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# Dynamical spectrum and thermodynamic functions of strange sets from an eigenvalue problem

T. Tél

*Institute for Theoretical Physics, Roland Eötvös University, Puskin utca 5-7, H-1088 Budapest VIII, Hungary*

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The dynamical multifractal spectrum of hyperbolic systems is found to be the fundamental equation in a kind of statistical-mechanics formalism for both permanent and transient chaos. It is shown that the free energy may appear in an eigenvalue problem, the solution to which provides a new method for calculating dynamical spectra. Explicit examples are given and the possibility of extending the method for higher-dimensional systems is discussed.

Fluctuations in the divergence of nearby trajectories is an essential feature of chaotic motion. Different quantities related to these fluctuations have been proposed to characterize the dynamical properties of chaotic systems.[1-8] Recently, Eckmann and Procaccia have pointed out[7] that, in analogy with the static multifractal spectrum,[9,10] there exists a spectrum $[g(\Lambda)]$ of dynamical scaling indices $(\Lambda)$. It is, therefore, of interest how $g(\Lambda)$ is related to other quantities and how it can be determined in practice. These questions are relevant also for transient chaotic phenomena extensively discussed in the literature.[11-24]

In this Rapid Communication we first find a link between $g(\Lambda)$ and the statistical-mechanics formalism worked out in Ref. 3, and some other quantities, for both permanent and transient chaos. Then it is shown that the free energy may appear as an eigenvalue of a linear equation. From the solution to this new eigenvalue problem, which is found very fast numerically, the dynamical spectrum follows.

We shall be mainly interested in one-dimensional (1D) hyperbolic maps $x'=f(x)$ modeling systems with extremely strong dissipation.[25] The dynamical scaling index $\Lambda$ is defined[7] by writing the probability of a chaotic trajectory of length $n\gg 1$ (using a finite resolution in space) as $\exp(-n\Lambda)$. For 1D maps the scaling index turns out to be[7,8]

$$\Lambda = n^{-1}\ln|f^{(n)\prime}(x_1)| - \alpha \ , \tag{1}$$

where $f^{(n)}$ stands for the $n$th iterate of the map, the prime denotes the derivative, and $x_1$ is the starting point of the trajectory. It seems natural to assume that for long trajectories $\alpha$ is a constant, the escape rate governing the exponential decay of the transients.[14-16] For permanent chaos $\alpha=0$.

The number of times the scaling index takes on a value between $\Lambda$ and $\Lambda+d\Lambda$ is[7] $\exp[ng(\Lambda)]d\Lambda$, from which a relation follows between the dynamical spectrum $g(\Lambda)$ and the generalized entropies $K_q$:[4,18,26]

$$(q-1)K_q = q\Lambda(q) - g(\Lambda(q)) \ , \tag{2}$$

with

$$g'(\Lambda(q)) = q \ . \tag{3}$$

Consequently, $\Lambda(1)=K_1$, where for 1D maps $K_1=\lambda-\alpha$ (Ref. 17) and $\lambda$ denotes the Lyapunov exponent.

The statistical-mechanics formalism of Takahashi and Oono[3] is an intuitive extension of the statistical formalism worked out for axiom-$A$ systems.[27-29] The central quantity of Ref. 3 is the partition function

$$Z_n(\beta) = \sum_{y\in\mathcal{F}(f^{(n)})} \exp[-\beta\ln|f^{(n)\prime}(y)|] \ , \tag{4}$$

where $\mathcal{F}(f^{(n)})$ denotes the set of fixed points of $f^{(n)}$ and $\beta\in(-\infty,\infty)$ is a free parameter, the inverse temperature. For large $n$ the free energy $F(\beta)$ is defined by

$$Z_n(\beta) = \exp[-\beta F(\beta) n] \ . \tag{5}$$

Standard thermodynamic relations yield the internal energy $E(\beta)$ and the entropy $S(\beta)$. It has been claimed[3] that for 1D maps with chaotic attractors

$$S(0)=K_0,\ S(1)=K_1,\ F(1)=0,\ E(1)=\lambda \ . \tag{6}$$

When connecting this formalism with the $g(\Lambda)$ spectrum we first observe that for large $n$ $Z_n(\beta)$ appears as an ensemble average (see also, Ref. 5). The stationary distribution on the invariant set associated with the dynamics is based in hyperbolic systems on the fixed points[30] of $f^{(n)}$. In the case with a chaotic attractor the weighting factor at the fixed point $y$ is just $|f^{(n)\prime}(y)|^{-1}$. For transient chaos $|f^{(n)\prime}|$ is to be multiplied by the probability that the trajectory has not yet escaped. The weighting factor is, therefore, $\exp(\alpha n)|f^{(n)\prime}(y)|^{-1}$ in this more general case.[8,17] Thus, by means of (1) we can write

$$Z_n(\beta) = \exp(-\alpha\beta n)\langle\exp[(1-\beta)\Lambda n]\rangle \ . \tag{7}$$

Note that the property $Z_n(1)\sim\exp(-\alpha n)$ found in Ref. 14 is inherent in this equation.

