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Phase transitions associated with dynamical properties of chaotic systems

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A new type of phase transition associated with a singularity in the spectrum of generalized entropies, or in a corresponding free energy, is shown to exist in dynamical systems. Candidates for exhibiting such transitions are intermittent chaotic systems where certain generalized entropies vanish. This new transition may coexist with one associated with the static properties of the system.

Nonanalytic behavior in the spectrum of generalized dimensions D_q (Refs. 1–3) or in the static multifractal spectrum $f(\alpha)$ (Refs. 4–6) has attracted recent interest. One of the reasons is that such behavior might be interpreted as a phase transition[7–14] in the framework of different kinds of thermodynamic formalism worked out for dynamical systems.[15–20] A typical example is the one-dimensional quadratic map in its fully developed chaotic state where there exists a break in the D_q spectrum at $q=2$.[21] Here, just as in all other examples investigated, the phase transition is not accompanied by a singular behavior in the spectrum of the generalized entropies K_q.[22,23] The latter can be regarded as the dynamical counterparts of D_q.

We show in this paper that singularities may occur also in the K_q spectrum and that this phenomenon, too, can be interpreted as a phase transition. Candidates for exhibiting such phase transitions are chaotic systems with intermittent behavior. Note that in these systems critical slowing down shows up not only in an abstract space but also in real-time dynamics. It will be pointed out that the new transition may simultaneously occur with a singularity in the D_q spectrum. We also discuss how the thermodynamic formalism introduced in this paper and that based on the ensemble of the unstable trajectories of the dynamics are related.

We consider one-dimensional maps $x'=f(x)$ modeling higher-dimensional systems with strong dissipation. For the sake of simplicity, $f(x)$ is assumed to be a single-humped function describing a fully developed chaotic state,[24] when the attractor is mapped two-to-one onto itself. These maps have an absolutely continuous invariant measure μ. The maximum point of $f(x)$ divides the attractor into two intervals $I_0^{(1)}$ and $I_1^{(1)}$. A common refinement[25] of this generator partition, made by taking subsequent preimages of the attractor, leads after n steps to a set of intervals $I_l^{(n)}$, $l=0,1,\ldots,2^n-1$, which completely covers the attractor. Let $\mu(I_l^{(n)})$ denote the natural measure of the interval $I_l^{(n)}$, i.e., the probability that a randomly chosen point (with respect to the stationary distribution) on the attractor falls into $I_l^{(n)}$.

We define a partition function $Z_{\mu,n}(\beta)$ by

$$Z_{\mu,n}(\beta)\equiv\sum_l[\mu(I_l^{(n)})]^\beta \ , \tag{1}$$

where $\beta\in(-\infty,\infty)$ is a parameter, the inverse temperature. A free-energy density $F_\mu(\beta)$ is then obtained for large n, as

$$\beta F_\mu(\beta)=-\frac{1}{n}\ln Z_{\mu,n}(\beta) \ . \tag{2}$$

Before showing that $F_\mu(\beta)$ is intimately related to the generalized entropies of the dynamics, a partition function based on unstable fixed points is introduced. This $Z_n(\beta)$ is defined[19] as

$$Z_n(\beta)\equiv\sum_{y\in\{f^{(n)*}\}}|f^{(n)\prime}(y)|^{-\beta}\sim\exp[\beta F(\beta)n] \ , \tag{3}$$

where $\{f^{(n)*}\}$ stands for the set of fixed points of the nth iterate $f^{(n)}$ of f, and prime denotes differentiation. If the stationary density goes to infinity or takes a zero value in certain points (this is the case if the map possesses a critical point or a cusp) $F(\beta)$ may differ from $F_\mu(\beta)$. Maps for which $F(\beta)=F_\mu(\beta)$ will be called dynamically simple.

We note in passing that other definitions of free energies have also been used in the literature. In Refs. 8 and 14, the intervals $I_l^{(n)}$ defined above are considered, but a partition function and a free energy with respect to the length these intervals are introduced. Another free energy, connected with the eigenvalue of a generalized Frobenius-Perron equation is studied in Ref. 12. The latter free energies and $F(\beta)$ are in certain cases equivalent.[26] These free energies are interesting quantities in themselves and are suitable to exhibit the existence of phase transitions, but they are, in general, different from $F_\mu(\beta)$. For the sake of completeness we mention that in the thermodynamic formalism applied for studying the D_q spectrum a partition is used[11,13,20] in which all intervals have the same natural measure.

Let us turn back to the formalism of Eqs. (1) and (2). Since $\mu(I_l^{(n)})$ represents the probability of trajectories with a certain binary code of length n,[25] the generalized entropy K_q (Refs. 22 and 23) can be expressed in terms of $\mu(I_l^{(n)})$ as

$$K_q = \frac{1}{1-q}\frac{1}{n}\ln\sum_l [\mu(I_l^{(n)})]^q \tag{4}$$

for $n\to\infty$. A comparison of Eqs. (1) and (4) immediately leads to a relation between $F_\mu(\beta)$ and K_q in the form

$$K_q = qF_\mu(q)/(q-1) \ . \tag{5}$$

Note that the free energies of Refs. 8 and 12 are defined independently from the generalized entropies and a relationship similar to (5) can be valid only for a certain class of maps.

