Nonequilibrium potentials and their power-series expansions

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(Received 26 April 1989)

The analytic properties of nonequilibrium potentials are studied in a class of two-variable models undergoing a bifurcation of codimension higher than 1. Several methods for the construction of nonequilibrium potentials are given. Cases are exhibited explicitly where a polynomial expansion does not exist due to logarithmic terms, even though the potential remains smooth. It is concluded (i) that the nonexistence of polynomial expansions near bifurcation points of higher order, recently reported by several authors, does not imply the nonexistence of a smooth potential, and (ii) that even in cases where the Hamilton-Jacobi equation has a particular solution in the form of a power series, that particular solution may still fail to represent the nonequilibrium potential by failing to satisfy the necessary boundary condition at the attractor.

I. INTRODUCTION

Nonequilibrium potentials have generally been accepted as important characteristics of stationary processes taking place far away from thermal equilibrium (for reviews see Refs. 1–3). Being analogs of the (coarse-grained) free energy they determine the stationary distribution of the system in the weak-noise limit and play the role of a Lyapunov function characterizing the stability and the average lifetime of different metastable states (if they coexist).

Recently, several authors have dealt with the problem of high codimensional bifurcations when the real parts of more than one eigenvalue of the linearized process disappear at the bifurcation point.[4–8] In Refs. 4, and 6–8 the conclusion has been reached that, generally, the nonequilibrium potential cannot be of a polynomial type in these cases. More or less explicitly it has also been suggested that no smooth potential can exist at all at the aforementioned bifurcation points. The aim of this paper is to show that even in such cases a *smoothly differentiable* nonequilibrium potential exists (although not in a polynomial form around its minimum).

The nonexistence of a polynomial solution, of course, implies the nonanalticity of the potential in the sense that it cannot be expanded into a Taylor series around its minimum. For the sake of clarity we mention that this type of nonanalyticity is completely different from that discussed earlier in the literature.[1,9–17] The latter implies *discontinuous* first derivatives of the potential and occurs due to either chaos (nonintegrability) of a Hamiltonian system associated with the weak-noise limit of the stochastic process,[9–12] or nontrivial (e.g. toroidal) topology of the phase space and coexistence of metastable states (i.e., attractors of the deterministic system).[1,13–17] The regions of nondifferentiability lie always far away from the local minima of the potential [in the case of a chaotic Hamiltonian dynamics, e.g., they are situated just around the maxima (repellers of the deterministic process) (Ref. 9)]. In contrast, the nonanalyticity appearing at bifurcation points of codimension higher than 1 is of a much weaker type and can be observed at the minimum of the nonequilibrium potential.

Bifurcations of higher codimensions have been considered in the framework of both master equations[6,8] and Fokker-Planck equations.[4,5,7] For concreteness we restrict our attention to the latter, but the general properties of the nonequilibrium potential are expected to be similar for master-equation dynamics, too. We consider systems, the states of which are specified by n stochastic variables q^ν, $\nu=1,\ldots,n$. Their dynamics is governed by a Fokker-Planck equation

$$\frac{\partial P(q)}{\partial t}=-\frac{\partial}{\partial q^\nu}K^\nu(q)P(q)+\frac{\eta}{2}Q^{\nu\mu}\frac{\partial^2}{\partial q^\nu \partial q^\mu}P(q)\ , \qquad (1.1)$$

where $K(q)$ is a drift vector. Here, and in the following, repeated lower and upper indices imply summation. For simplicity, the diffusion matrix $Q^{\nu\mu}$ is assumed to be constant. The dimensionless number η is a measure of the noise intensity. The nonequilibrium potential $\Phi(q)$ can be read off the stationary distribution $P_{st}(q)$ in the weak-noise limit $\eta\to 0$, where

$$P_{st}(q)\sim \exp[-\Phi(q)/\eta] \qquad (1.2)$$

holds. Our aim is to study $\Phi(q)$ for bifurcations of codimension higher than 1.

The paper is organized as follows. In Sec. II a mechanical analogy is presented and it is shown that the nonequilibrium potential must always be smooth around its local minima. A general expression is derived for Φ in the framework of a perturbation expansion when the system slightly deviates from one possessing an exactly known potential. A method for specifying a polynomial approximant to $\Phi(q)$ is described in Sec. III for two variable processes indicating that no polynomial form exists at bifurcation points where (the real parts of) both eigenvalues vanish. An illustrative example is worked out in Sec. IV. We can here follow how the radius of convergence of a Taylor-series expansion around the origin shrinks as the bifurcation point is approached. At the bifurcation point logarithmic terms survive making the potential not expandable in a power series but remaining smooth. Beyond the bifurcation the actual minimum is shifted away from the origin and the radius of convergence around the minimum increases with the control parameter. A novel feature of the model is the nonanalyticity of the potential at the origin which is a repeller beyond the bifurcation, although the Hamiltonian dynamics is nonchaotic in this case. We conclude in Sec. V by mentioning the shortcomings of a direct search for polynomial approximants to the nonequilibrium potential and suggest that such investigations always be completed by other methods.