The average to be taken in (7) can be evaluated as an integral over $\Lambda$, provided ergodicity holds, since the probability distribution of $\Lambda$ is known:

$$Z_n(\beta) \sim \exp(-\alpha\beta n)\int\exp[-\beta\Lambda n + ng(\Lambda)]d\Lambda \ . \tag{8}$$

For large $n$, the application of the saddle point method (used as in Ref. 7) leads to

$$\beta F(\beta) = \beta\Lambda(\beta) - g(\Lambda(\beta)) + \alpha\beta \ , \tag{9}$$

 

where $\Lambda(\beta)=\Lambda(q=\beta)$ is given by (3). Thus, we have obtained a direct relation between the spectrum $g(\Lambda)$ and the free energy $F(\beta)$. As a consequence of Eqs. (9) and (2) we find in hyperbolic systems

$$K_{q=\beta}=K(\beta)\equiv\frac{\beta[F(\beta)-\alpha]}{\beta-1} . \tag{10}$$

Furthermore, since

$$E(\beta)\equiv\frac{\partial[\beta F(\beta)]}{\partial\beta}=\Lambda(\beta)+\alpha , \tag{11}$$

the special scaling index $\Lambda(q)$ of Eq. (3) turns out to be, up to an additive constant, the internal energy taken at the temperature $1/q$. Finally, for the entropy, we have

$$S(\beta)\equiv\beta[E(\beta)-F(\beta)]$$
$$=g(\Lambda(\beta))=\beta\Lambda(\beta)+(1-\beta)K(\beta) . \tag{12}$$

Consequently,

$$g(\Lambda)=S(E)|_{E=\Lambda+\alpha} . \tag{13}$$

The dynamical spectrum $g(\Lambda)$ is, thus, essentially the fundamental equation $S(E)$ of the statistical formalism.

Other important relations follow from (7) by considering $\ln|f^{(n)'}|$ as a random variable. Its cumulants were shown[4] to be linear in $n$. Provided that the cumulant expansion converges, one finds

$$g(\Lambda(\beta))-\beta\Lambda(\beta)=(1-\beta)(\lambda-\alpha)+\sum_{l(>1)}(1-\beta)^l Q_l/l! , \tag{14}$$

where the $l$th cumulant has been denoted by $Q_l n$, $Q_1\equiv\lambda$. $Q_l$ thus appears as the $l$th derivative of $(-1)^{l+1}\beta F(\beta)$ taken at $\beta=1$.

Furthermore, since the partition function is expected[13,14,28] to tend toward a constant for $\beta=D_0$ where $D_0$ is the fractal dimension of the strange invariant set (repeller), the well-known relation[17,19]

$$\alpha=\sum_{l(>0)}(1-D_0)^l Q_l/l!$$

is recovered.

The second cumulant $Q_2$ has a special meaning since it can be interpreted as a diffusion coefficient.[1] From (14)

$$Q_2=-\Lambda'(\beta=1)=-1/g''(\Lambda(1)) ,$$

i.e., $Q_2$ is the specific heat taken at $\beta=1$ in the statistical formalism. In a local parabolic approximation around the maximum of the spectrum the half-width[8] $\Gamma\equiv\Lambda(0)-\Lambda(1)$ turns out to be $Q_2$ which equals $2(K_0-K_1)$ in this approximation.

It is worth noting that a slightly different dynamical spectrum has been introduced and investigated by Fujisaka[1] and Benzi, Paladin, Parisi, and Vulpiani[5] for permanent chaos. They both considered $n^{-1}\ln\langle|f^{(n)'}|^q\rangle$, $n\gg1$ which they denoted by $q\lambda_q$ and $L(q)$, respectively. It is obvious from our considerations that

$$q\lambda_q=L(q)=g(\Lambda(1-q))+(q-1)\Lambda(1-q)+\alpha q ,$$

including the case of transient chaos as well.

The results (9)–(12) for $\alpha=0$ are generalizations of Eq. (6). They are, however, valid also for $\alpha>0$, where they contradict the conjectures $S(D_0)=K_1$ and $E(D_0)=\lambda$ formulated in Ref. 3 and in the addendum of Ref. 5, respectively. It seems that the discrepancy originates from another choice of the weighting factor when writing the partition function as an average. A simple example below illustrates that this choice may lead to inconsistencies.