It is worth establishing a connection with the dynamical multifractal spectrum $g(\Lambda)$.[27-29] A dynamical scaling index Λ was introduced[27] by writing the probability of a path of length $n \gg 1$ as $\exp(-n\Lambda)$. The set of the Λ values is, thus, given in our case by $\{-n^{-1}\ln[\mu(I_l^{(n)})]\}$. (Note that the path probability as defined in Ref. 27 requires a partitioning of the space-time into uniform boxes, while we use a generator partition.) The dynamical spectrum $g(\Lambda)$ is the topological entropy of trajectories with the same Λ. Their number is, therefore, $\exp[ng(\Lambda)]$. An evaluation of the sum in (4) for $n\to\infty$ leads to the result

$$\beta F_\mu(\beta) = \beta\Lambda(\beta) - g(\Lambda(\beta)) \ , \tag{6}$$

where $\Lambda(\beta)$ is obtained from $dg/d\Lambda|_{\Lambda(\beta)}=\beta$. This means that Λ and $g(\Lambda)$ can be considered to be an energy and an entropy function, respectively. The dynamical multifractal spectrum, thus, has an important thermodynamical meaning:

$$g(\Lambda) = S_\mu(E)|_{E=\Lambda} \ , \tag{7}$$

where $S_\mu(E)$ denotes the fundamental equation. [A similar relation with S_μ replaced by the entropy of the formalism of Eq. (3) has been conjectured,[12] which is valid for dynamically simple maps.] As a consequence of Eqs. (5) and (6) the quantity $(1-\beta)K_\beta = -\beta F_\mu(\beta)$ is the Legendre transform of $S_\mu(E)$.

If a nonanalytic behavior is present in the K_q spectrum this implies singularities in the free energy $F_\mu(\beta)$ or in the entropy $S_\mu(E)$, which is a sign of a phase transition. Such a transition we shall call a phase transition associated with dynamical properties of the system.

We now apply the general formalism to the so-called critical maps[24,30-35] of the interval (0,1), which are fully developed chaotic maps with an intermittent point[36-38] in the origin: $f'(0)=1$. Because of this marginally stable point there is a critical slowing down in the dynamics, and the correlation function decays slower than exponentially.[33] We shall point out that in such systems phase transitions associated with the dynamical properties also occur. A particular example for critical maps is the family defined by[34]

$$f(x) = 1 - |x^r - (1-x)^r|^{1/r} \ , \tag{8}$$

with $r>1$. In this class the invariant density is known[34] to be $P(x)=r(1-x)^{r-1}$.

First, we show that the generalized entropies K_q vanish for $q>1$ in critical maps (see also Ref. 29). Since $\sum_l [\mu(I_l^{(n)})]^q$ contains only positive terms, an upper bound is found for K_q, $q>1$ by keeping the contribution of the leftmost interval only. From Eq. (4)

$$K_q \le \frac{1}{1-q}\lim_{n\to\infty}\frac{1}{n}\ln[\mu(I_1^{(n)})]^q \ . \tag{9}$$

It can be seen[34,35] that $\mu(I_1^{(n)})$ exhibits a power-law behavior for large n: $\mu(I_1^{(n)}) \sim n^{-s}$, $s>0$. Since K_q cannot be negative, $K_q=0$ follows for $q>1$. Note that, by definition, fully developed chaotic maps in the critical case should have an absolutely continuous measure. Consequently, the Kolmogorov entropy $K_1>0$. In the special case $r=2$ of the family (8), for example, $K_1=\frac{1}{2}$ (Ref. 24).

For $q<1$ no analytic estimates exist for the generalized entropies. In order to see the behavior of K_q for $q<1$, the entropies are to be determined by using definition (4). The family (8) is particularly well suited for this purpose since the stationary density $P(x)$ is known. We have calculated the preimages of the unit interval up to the fourteenth generation and $\mu(I_l^{(n)})$ has been evaluated by integrating $P(x)$ over $I_l^{(n)}$. By using the asymptotic behavior

$$n^{-1}\sum_l [\mu(I_l^{(n)})]^q = (1-q)K_q + (A+B\delta^n)/n \ ,$$

where A, B, and δ are constants,[39] the entropies have been obtained with a high accuracy for $q<1$. The results suggest a smooth behavior up to $q=1$. Figure 1 shows K_q vs q for the case $r=2$. For critical maps at $q=1$ the truncated entropy exhibits a power-law decay.[34] This means that $\delta\to 1$ when $q\to 1-0$, i.e., a critical slowing down sets in.