II. THE NONEQUILIBRIUM POTENTIAL AS AN ACTION

A. A general setup

A Hamiltonian of type

$$H(q,p)=\tfrac{1}{2}Q^{\nu\mu}p_\nu p_\mu+K^\nu(q)p_\nu \ , \tag{2.1}$$

a so-called Fokker-Planck Hamiltonian, can be associated with any stochastic process (1.1).[1-3] The mechanical motion described by the canonical equations

$$\dot q^\nu=Q^{\nu\mu}p_\mu+K^\nu(q) \ , \tag{2.2a}$$

$$\dot p_\nu=\frac{\partial K^\mu}{\partial q^\nu}p_\mu \ , \tag{2.2b}$$

at total energy

$$E=0 \ , \tag{2.3}$$

can be used to construct the nonequilibrium potential $\Phi(q)$. Based on the path-integral solution of the Fokker-Planck equation, or on the above-mentioned mechanical picture,[9] one can show that $\Phi(q)$ appears as the action along a special Hamiltonian trajectory. This special trajectory has the property that it ends at a certain time t_e at q and starts at an infinitely earlier time, i.e., for $t_0\to-\infty$, on the attractor A of the deterministic system $\dot q^\nu=K^\nu(q)$.[9] The attractor A must lie on the $p\equiv 0$ hyperplane of the Hamiltonian phase space, as follows from (2.2). In other words, the trajectory ends at q and must belong to the unstable manifold of the hyperbolic object $(q\in A,p\equiv 0)$. If the end point $q=q^*$ is a repeller or a saddle of the deterministic dynamics it can be reached only asymptotically for $t_e\to\infty$. With the exception of such isolated points, t_e is finite and, in autonomous systems, can be chosen to be zero. The deterministic motion possesses at least one attractor; otherwise the stationary distribution is not normalizable. For the sake of simplicity, we assume that there exists a single attractor only. The nonequilibrium potential can then be expressed as[9]

$$\Phi(q)=\min\int_{t_0\to-\infty,\ q(t_0)\in A}^{t=0,\ q(0)=q}L(q(t),\dot q(t))dt \ . \tag{2.4}$$

$L(q,\dot q)$ is here the Lagrangian associated with $H(q,p)$. The min selects the absolute minimum of the action integral if it is multivalued. Explicitly,

$$L=\tfrac{1}{2}Q^{-1}_{\nu\mu}[\dot q^\nu-K^\nu(q)][\dot q^\mu-K^\mu(q)] \ , \tag{2.5}$$

where we made use of the fact that $Q^{\nu\mu}$ is positive definite and is, consequently, invertible. [For positive semidefinite diffusion matrices, nevertheless, a positive Lagrangian can be found which depends then also on higher than first-order temporal derivatives of $q(t)$ (Ref. 14).] As a consequence of (2.4) and (2.5) the potential is minimal on the attractor $q\in A$.

By using the fact that the motion takes place on the hyperplane (2.3), we find $L=p_\nu\dot q^\nu$. Therefore, we can write, equivalently,

$$\Phi(q)=\min\int_{q\in A}^{q}p_\nu(q)dq^\nu \ , \tag{2.6}$$

where $p_\nu(q)$ is the equation specifying the unstable manifold of the object $(q\in A,p\equiv 0)$.

The accumulation of knowledge on nonlinear (chaotic) dynamical systems[18] has led also to a more detailed understanding of general properties of nonequilibrium potentials.[9-12] In phase spaces of higher than three dimensions (in stochastic processes with more than one variable) Hamiltonians [e.g., (2.1)] are generally nonintegrable. The existence of wildly oscillating stable and unstable manifolds is generic.[18] Consequently, the action $\int L\,dt=\int p_\nu dq^\nu$ is multivalued and the min in equations (2.4) and (2.6) leads to a piecewise differentiability of the potential.[9] This type of behavior is characteristic for regions away from the attractor A (generally close to some repeller R), since even in generic cases the unstable manifold $p_\nu(q)$ is smooth around $(q\in A,p\equiv 0)$. Therefore, the nonequilibrium potential $\Phi(q)$ is locally always *smooth* around the attractor.

B. Perturbation theory

As known from classical mechanics, the action cannot always be evaluated explicitly. The formalism is, however, well suited for perturbative calculations.

