{\partial\Gamma}{\partial v} , \tag{6.10}$$

one finds

$$\frac{d\tilde\phi}{d\Gamma} = \left[1-\frac{\Gamma}{\gamma}\right]v\frac{\partial v}{\partial\Gamma} , \tag{6.11}$$

where (6.6) and (6.9) have been used and x and Γ have been considered as independent variables. From (6.11) one obtains

$$\tilde{\phi}(\Gamma)=\frac{1}{2}\int_{\gamma}^{\Gamma}\left[1-\frac{\Gamma}{\gamma}\right]\frac{\partial v^2(x,\Gamma)}{\partial\Gamma}d\Gamma+\text{const} . \tag{6.12}$$

Since $\tilde{\phi}$ is by assumption a function of Γ only, the x dependence of the right-hand side is only formal. The result has several attractive features: The minimum of the potential is located on the limit cycle of the original system, $\Gamma=\gamma$, as it should be, and $\tilde{\phi}$ is increasing with Γ. The solution can, therefore, be expected to be correct at least in a small neighborhood of the limit cycle. The equipotential curve associated with a critical value $\Gamma=\gamma_c(F)$ becomes the separatrix of the saddle point in the dissipative system with γ_c (cf. Sec. II).[16]

An interesting relation follows from the energy balance of the dynamics (6.8). We find

$$\frac{d}{dt}\left[\frac{v^2}{2}+g(x)\right]=Fv-\Gamma v^2 . \tag{6.13}$$

Due to periodicity the integral on the left-hand side vanishes in an interval over the time period T. Therefore,

$$\Gamma=\frac{F}{\bar{v}(\Gamma)} , \tag{6.14}$$

where $\bar{v}(\Gamma)$ denotes the velocity averaged over x along the periodic solution of (6.8).

However, we must now investigate to what extent the basic assumption leading to these results is satisfied, i.e., we analyze the condition for $D(x,y)$ being a function of ϕ only. If $D(x,v)=\tilde{D}(\phi)$, it follows from (6.6) that

$$\frac{\partial}{\partial v}\tilde{H}(\phi)=v , \tag{6.15}$$

where $\tilde{H}$ is defined by $\tilde{D}(z)=d\tilde{H}(z)/dz$. By integrating (6.15)

$$\phi=\tilde{H}^{-1}\left[\frac{v^2}{2}+G(x)\right] \tag{6.16}$$

with a yet undetermined function $G(x)$. From the equation (6.16) it follows that

$$\frac{\partial\phi}{\partial x}=\frac{G'(x)}{\tilde{D}(\phi)} \tag{6.17}$$

which reduces Eq. (6.7) to the form

$$\dot{v}=-G'(x) . \tag{6.18}$$

The comparison of this equation with Eq. (6.5) yields

$$\gamma\left[\phi-\frac{v^2}{2}\right]=v[f(x)-F-G'(x)]+K(x) , \tag{6.19}$$

where $K(x)$ is yet another undetermined function of x. For $\gamma\neq 0$ this equation is only compatible with Eq. (6.16) if

$$G'(x)=f(x)-F,\quad K(x)=\gamma G(x) ,$$
$$\tilde{H}^{-1}(z)=z+\text{const} , \tag{6.20}$$

which yields the solution

$$\phi=\frac{v^2}{2}+g(x)-Fx+\text{const} , \tag{6.21}$$

which is known to hold only in the vicinity of the attracting fixed point. If ϕ is to be periodic in x, Eq. (6.19) is compatible with Eq. (6.16) only for $F\to 0$. In order to have coexistence of attractors for $F\to 0$ it is then also necessary to take $\gamma\to 0$ such that $F_0=F/\gamma$ remains finite. This is the limit we have considered in Sec. III A. Indeed, Γ defined by Eq. (6.9) then becomes a function of the energy E [Eq. (3.2)], therefore, $\bar{v}(\Gamma)=\bar{v}(E)$ and from (6.14)

$$\Gamma=\frac{F}{\bar{v}(E)} . \tag{6.22}$$

Since $v^2=2E-2g(x)$, (6.12) yields $\tilde{\phi}(\Gamma)=\phi^+(E)$, where $\phi^+(E)$ is given by (3.8). Thus, we recover the result of Sec. III A obtained there in leading order in the friction coefficient. The dashed line in Fig. 7 shows the equipotential curve $\phi=0$ following from this method in a case where F cannot be considered to be small. Although the deviation from the numerically calculated curve is not drastic, there is an important qualitative difference between the two curves. The method of Ref. 16, therefore, appears to be valid only in the limit of small γ, or in a small neighborhood of the limit cycle.