It is to be mentioned that the formula for $K_q$ obtained by the method of Ref. 8 can be shown for everywhere-expanding maps to be equivalent to Eq. (10) for $\alpha=0$. An extension of the calculation for transient chaos would lead to results which are in accord with Eq. (10) for $\alpha>0$.

After finding the link between $g(\Lambda)$ and the free energy, it is now shown that $F(\beta)$ may appear as an eigenvalue in a class of equations. By extending the ideas of Ref. 19, let us consider the recursion

$$Q_{n+1}^{(\beta)}(x')=R(\beta)\sum_{x\in f^{-1}(x')}\frac{Q_n^{(\beta)}(x)}{|f'(x)|^\beta} , \tag{15}$$

where $\beta\in(-\infty,\infty)$ and $R(\beta)$ is positive. We have investigated Eq. (15) for single humped maps beyond crisis,[15] where chaotic transients are present, and in crisis (or fully developed chaos[31]) cases, where a chaotic attractor shows up; however, the properties discussed below may be valid in more general cases too. These studies suggest (in another context, see Ref. 24) that for any $\beta$ there exists one single prefactor $R^*(\beta)$ so that iteration (15) leads to a unique nontrivial limit solution $Q^{(\beta)}(x)=\lim_{n\to\infty}Q_n^{(\beta)}(x)$ for smooth positive initial functions $Q_0^{(\beta)}(x)$. In cases with a chaotic attractor, Eq. (15) for $\beta=1$ $[R^*(\beta)=1]$ is the well-known Frobenius-Perron equation.[25,28] For transient chaos, $R^*(D_0)=1$ and $Q^{(D_0)}(x)$ is the invariant density on the coarse grained repeller,[19] while $Q^{(1)}(x)$ represents the density of the conditionally invariant measure[12] and $R^*(1)=\exp(\alpha)$.[21]

Since the limit solution is independent of the initial function we choose the latter to be unity. Then, the iteration of (15) yields

$$Q_n^{(\beta)}(x')=R^n(\beta)V_n(\beta,x') ,$$

where

$$V_n(\beta,x')=\sum_{x\in f^{-n}(x')}\exp[-\beta\ln|f^{(n)'}(x)|] . \tag{16}$$

With the eigenvalue $R^*(\beta)$ we have for large $n$,

$$V_n(\beta,x')=[R^*(\beta)]^{-n}Q^{(\beta)}(x') . \tag{17}$$

It is reasonable to assume that the $n$th preimages of $x'$ form an ensemble which is as representative as the set of fixed points of $f^{(n)}$ if $n\gg1$. By accepting the equivalence, we have $V_n(\beta,x')\sim Z_n(\beta)$ and $Q^{(\beta)}(x')$ plays the role of an $x'$-dependent normalization constant. Thus we obtain

$$R^*(\beta)=\exp[\beta F(\beta)] \tag{18}$$

and

$$Q^{(\beta)}(x')=V_n(\beta,x')/Z_n(\beta) .$$

In fact, the results of Refs. 19 and 21 have shown, in comparison with the properties $F(D_0)=0$ and $F(1)=\alpha$, that Eq. (18) is valid for $\beta=D_0$ and $\beta=1$, respectively.

As an example in fully developed chaotic state of nonhyperbolic systems let us shortly consider the case $f(x)=1-2x^2$. The slope of all fixed points of $f^{(n)}$ is then $2^n$ with the exception of $x=-1$, where the slope is $4^n$. Thus, one finds from (4) and (5) that $\beta F(\beta)=(\beta-1)\ln 2$ if $\beta\geq -1$, but, since the leftmost fixed point becomes dominating for $\beta<-1$, $F(\beta)=\ln 4$ in this range. A direct solution of Eq. (15) yields the same result for $\ln R^*(\beta)$ illustrating that relation (18) can hold also for nonhyperbolic systems. [The deviation from a uniform distribution in $K(\beta)\equiv\beta F(\beta)/(\beta-1)$ seems to be the analog of the deviation from unity[32] in the spectrum of generalized dimensions.[33]] Note that $K_{q=\beta}\neq K(\beta)$ in this case.

Since the iterative solution of Eq. (15) converges exponentially fast,[24] a numerical determination of the eigenvalue $R^*(\beta)$ may provide a powerful method, via Eq. (15), for calculating the free energy. Further quantities like $K_q$ and $g(\Lambda)$ can then be derived in hyperbolic cases through relations (10), (12), and (13). This method may be of special importance for transient chaotic phenomena with relatively large values of $\alpha$ when it is difficult to find long trajectories in the vicinity of the repeller [and a direct calculation of $Z_n(\beta)$ is also cumbersome].