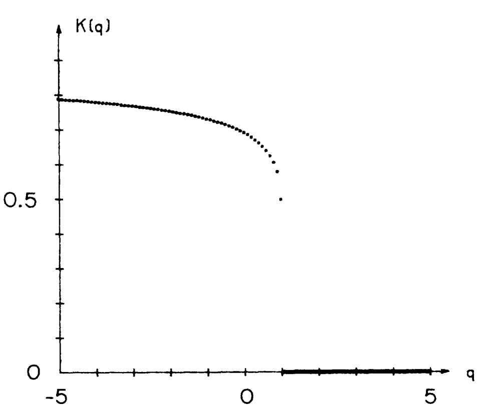

FIG. 1. The K_q spectrum for the $r=2$ case of the family (8) obtained via Eq. (4).

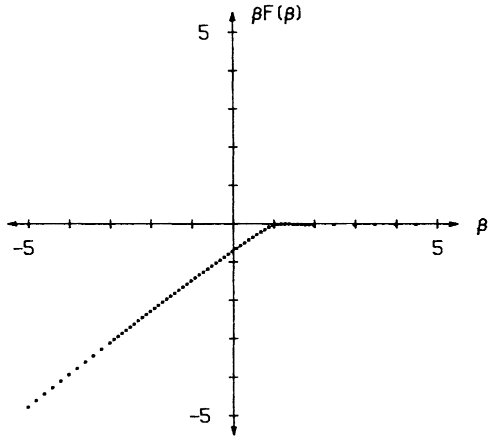

FIG. 2. The $\beta F(\beta)$ spectrum for the same system as in Fig. 1, obtained via Eq. (3).

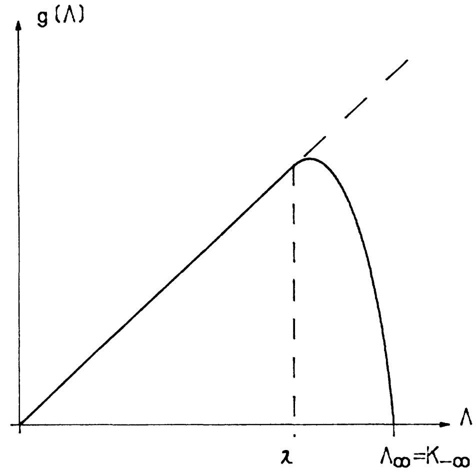

FIG. 3. Qualitative plot of the $g(\Lambda)$ spectrum for a typical critical map. λ denotes the Lyapunov number.

Since critical maps are not everywhere hyperbolic and possess a cusp, it is a basic question whether they belong to the class of dynamically simple maps. In view of this, we have also calculated the partition function (3) and the corresponding free energy $F(\beta)$. The fixed points of $f^{(n)}$ have been determined, up to $n=12$, by iterating the map backward. $\beta F(\beta)$ has been obtained by comparing $Z_n(\beta)$ of the last two generations. The results for the case $r=2$ are plotted in Fig. 2. A comparison of $F_\mu(\beta)$, determined via Eq. (5), and $F(\beta)$ shows that these quantities coincide within numerical error. This suggests that the thermodynamic formalisms based on the dynamical behavior and on the fixed points are equivalent in this case.

We are now in a position to sketch the qualitative form of the $g(\Lambda)$ spectrum of typical critical maps. Since the Kolmogorov entropy is finite, there is a jump in K_q at $q=1$. Consequently, $\beta F_\mu(\beta)$ behaves like $(\beta-1)K_1$ for $\beta \rightarrow 1-0$ and $F_\mu(\beta)\equiv 0$ for $\beta \geq 1$. When performing the Legendre transform of $-\beta F_\mu(\beta)=(1-\beta)K_\beta$ [see Eq. (6)] we find, therefore, that $\Lambda(\beta=1)$ can take any value between K_1 and 0. Thus, $g(\Lambda)=\Lambda$ in the range $0\leq\Lambda\leq K_1$. This part of the $g(\Lambda)$ curve then joins a single humped curve with a continuous first derivative (see Fig. 3). Phase transition occurs at $\Lambda=K_1$. In the language of the thermodynamics, the internal energy $E(\beta)$ jumps from K_1 to 0 at $\beta=1$.

Finally, we note that phase transitions associated with static properties may coexist with those associated with dynamical properties. In the case of family (8), for example, one obtains[4] for the generalized dimension $D_q=rq/(q-1)$ for $q<q^*=1/(1-r)$ and $D_q\equiv 1$ for $q>q^*$, since the stationary density has a singularity of order r at $x=1$. The phase transition associated with this singularity differs from that found in the thermodynamic formalism based on $Z_{\mu,n}(\beta)$, a sign of which is the fact that the critical point of the former, q^*, depends on the parameter r, while the second occurs always at $\beta=1$, independently of r. The connection between these two different types of phase transition remains to be clarified by further studies.

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