Escape rate from strange sets as an eigenvalue

T. Tél*
International Centre for Theoretical Physics, Trieste, Italy
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A new method, based on the eigenvalue problem of the master equation of discrete dynamical systems, is applied to the calculation of the escape rate from chaotic repellers or semiattractors. The corresponding eigenfunction is found to be smooth along unstable directions and to be, in general, a fractal measure. Examples of one- and two-dimensional maps are investigated.

Chaotic behavior which can be observed on a long but finite time scale has attracted recent interest from both theoretical[1–15] and experimental[16–19] points of view. This transient chaos is related to strange invariant objects (Cantor sets) of the phase space, called chaotic repellers[8] or semiattractors[12] depending on whether the strange set is repelling in all or in a few directions only. On the other hand, Cantor sets (being, e.g., supports of the density of state) have recently been found to play an essential role in certain localization problems[20] in the dynamics on fractal lattices,[21] in models of amorphous solids[22] and of quasicrystals.[23] Since these Cantor sets may be interpreted as chaotic repellers or semiattractors, the investigation of the latter may be of relevance also for the theory of disordered systems.

An important characteristic of the dynamics in the vicinity of such strange sets is the escape rate α,[7,8] which measures how fast the repulsion occurs. For discrete dynamical systems, which we shall investigate here, the definition is the following. Let us distribute a large number of points in some neighborhood Γ of the repeller (semiattractor). The probability that a randomly chosen point has not yet escaped Γ after n steps is denoted by W_n. As n gets large ($n \to \infty$) one observes, in general, an exponential decay,[6,8,11,13]

$$W_n \sim \exp(-\alpha n) , \tag{1}$$

where $\alpha > 0$ is the escape rate (known to be independent of Γ). The long-time behavior, however, should then be governed by the largest eigenvalue $1/q = \exp(-\alpha)$ of the master equation describing the evolution of a probability distribution on Γ, which has now no nontrivial stable solution (see also Ref. 11). This eigenvalue develops from the unit eigenvalue associated with the stationary distribution of a chaotic attractor when the latter ceases to exist and is replaced by a repeller or semiattractor as the control parameter is raised. Our aim is here to illustrate that the concept of the escape rate as an eigenvalue may provide a rather rapid and simple method for determining α.

First, we consider chaotic transients in one-dimensional maps $x' = f(x)$, where $f(x)$ is a single humped function. Γ is then an interval containing the repeller. Let us investigate the iteration scheme

$$C_{n+1}(x') = q \sum_{x \in f^{-1}(x')} \frac{C_n(x)}{|f'(x)|} \tag{2}$$

starting with a smooth positive function $C_0(x)$ on Γ, where f' denotes the derivative of f, and $q > 1$. According to our numerical studies there is, in a broad class of maps, a single value of q, $q = \exp(\alpha)$, for which the functions $C_n(x)$ converge towards a finite $C(x)$ for $n \to \infty$. The independence of $C(x)$ from the initial condition $C_0(x)$ has been proved in Ref. 3 under general conditions. Since $C(x) = \lim_{n\to\infty} C_n(x)$ fulfills the equation

$$C(x') = q \sum_{x \in f^{-1}(x')} \frac{C(x)}{|f'(x)|} , \tag{3}$$

it can be considered as an eigenfunction of the Frobenius-Perron equation[24] associated with the eigenvalue $1/q = \exp(-\alpha)$. The quantity $\Gamma^x \int C(x) dx$ is called the conditionally invariant measure which has been invented and interpreted by Pianigiani and Yorke (first item of Ref. 3).

It is worth mentioning here that the fractal dimension of the repeller also appears as eigenvalue: there exists another iteration scheme [Eq. (4) of Ref. 14] which converges if and only if a certain exponent in it takes on the value of the fractal dimension. The corresponding eigenfunction which was shown[14] to yield the stationary distribution on the repeller, accessible in numerical simulations or in experiments, is to be distinguished from $C(x)$, the density of the conditionally invariant measure. In what follows we show that Eq. (2) just like Eq. (4) of Ref. 14, may converge rather rapidly, and this fact makes it useful for practical calculations.

In order to illustrate the rapid convergence of the procedure with a simple example we consider the map

$$f(x) = \begin{cases} 1 - a_1 x, & x > 0 \\ 1 + a_2 x, & x < 0 \end{cases} \tag{4}$$

where $a_1, a_2 > 1$ and $a_1^{-1} + a_2^{-1} < 1$ so that a chaotic repeller shows up. The evolution of a linear initial function $C_0(x) = \gamma_0 x + \beta_0$ on an interval Γ, say $(-1,1)$, can then be followed exactly. The result is $C_n(x) = \gamma_n x + \beta_n$ with

$$\gamma_{n+1} = q(a_2^{-2} - a_1^{-2})\gamma_n , \tag{5a}$$

$$\beta_{n+1} = q(a_1^{-1} + a_2^{-1})\beta_n - \gamma_{n+1} . \tag{5b}$$

Since $a_1, a_2 > 1$, γ_n tends towards zero. A nontrivial limit for $n \to \infty$ exists only if β_n remains constant. This specifies the eigenvalue to be

$$1/q=\exp(-\alpha)=a_1^{-1}+a_2^{-1} \; . \quad (6)$$

It follows from Eq. (5a) that the convergence to the constant limit solution is then exponentially fast (with a critical slowing down for $a_{1(2)}\rightarrow\infty$, $a_{2(1)}\rightarrow 1$).