We have also investigated the quadratic map $f(x)=1-ax^2$ for $a>2$. By considering $R(\beta)$ as a free parameter at fixed $\beta$, Eq. (15) was iterated with a constant initial function. The rapid convergence was used to find an approximate value for $R^*(\beta)$ as the value where $|Q_6^{(\beta)}(x)-Q_5^{(\beta)}(x)|<\varepsilon$ with a small $\varepsilon$ and arbitrary $x$. Figure 1 exhibits the results obtained for $a=2.1$ by calculating $F(\beta)$ in this way and by using the general formulas.

The map defined by $f(x)=1-a_1x$ for $x\geq 0$ and $f(x)=1+a_2x$ for $x<0$ provides an exactly solvable example in the range $a_1^{-1}+a_2^{-1}\leq 1$. The validity of relations (9)–(13) can now be checked in an analytic way and we can see that the results contradict the conjectures $S(D_0)=K_1$ and $E(D_0)=\lambda$. The dynamical spectra $K_q$ or $g(\Lambda)$ calculated via Eq. (18) coincide with the results of Ref. 8. Instead of citing them explicitly, we mention that there is a nontrivial connection in this system between the dynamical and static spectra. Namely,

$$\beta[F(\beta)-\alpha]\,|_{\beta=q-\tau(q)}=[\beta\Lambda(\beta)-g(\Lambda(\beta))]\,|_{\beta=q-\tau(q)}=\alpha\tau(q)\ , \qquad (19)$$

where

$$\tau(q)\equiv(q-1)D_q=q\alpha(q)-f(\alpha(q))$$

(see Ref. 10), and $D_q$ has been given in Ref. 24. This shows that certain features of the $f(\alpha)$ spectrum may contain information about the dynamics as claimed in Ref. 34. Of course, the dynamical spectrum is the more general one from which $\tau(q)$ follows. In the permanent chaos case $a_1^{-1}+a_2^{-1}=1$, $g(\Lambda)$ is nontrivial but all generalized dimensions are unity in this example.

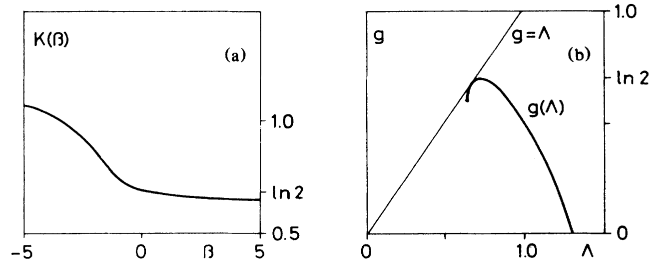


FIG. 1. (a) The spectrum $K(\beta)=\beta[F(\beta)-F(1)]/(\beta-1)$ obtained via Eq. (18) for $f(x)=1-2.1x^2$ in the range $-5\leq\beta\leq 5$. (b) The spectrum $g(\Lambda)$ determined through Eqs. (11)–(13) and (18). $S(\beta)$ seems to reach a finite constant for $\beta>3$ inducing a left end point in $g(\Lambda)$. The range of investigation was $|\beta|\leq 10$.

Finally, it is mentioned that the eigenvalue problem (15) can be generalized[19,21,24] for invertible maps of the plane $x'=T(x')$. The derivative $f'(x)$ is to be replaced then by $\lambda_1(x)$, the local coefficient of expansion[35] along the unstable direction, and it is convenient to formulate the recursions in terms of certain measures $\mu^{(\beta)}$. They read

$$(\mu_j^{(\beta)})'=R(\beta)T(\mu_j^{(\beta)}|\lambda_1(x_j)|^{1-\beta})\ , \qquad (20)$$

where $\mu_j^{(\beta)}$ stands for the $\beta$ measure of a tiny region around $x_j$, and the induced map in the space of measures has been denoted by $T$ too. It was argued in Ref. 24 that a nontrivial limiting measure can be reached only at a unique choice $R^*(\beta)$ of the prefactor at any fixed $\beta$. [$R^*(1)=\exp(\alpha)^{21}$ and $R^*(D_0^{(1)})=0$ (Ref. 19), where $D_0^{(1)}$ is the fractal dimension along the unstable direction.] The definition $\beta F(\beta)=\ln R^*(\beta)$ may then provide an extension of the statistical formalism of Ref. 3 for higher dimensional systems but the relation to $g(\Lambda)$ remains to be clarified.

*Note added in proof.* After submitting this paper the author became aware of the manuscript by Bohr and Rand dealing with similar aspects from a different point of view. The quantity $\alpha$ of Ref. 36 plays the role of the energy $E$, and their $S(\alpha)$ is analogous with our $S(E)$. Further related papers are Refs. 37 and 38, where certain dynamical spectra of higher dimensional chaotic attractors are considered.

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