PHYSICAL REVIEW A VOLUME 35, NUMBER 1 JANUARY 1, 1987

Dynamical fractal properties of one-dimensional maps

P. Szépfalusy
Institute for Solid State Physics, Eötvös University, P.O. Box 327, H-1445 Budapest, Hungary and Central Research Institute for Physics, P.O. Box 49, H-1525 Budapest, Hungary

T. Tél
Institute for Theoretical Physics, Eötvös University, P.O. Box 327, H-1445 Budapest, Hungary
(Received 28 August 1986)

The interpretation of the dynamical scaling indices for transient chaos is given. The spectrum of these scaling indices is calculated in an exactly solvable example of chaotic repellers and, by means of a perturbative method, in a class of chaotic attractors in crisis. An approximate form of the universal spectrum related to the universal chaos function is derived. We point out a degeneracy in the spectrum appearing in an intermittent situation.

It is a recent observation that strange sets covered by a stationary probability distribution possess, in general, several coexisting singularities, and are, consequently, characterized by a spectrum of scaling indices.[1-3] A multifractal analysis investigating this spectrum has already been applied to several systems[1-9] ranging from fully developed turbulent flows and diffusion-limited aggregates to strange attractors. Furthermore, the concept has been generalized for dynamical properties of chaotic systems.[10,11] In this paper, our aim is to study the spectrum of dynamical scaling indices in one-dimensional maps $x_{n+1}=f(x_n)$, modeling higher dimensional systems in the presence of strong dissipation,[12] and to carry out explicit calculations. We also discuss how to extend the formalism to the problem of transient chaos.[13-20]

The dynamical scaling indices Λ are defined (Eckmann and Procaccia[11]) by writing the probability of very long but finite paths of length n as $\exp(-n\Lambda)$, and the path probabilities are to be calculated by partitioning the space into small boxes and using discrete time (inherent in maps). Different paths may, of course, have the same scaling index. The number of times Λ takes on a value between Λ' and $\Lambda'+d\Lambda'$ is proportional to

$$\exp[ng(\Lambda')]d\Lambda' \ , \tag{1}$$

where n is fixed and $g(\Lambda)$ is a smooth function.[11] From the scaling form (1) then follows that the generalized entropies[18,21,22] can be expressed through the spectrum $g(\Lambda)$ as

$$K_q=[q\Lambda(q)-g(\Lambda(q))]/(q-1) \ , \tag{2}$$

where $\Lambda(q)$ is defined by the relation

$$\left.\frac{dg(\Lambda)}{d\Lambda}\right|_{\Lambda(q)}=q \ . \tag{3}$$

Consequently, $g(\Lambda)$ is related to $(q-1)K_q$ by means of a Legendre transformation.

First, we deal with the interpretation of the scaling indices in the case of transient chaos. Since the number of boxes the system can visit is increased by a factor $|f'(x_j)|$ after the jth step, the total number of such boxes is proportional to $\prod_{j=0}^{n-1}|f'(x_j)|$. Here and in the following, derivative is denoted by a prime. Without any escape, the reciprocal value of this number would be proportional[11,23] to the probability of the path $\{x_j\}_0^{n-1}$. Transient chaos, however, means that trajectories escape from any finite interval with the exception of a set of measure zero. Those points from where no escape occurs form a Cantor set, the so-called repeller.[16,17] Long chaotic trajectories spend a long time in a small neighborhood of it. The number of available boxes for such trajectories is obtained, therefore, by multiplying $\prod_{j=0}^{n-1}|f'(x_j)|$ with the probability that the trajectory has not yet escaped after n steps. The latter quantity we write as $\exp(-\alpha_n n)$, and call α_n the escape rate. (For $n\to\infty$, α_n goes over to α, the escape rate defined in Refs. 15 and 16.) The probability of a path staying close to the repeller is thus proportional to $[\prod_{j=0}^{n-1}|f'(x_j)|]^{-1}\exp(\alpha_n n)$ and, consequently,

$$\Lambda=\lambda_n-\alpha_n \ , \tag{4}$$

where $\lambda_n=n^{-1}\sum_{j=0}^{n-1}\ln|f'(x_j)|$, the finite time Lyapunov exponent. (The Lyapunov exponent λ is obtained as $\lim_{n\to\infty}\lambda_n$.) Since $\lambda-\alpha$ is the Kolmogorov entropy for a motion around a repeller,[17] the scaling index Λ can be considered as a quantity measuring the fluctuation of the metric entropy. In case of an attractor $\alpha_n=0$, and the results of Ref. 11 are recovered.