Our next example is the quadratic map defined by $f(x)=1-ax^2$ in the region $a>2$. In order to determine the escape rate we considered the iterates of $C_0(x)\equiv 1$. An explicit expression for $C_n(x)$ follows then from (2) in the form

$$C_n(x')=q^n \sum_{x\in f^{-n}(x')} |[f^n(x)]'|^{-1} \; , \quad (7)$$

where f^n and f^{-n} represents the nth iterate of f and its inverse, respectively. We treated $q>1$ as a free parameter to be adjusted so that a nontrivial limit solution exists. As long as q is too small (large) $C_n(x)$ monotonously decreases (increases) with n at a fixed x_0. If, however, q is appropriately chosen a convergence is found with an accuracy of less than 1% at the fourth iterate. We used this property to obtain a lower (upper) bound for the escape rate as the value $\alpha=\ln q$, where $C_5(x_0)\ -C_6(x_0)<\epsilon$ $[C_6(x_0)-C_5(x_0)>\epsilon]$ with a small positive ϵ was first realized when increasing q. By this way a fast algorithm has been found. x_0 can arbitrarily be chosen in this procedure since not the particular value of $C_n(x_0)$ determines the escape rate but rather the convergence of this series. We have taken $x_0=0.5$. Figure 1 displays the plot α versus a in the interval $2<a<3$ calculated at 300 different values of a ($\epsilon=10^{-3}$). Error bars typically correspond to the thickness of the curve $\alpha(a)$.

To find a comparison with the results obtained by means of other methods we applied the present procedure at those special values of a (also for $a>3$), where the escape rate has been determined by Widom *et al.*[7] Table I reflects a quantitative agreement. By a more detailed analysis of the convergence of $C_n(x)$ and by going beyond $n=6$, q could have been calculated more accurately but this refinement is not the subject of the present paper.

Due to the rapid convergence of the iteration the eigenfunction $C(x)$ can be safely approximated by $C_6(x)$ of Eq. (7). Figure 2 shows the eigenfunction obtained at different values of a. (It is to be noted that the shortest interval covering the repeller is $(-x^*,x^*)$ with $x^*=[1+(1+4a)^{1/2}]/(2a)<1$). All $C(x)$ are found to be smooth in $(-1,1)$ since for $a>2$ the singularity appearing at $x=1$ is mapped outside this region. The figure makes it evident that the functions $C(x)$ tend to a finite solution for $a\rightarrow\infty$.

TABLE I. A comparison with the results of Ref. 7. The second column contains the reciprocal eigenvalue q obtained with the procedure described in the text. At these values of a the convergence of the series $C_n(x_0)$, $n>3$ seems to be monotonous; therefore, an ϵ as small as 10^{-4} could have been chosen. The data of the last two columns are taken over from Ref. 7 and show the results of a theoretical and an "experimental" method.

a	q	q(theor)	q (expt)
2.4725	1.3019 ± 2.10^{-4}	1.3020	1.3022
3.75	1.7381 ± 2.10^{-4}	1.7384	1.7380
20.0	4.3888 ± 3.10^{-4}	4.3888	4.3887
48.75	6.9286 ± 5.10^{-4}	6.9278	6.9285
90.0	9.4473 ± 7.10^{-4}	9.4458	9.4473

This asymptotic form can be easily deduced from Eq. (3). Since $x=\pm[(1-x')/a]^{1/2}$ and $C(x)$ is smooth, the argument of $C(x)$ on the right-hand side can be set zero for $a\rightarrow\infty$ and one obtains

$$C(x')=\frac{qC(0)}{a^{1/2}\,|1-x'|^{1/2}} \; . \quad (8)$$

The requirement of a finite solution then yields

$$\exp(\alpha)=q=a^{1/2} \; . \quad (9)$$

The same result was obtained by means of a different method in Ref. 7. Curve 4 of Fig. 2 corresponding to the eigenfunction at $a=90$ is, in fact, hard to distinguish from that corresponding to Eqs. (8) and (9).

For maps defined by $f(x)=1-a|x|^z$, $z>0$ we obtain in a similar way

$$\exp(\alpha)=q=(z/2)a^{1/z} \; , \quad (10)$$

$$C(x)=C(0)\,|1-x|^{1/z-1} \; , \quad (11)$$

as the asymptotic results for $a\rightarrow\infty$.