ACKNOWLEDGMENT

We would like to thank Stefan Isermann for assistance in the numerical work concerning Figs. 4, 7, 8, and 9.

APPENDIX: $1/\gamma$ EXPANSION OF THE POTENTIAL AROUND A PERIODIC TRAJECTORY

We perform here a systematic expansion of the potential ϕ in powers of $1/\gamma$. It is convenient to write the Hamilton-Jacobi equation (2.6) in the form

$$\left[\frac{\partial\phi}{\partial v}\right]^2-\left[\frac{\partial\phi}{\partial v}\right]v-\frac{1}{\gamma}\frac{\partial\phi}{\partial v}[f(x)-F]+\frac{1}{\gamma}\frac{\partial\phi}{\partial x}v=0 . \tag{A1}$$

By inserting the ansatz

$$\phi(x,v)=\sum_{n=0}^{\infty}\gamma^{-n}\phi_n(x,v) \tag{A2}$$

one finds a hierarchy of equations in different powers of γ^{-1}. From the γ-independent part

$$\phi_0=v^2/2 \tag{A3}$$

follows. The next equation yields

$$\phi_1=[f(x)-F]v+g_1(x) , \tag{A4}$$

where $g_1(x)$ is unspecified at this point. From order γ^{-2} we obtain

$$\phi_2=-f'(x)v^2/2-g_1'(x)v+g_2(x) \tag{A5}$$

with another free function $g_2(x)$. The function ϕ_3 is specified by

$$v\frac{\partial\phi_3}{\partial v}=(f-F)f'v+(f-F)g_1'-v(-f''v^2/2-g_1''v+g_2') \ . \tag{A6}$$

In order to have a bounded solution at $v=0$ we have to take $g_1=$const which is chosen to be zero. Similarly, the x-dependent function g_n appearing in nth order is determined by the requirement that logarithmic singularities at $v=0$ are to be avoided in the $(n+2)$th equation. Thus, for example, g_2 turns out to be

$$g_2(x)=[f(x)-F]^2/2 \ . \tag{A7}$$

Below we list the explicit form of ϕ_n up to $n=6$:

$$\phi_3=f''v^3/6 \ , \tag{A8}$$

$$\phi_4=-(f-F)f''v^2/4-f'^2v^2/2-f'''v^4/24 +(f-F)^2f'/2 \ , \tag{A9}$$

$$\phi_5=(f-F)f'''v^3 5/36+f'f''v^3 3/4+f^{(IV)}v^5/120 \ , \tag{A10}$$

$$\begin{aligned}\phi_6=&-(f-F)f'f''v^2 13/8-f'^3v^2\\&-(f-F)^2f'''v^2 5/24-(f-F)f^{(IV)}v^4 13/288\\&-f'f'''v^4 11/36-(f'')^2v^4/4-f^{(V)}v^6/720\\&+(f-F)^2f'^2+(f-F)^3f''5/12 \ .\end{aligned} \tag{A11}$$

Note that ϕ obtained in this way is automatically 2π periodic in x due to the periodicity of $f(x)$. A straightforward calculation shows that the minimum of ϕ lies on the curve

$$\tilde{v}(x)=-\frac{1}{\gamma}\left\{f-F+\frac{1}{2\gamma^2}[(f-F)^2]'+\frac{1}{\gamma^4}[(f-F)^2f']' +\frac{1}{\gamma^6}[\tfrac{5}{2}(f-F)^2f'^2+(f-F)^3f'']'\right\} \tag{A12}$$

which, at the same time, is the periodic solution of

$$\frac{dv}{dx}=-\gamma+\frac{F-f(x)}{v} \tag{A13}$$

in an expansion in $1/\gamma$. Consequently, $v=\tilde{v}(x)$ is a 2π-periodic trajectory of the deterministic dynamics (2.4) with $\xi=0$. Thus, the potential obtained above is a global potential for $F>1$ where the limit cycle is the only attractor, and it is a local solution around the unstable manifolds of the saddle point for $F<1$ [see Fig. 1(e)] where no limit cycle exists. Since the curve (A12) agrees with the numerical solution of (A13) for $\gamma>2.5$ within a 1% accuracy, one may expect that (A3)—(A11) yields also a very good approximation for the potential in this regime.

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