Let us assume that the drift $K^\nu(q)$ appears as a sum

$$K^\nu(q)=K^\nu_0(q)+\epsilon K^\nu_1(q) \ , \tag{2.7}$$

where the nonequilibrium potential $\Phi_0(q)$ associated with $K^\nu_0(q)$ is explicitly known and is smoothly differentiable everywhere. ϵ is here a dimensionless small parameter. Up to first order in ϵ the Lagrangian (2.5) can be written as

$$L(q,\dot q)=L_0(q,\dot q)+\epsilon L_1(q,\dot q) \ , \tag{2.8}$$

with

$$L_0(q,\dot q)=\tfrac{1}{2}Q^{-1}_{\nu\mu}[\dot q^\nu-K^\nu_0(q)][\dot q^\mu-K^\mu_0(q)] \ , \tag{2.8a}$$

$$L_1(q,\dot q)=-Q^{-1}_{\nu\mu}[\dot q^\nu-K^\nu_0(q)]K^\mu_1(q) \ . \tag{2.8b}$$

The nonequilibrium potential is then, up to the same order,

$$\Phi(q)=\Phi_0(q)+\epsilon\Phi_1(q) \ , \tag{2.9}$$

where

$$\Phi_0(q)=\int_{t_0\to-\infty,\, q(t_0)\in A_0}^{t=0,q(0)=q} L_0(q(t),\dot q(t))dt \ . \tag{2.9a}$$

A_0 denotes the attractor of the unperturbed deterministic equations $\dot q^{\nu}=K_0^{\nu}(q)$. Our aim is to express $\Phi_1(q)$ in terms of $K_1^{\nu}(q)$ and the unperturbed motion.

First, let us note that the Hamiltonian trajectory in zeroth order is a solution of a first-order differential equation. According to (2.6), $p_\nu=\partial\Phi_0/\partial q^{\nu}$ holds for the special trajectory we need in (2.9a) and, consequently from (2.2a)

$$\dot q^{\nu}(t)=Q^{\nu\mu}\frac{\partial\Phi_0}{\partial q^{\mu}}+K_0^{\nu}(q) \ . \tag{2.10}$$

This equation is to be solved with the condition $q(t=0)=q$. Since by $p_\nu=\partial\Phi_0/\partial q^{\nu}$ the trajectory is on the unstable manifold of $(q\in A_0, p\equiv 0)$, this trajectory automatically has the property that it started on the attractor, i.e., for $t_0\to-\infty\, q^{\nu}(t_0)\in A_0$.

Denoting by $\epsilon\delta q^{\nu}(t)$ the difference between the solutions of (2.2) and (2.10), we can write Φ_1 as

$$\Phi_1(q)=\int_{t_0\to-\infty}^{t=0}\left[\frac{\partial L_0}{\partial q^{\nu}}-\frac{d}{dt}\frac{\partial L_0}{\partial \dot q^{\nu}}\right]\delta q^{\nu}\Big|_t\, dt + \left[\frac{\partial L_0}{\partial \dot q^{\nu}}\delta q^{\nu}\right]_{t_0\to-\infty}^{t=0} + \min\int_{t_0\to-\infty,\, q(t_0)\in A_0}^{t=0,\, q(0)=q} L_1(q(t),\dot q(t))dt \ , \tag{2.11}$$

where (2.9) and the calculus of variation have been used. The first two terms represent the difference between the action of L_0 calculated along the first order and the unperturbed trajectory, while the last term is the action of L_1 taken along the unperturbed trajectory. The first integral vanishes according to Hamilton's principle. The second term does not contribute at $t=0$, since both trajectories end at q. There is a finite δq^{ν}, however, at $t_0\to-\infty$ due to the difference of the attractor in zeroth and first order. Nevertheless, this contribution also vanishes since $\partial L_0/\partial\dot q^{\nu}\equiv p_\nu$, which is zero on the attractor A_0. Finally, by taking into account (2.8b) and (2.10) we find

$$\begin{aligned}\Phi_1(q)&=\min\int_{t_0\to-\infty,\, q(t_0)\in A_0}^{t=0,q(0)=q} L_1(q(t),\dot q(t))dt\\ &=\min\int_{t_0\to-\infty}^{t=0}\left[-\frac{\partial\Phi_0(q)}{\partial q^{\nu}}\right]K_1^{\nu}(q)\Big|_{q^{\nu}=q^{\nu}(t)} dt\end{aligned} \tag{2.12}$$

where the integral is to be taken along the solution of (2.10). $\Phi_1(q)$ is thus nothing other than the action of L_1 evaluated along the unperturbed trajectory. Although the principle of perturbation theory has often been used in calculating the nonequilibrium potential,[9,10] the compact formula (2.12) has not yet appeared in the literature. It is worth stressing again that, in spite of the smoothness of Φ_0 and K_1^{ν}, the integral might be multivalued far away from the attractor.