In order to calculate the path probability for a motion around a repeller, it is essential to know that there exists an invariant distribution for this strange set. It was shown in Ref. 19 that the stationary density $P(x)$ for a coarse grained repeller is the solution of the iteration scheme

$$P_{n+1}(x')=\sum_{x\in f^{-1}(x')}\frac{P_n(x)}{|f'(x)|^{D_0}} \ , \tag{5}$$

obtained in the limit $n\to\infty$ with any smooth initial function $P_0(x)$. The exponent D_0 is the fractal dimension of the repeller.

As an example, where the spectrum $g(\Lambda)$ can be explicitly calculated, we consider the map producing chaotic transients defined by

$$f(x)=\begin{cases}1-a_1x, & x>0 ,\\ 1+a_2x, & x<0 ,\end{cases} \tag{6}$$

and $1<a_2<a_1$, $a_1^{-1}+a_2^{-1}<1$. The stationary distribution $P(x)$ turns out to be constant, as follows from (5) with $a_1^{-D_0}+a_2^{-D_0}=1$.[19] Consequently, one can easily show that the ratio of probabilities of finding a point moving on the repeller in the right $(x>0)$ or left $(x<0)$ region is a_2/a_1. Therefore, the probability of a path of length $n\gg 1$, which is close to the repeller and which happens to be m times in the right region, is given by

$$a_1^{-m}a_2^{m-n}(a_1^{-1}+a_2^{-1})^{-n} . \tag{7}$$

The corresponding scaling index is then

$$\Lambda=\left[\frac{m}{n}\ln a_1+\left(1-\frac{m}{n}\right)\ln a_2\right]+\ln(a_1^{-1}+a_2^{-1}) . \tag{8}$$

By taking into account that the escape rate α is known to be $-\ln(a_1^{-1}+a_2^{-1})$ in this system,[17,20] Eq. (8) appears as a special form of the relation (4). Furthermore, the number of paths having the same probability, i.e., the same Λ, is $\binom{n}{m}$ for a fixed n. A use of Stirling's formula and of Eq. (1) then yields g as a function of m. After eliminating m through (8) we find

$$g(\Lambda)=\ln\Delta-\Delta^{-1}[(\Lambda-\Lambda_{\min})\ln(\Lambda-\Lambda_{\min}) + (\Lambda_{\max}-\Lambda)\ln(\Lambda_{\max}-\Lambda)] , \tag{9}$$

where

$$\Delta\equiv\Lambda_{\max}-\Lambda_{\min} ,\quad \Lambda_{\min}=\ln(1+a_2/a_1) ,\quad \Lambda_{\max}=\ln(1+a_1/a_2) . \tag{10}$$

Figure 1 displays g vs Λ at different values of the ratio a_1/a_2. The maximum of $g(\Lambda)$ is always ln2, corresponding to a topological entropy ln2. The graph of $g(\Lambda)$ touches the line $g=\Lambda$ at $\Lambda=\lambda-\alpha=K_1$ [cf. Eqs. (2) and (3)] which depends on a_1/a_2. From $g(\Lambda)$ then immediately follows, via Eq. (2), the complete set of generalized entropies:

$$K_q=(1-q)^{-1}\ln[(a_1^q+a_2^q)/(a_1+a_2)^q] .$$

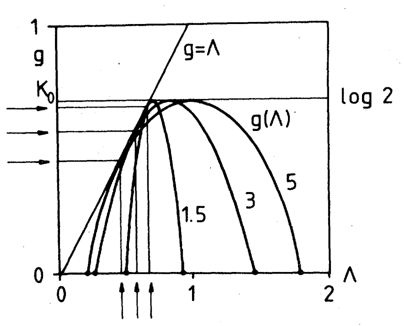

FIG. 1. Plot of the $g(\Lambda)$ spectrum (9) (bold lines). The numbers at these lines denote the ratio a_1/a_2. Arrows mark the values $\Lambda(1)=K_1=\lambda-\alpha$, where $g(\Lambda)=\Lambda$.

It is worth mentioning that for $a_1^{-1}+a_2^{-1}\to 1$ the repeller goes over into a chaotic attractor. In this limit, Eqs. (9) and (10) go over into those valid for the attractor, and formally coincide with the results valid for the baker transformation.[11] [Note that the quantity g in the example of Ref. 11 corresponds to $g(\Lambda)-\Lambda$ in our notation.]