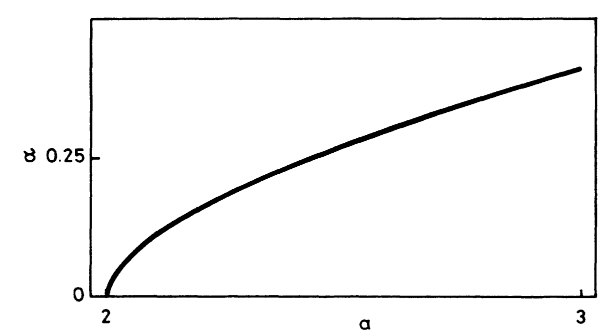

FIG. 1. Escape rate $\alpha=\ln(q)$ obtained as an eigenvalue of Eq. (2) for the map $x'=1-ax^2$, $2<a<3$. Near $a=2$ a power law behavior with exponent $\frac{1}{2}$ is present (see, e.g., Ref. 15).

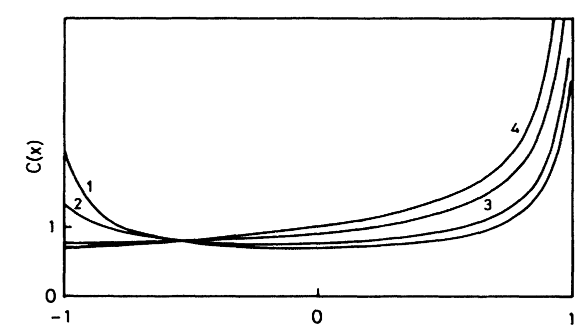

FIG. 2. Eigenfunction $C(x)$ (not normalized). The curves 1, 2, 3, and 4 belong to control parameter value $a=2.01$, 2.10, 3.75, and 90.0, respectively. Curves with $a\geq 90$ are practically indistinguishable from $C(x)=|1-x|^{1/2}$.

Finally, we investigate invertible maps of the plane $\mathbf{x}'=T(\mathbf{x})$ producing chaotic transients. Now, the eigenvalue problem can be most conveniently formulated in terms of a certain measure, which we will call the c measure [the analogue of ${}^{x}\int C_n(x)dx$ of the one-dimensional case]. As an extension of Eq. (2), its evolution is defined by

$$(\mu_j^c)'=qT(\mu_j^c)\ , \tag{12}$$

where μ_j^c denotes the c measure of a region B around a point $\mathbf{x}_j$ lying in a neighborhood Γ of the semiattractor, and $T(\mu_j^c)$ represents the action of the dynamics on μ_j^c. The transformed c measure $(\mu_j^c)'$ belongs to the region $T(B)$ around x_j'. Starting, e.g. with the Lebesque measure on Γ, subsequent applications of (12) with $q=\exp(\alpha)$ lead to the conditionally invariant measure.[3] Note that the conditionally invariant measure again differs from the "natural" invariant measure defined in Ref. 14.

This procedure will be illustrated on a generalized version of the baker transformation[25] introduced in Ref. 14. The dynamics is given by

$$y'=\begin{cases} sy, & y<c\\ 1-t(1-y), & y>c,\end{cases}\quad x'=\begin{cases} ax, & y<c\\ 1/2+bx, & y>c\ ,\end{cases} \tag{13}$$

where $0<a,b,c<1/2$ and $sc,t(1-c)>1$. The latter condition ensures the escape along the unstable direction y. Although (13) is a piecewise linear model it seems to reflect the most typical features of chaotic semiattractors. By means of this example, general properties of conditionally invariant measures can be studied which have not been discussed in the literature.

Starting with the Lebesque measure on the unit square (Γ) Eq. (12) leads to a c measure q/s and q/t on the strips, $0<x<a$, $0<y<1$ and $1/2<x<1/2+b$, $0<y<1$, respectively. After n steps, the c measure on strips of width $a^m b^{n-m}$, $m=0,1,\ldots,n$ is $(q/s)^m(q/t)^{n-m}$. The total c measure of the unit square is then $q^n(s^{-1}+t^{-1})^n$. A nontrivial limit exists only if this quantity remains finite for $n\to\infty$, from which

$$\exp(\alpha)=q=(s^{-1}+t^{-1})^{-1} \tag{14}$$

follows for the escape rate.

The measure obtained for $n\to\infty$ has then a smooth density along the y direction, it is, however, a fractal measure.[26] By means of the method used in Ref. 25 one obtains for the fractal dimension d_0 (Ref 27) and the information dimension d_1 (Ref. 26) of this measure the equations

$$a^{d_0-1}+b^{d_0-1}=1\ , \tag{15}$$

and

$$d_1=1+\frac{t\ln(q/s)+s\ln(q/t)}{t\ln(a)+s\ln(b)}\ , \tag{16}$$

respectively. The difference between conditionally and natural invariant measures is reflected also in the difference of their dimensions (c. Ref. 14).

Finally, we mention that the smoothness along the unstable direction, the fractal structure along the stable direction, and the fractal measure property seem to be general features of conditionally invariant measures of semiattractors.

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*On leave of absence from the Institute for Theoretical Physics, Eötvös University, H-1088 Budapest, Hungary.

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