We now turn to the investigation of the spectrum in fully developed chaotic (FDC) single humped maps, specified by the condition that the attractor is mapped two to one into itself. We consider the generator partition which divides, at the maximum point of $f(x)$, the attractor into two intervals I_0,I_1. The paths in the corresponding symbolic dynamics consist of binary sequencies. A common refinement[12] of this bipartition leads, after n steps, to a set of intervals $I_l^{(n)}$, $l=0,\dots,2^n$. The probabilities of paths of length n are then given by the invariant measure of the interval they start from.[12] It is convenient to introduce the transformed map $\tilde{f}(x)$, where

$$\tilde{f}(x)=\mu[f(\mu^{-1}(x))] , \tag{11}$$

since $\tilde{f}$ turns out to be an everywhere expanding map and since the invariant measure of it is the Lebesque one[24] [$\mu(x)$ in (11) stands for the invariant measure of the interval $(0,x)$]. Consequently, the order-q entropies[18,21,22] can be expressed as

$$K_q=\frac{1}{1-q}\lim_{n\to\infty}\frac{1}{n}\ln\sum_l(\tilde{I}_l^{(n)})^q , \tag{12}$$

where $\tilde{I}_l^{(n)}$ denotes the intervals obtained by the common refinement of the binary partition of $\tilde{f}$. For large n the intervals $\tilde{I}_l^{(n)}$ are short and one can write[23]

$$\sum_l(\tilde{I}_l^{(n)})^q=\int[\tilde{I}^{(n)}(x)]^{q-1}dx , \tag{13}$$

where $\tilde{I}^{(n)}(x)=\tilde{I}_l^{(n)}$ if $x\in\tilde{I}_l^{(n)}$. Furthermore, by using the Markov property of the partition one finds[23]

$$\tilde{I}^{(n)}(x)=\frac{\tilde{I}^{(n-1)}[\tilde{f}(x)]}{|\tilde{f}'(x)|}=\frac{\tilde{I}^{(n-2)}[\tilde{f}^{(2)}(x)]}{|\tilde{f}^{(2)\prime}(x)|}=\dots . \tag{14}$$

After substituting (14) into (12) and (13) and keeping only those terms which survive for $n\to\infty$ we obtain for the generalized entropies

$$K_q=\frac{1}{1-q}\lim_{n\to\infty}\frac{1}{n}\ln F_q^{(n)} , \tag{15}$$

$$F_q^{(n)}=\int|\tilde{f}^{(n)\prime}(x)|^{1-q}dx \tag{16}$$

(the integral is taken over the chaotic attractor). When repeating the calculation for the map f, or for any conjugated map of it, extra weighting factors appear in intermediate steps, which, however, drop when taking the limit $n\to\infty$ in (15). Thus, one finds that the formulas (15) and (16) hold with any conjugated function of f in (16), supposing the integral exists. Furthermore, note that a comparison with the results obtained for the entropy decay rate γ in analytic maps[23] yields the relation $\gamma=2K_3$ in this case. Equations (15) and (16) remain valid in any chaotic states; only the bipartition should be replaced by an appropriate Markov partition in the derivation.

Now we apply the general results (15) and (16) to a family of symmetric maps obtained by perturbing the tent map $f(x)=f_0(x)\equiv 1-|1-2x|$. The perturbed map we consider is

$$f(x)=f_0(x)+\varepsilon G(f_0(x)) ,\qquad G(x)=G(1-x) , \tag{17}$$

where ε is a small parameter. This type of perturbation, if combined with conjugation, is the most general one around double symmetric maps (symmetric maps with symmetric probability density).[24] Since $f(x)$ is everywhere expanding and is linear around its maximum, it can be substituted for any q value into (16). One can then work out a systematic expansion of K_q in powers of ε (as in Ref. 23) and one obtains in first nontrivial order

$$K_q=\ln 2-\varepsilon^2 qA/2 , \tag{18}$$

where $A\equiv\int_0^1 G'(x)^2dx$. By inverting the relations (2) and (3) the spectrum is found to be

$$g(\Lambda)=\ln 2-(\ln 2+\Gamma/2-\Lambda)^2/(2\Gamma) , \tag{19}$$

where $\Gamma=\varepsilon^2 A$ (see Fig. 2). Γ is defined as the halfwidth of the spectrum taken at the height K_1. The halfwidth decreases with the strength of perturbation, which for $\varepsilon\to 0$ is consistent with the fact that the spectrum of the unperturbed case consists of a single point $K_q=\Lambda=\ln 2$. On the other hand, the form (18) is valid only in the vicinity of $\Lambda=\ln 2$ since (18) applies only if $\varepsilon^2 qA$ is small. This illustrates that the end points $\Lambda_{\min}$ and $\Lambda_{\max}$ of the range of scaling indices cannot be determined by perturbative methods.

As an interesting application we consider the biquadratic map[23]

$$f(x)=1-(1-\varepsilon)(1-2x)^2-\varepsilon(1-2x)^4$$

which, after conjugation, becomes an element of the family (17). In this case the halfwidth is given by $\Gamma=\varepsilon^2/8$. Since maps with parabolic maximum possess a universal FDC map[25] which is well approximated by the biquadratic map with $\varepsilon=-0.2629$,[26] one obtains an approximate universal spectrum $g^*(\Lambda)$ in the form of (19), and $\Gamma^*=8.6396\times 10^{-3}$ as its universal halfwidth.

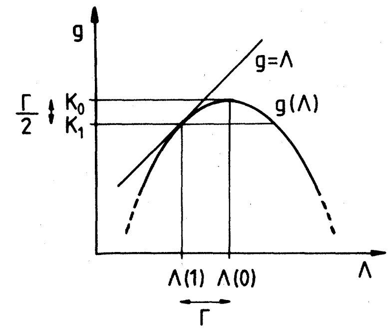

FIG. 2. Qualitative plot of the spectrum (19).

Among FDC maps, those in which the left unstable fixed point is marginally stable form a special class.[24,26–28] We show that in this class a global change occurs in the shape of the spectrum. Let us denote by $\tilde{I}_1^{(n)}$ the leftmost interval generated by the common refinement of the bipartition of $\tilde{f}$. By keeping the first term only in the sum of Eq. (12), we find for $q>1$ the inequality

$$K_q<\frac{1}{1-q}\lim_{n\to\infty}\frac{1}{n}\ln\tilde{I}_1^{(n)} . \tag{20}$$

It can be easily seen[23] that for $n\to\infty$ the size of $\tilde{I}_1^{(n)}$ tends to zero slower than exponentially (power-law behavior) because of the intermittent situation. Thus, all generalized entropies for $q>1$ vanish. Consequently, the spectrum $g(\Lambda)$ cannot have a continuous part for $\Lambda<\lambda=K_1$ in this case.

Finally, we emphasize that the parallelism between dynamic and static multifractality of chaotic systems has a deep reason. As pointed out by Farmer,[29] the path probabilities of paths of length n on a chaotic attractor can be represented in the space of symbolic dynamics by a distribution $\rho^{(n)}(x)$ defined on the interval (0,1). The resolution of this interval is then m^{-n}, where m denotes the number of symbols appearing in the symbolic dynamics. Thus, $\exp(-n\ln m)$ is the analog of the grid size l used in static cases.[3] Furthermore, since the generalized dimensions D_q (Refs. 30 and 31) of the distribution $\rho^{(\infty)}(x)$ coincide with $K_q/\ln m$, the quantity $g(\Lambda)/\ln m$ is analogous with the $f(\alpha)$ spectrum (defined by Halsey *et al.*[3]) of $\rho^{(\infty)}(x)$, the symbol sequence distribution.

[1]U. Frisch and G. Parisi, in *Turbulence and Predictability of Geophysical Flows and Climate Dynamics*, edited by N. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985).
[2]R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).
[3]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev A **33**, 1141 (1986).
[4]G. Paladin and A. Vulpiani, Lett. Nuovo Cimento **41**, 82 (1984).
[5]M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia, and J. Stavans, Phys. Rev. Lett. **55**, 798 (1985).
[6]L. P. Kadanoff, J. Stat. Phys. **42**, 395 (1986).
[7]D. Bensimon, M. H. Jensen, and L. P. Kadanoff, Phys. Rev. A **33**, 3622 (1986).
[8]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).
[9]C. Castellani and L. Peliti, J. Phys. A **19**, L429 (1986).
[10]R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **18**, 2157 (1985).
[11]J. P. Eckmann and I. Procaccia, Phys. Rev. A **34**, 659 (1986).
[12]P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhäuser, Basel, Switzerland, 1980).
[13]J. A. Yorke and E. D. Yorke, J. Stat. Phys. **21**, 263 (1979).
[14]C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983).

[15]M. Widom, D. Bensimon, L. P. Kadanoff, and S. J. Shenker, J. Stat. Phys. **32**, 443 (1983).
[16]L. P. Kadanoff and C. Tang, Proc. Nat. Acad. Sci. **81**, 1276 (1984).
[17]H. Kantz and P. Grassberger, Physica D **17**, 75 (1985).
[18]J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[19]P. Szépfalusy and T. Tél, Phys. Rev. A **34**, 2520 (1986).
[20]T. Tél (unpublished).
[21]P. Grassberger and I. Procaccia, Phys. Rev. A **28**, 2591 (1983); Physica D **13**, 34 (1984).
[22]P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).
[23]P. Szépfalusy and G. Györgyi, Phys. Rev. A **33**, 2852 (1986).
[24]G. Györgyi and P. Szépfalusy, Z. Phys. B **55**, 179 (1984).
[25]S. Chang and J. Wright, Phys. Rev. A **23**, 1419 (1981).
[26]G. Györgyi and P. Szépfalusy, J. Stat. Phys. **34**, 451 (1984).
[27]P. C. Hemmer, J. Phys. A **17**, L247 (1984).
[28]S. Grossmann and H. Horner, Z. Phys. B **60**, 79 (1985).
[29]D. Farmer, Z. Naturforsch. Teil A **37**, 1304 (1982).
[30]H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).
[31]P. Grassberger, Phys. Lett. **97A**, 227 (1983).