For the sake of completeness and in order to make further comparison possible we mention that the nonequilibrium potential, if it is smoothly differentiable, can be evaluated from the Hamilton-Jacobi equation. Since $\Phi(q)$ is an action and by using (2.1) and (2.3) one obtains

$$\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi(q)}{\partial q^{\nu}}\frac{\partial\Phi(q)}{\partial q^{\mu}}+K^{\nu}(q)\frac{\partial\Phi(q)}{\partial q^{\nu}}=0 \ , \tag{2.13}$$

which is to be solved with the boundary condition that $\Phi(q)$ be minimal on the attractor A,

$$\Phi(q)\Big|_{q\in A}=\text{minimal} \ . \tag{2.14}$$

The perturbation scheme can again be easily worked out. In first order in ϵ we obtain from (2.13) and (2.7) for Φ_1 the linear equation

$$\frac{\partial\Phi_1(q)}{\partial q^{\nu}}\left[Q^{\nu\mu}\frac{\partial\Phi_0(q)}{\partial q^{\nu}}+K_0^{\nu}(q)\right]=-\frac{\partial\Phi_0(q)}{\partial q^{\nu}}K_1^{\nu}(q) \ . \tag{2.15}$$

By noticing that the large parentheses denote just $\dot q^{\nu}$ for the unperturbed motion [see (2.10)], the left-hand side appears as $d\Phi_1(q)/dt$ along (2.10). Consequently,

$$\Phi_1(q,t_0)=-\int_{t_0}^{t=0}\frac{\partial\Phi_0(q)}{\partial q^{\nu}}K_1^{\nu}(q)\Big|_{q^{\nu}=q^{\nu}(t)}dt \tag{2.16}$$

is a solution of (2.15) for *any* choice of t_0. This is , however, only a particular solution and, in general, does not fulfill the boundary condition (2.14). The general solution of Eq. (2.15) is obtained by adding the general solution $\Phi_{1h}(q)$ of the homogeneous equation, i.e.,

$$\Phi_1(q)=\Phi_1(q,t_0)+\Phi_{1h}(q) \ . \tag{2.17}$$

In order to single out the solution satisfying the boundary condition (2.14), Φ_{1h} has to be chosen in such a way that $\Phi_0+\epsilon\Phi_1$ is minimal on the attractor (specified up to first order in ϵ). In particular, a comparison (2.12) with (2.16) and (2.17) shows that, for $t_0\to-\infty$, $\Phi_{1h}(q)$=const. Another interesting special choice of t_0 corresponds to $t_0\to\infty$. By time reversal, one immediately convinces oneself that

$$\Phi_0(q)+\epsilon\Phi_1(q,t_0\to\infty)$$

is an action generated by trajectories starting with $t_0\to-\infty$ on the repeller R of the deterministic system. However, this function is not minimal on the attractor A and, moreover, need not even be an approximant to the nonequilibrium potential in the vicinity of the repeller. In this case the addition of a $\Phi_{1h}(q)\neq$const is unavoidable, if one is interested in the nonequilibrium potential, and not, in other particular solutions of the Hamilton-Jacobi equation (2.13).

III. CONSTRUCTION OF NONEQUILIBRIUM POTENTIALS AS A POWER SERIES

If the potential is analytic near the attractor it can be obtained in this region by directly expanding the left-hand side of the Hamilton-Jacobi equation (2.13) in a power series. For the sake of concreteness, let us consider the case of two degrees of the freedom. We put

$$q_1 \equiv x, \quad q_2 \equiv y \ . \tag{3.1}$$

For simplicity and clarity, we assume

$$Q^{\nu\mu} = \delta^{\nu\mu} \ . \tag{3.2}$$

The approach can easily be extended to more general cases.

Suppose that the origin is an attractor, namely,

$$K^1(0,0) = K^2(0,0) = 0 \ . \tag{3.3}$$

We may then expand the drift and the potential as

$$\begin{aligned} K^1(x,y) &= \sum_{n,m \geq 0} a_{nm} x^n y^m \ , \\ K^2(x,y) &= \sum_{n,m \geq 0} b_{nm} x^n y^m \ , \\ \Phi(x,y) &= \sum_{n,m \geq 0} \Phi_{nm} x^n y^m \ , \end{aligned} \tag{3.4}$$

with

$$\Phi_{01} = \Phi_{10} = 0 \ . \tag{3.5}$$

Inserting (3.4) into (2.13), each term in the power-series expansion of the potential can be specified order by order. First of all, the coefficients of the quadratic terms of the potential can be obtained independently from higher-order terms as

$$\begin{aligned} \Phi_{20} &= (a_{10} + b_{01})[b_{10}(a_{01} - b_{10}) - a_{10}(a_{10} + b_{01})] \\ &\quad \times [(a_{10} + b_{01})^2 + (a_{01} - b_{10})^2]^{-1} \ , \\ \Phi_{02} &= -(a_{10} + b_{01} + \Phi_{20}) \ , \\ \Phi_{11} &= -2b_{10} + 2\Phi_{20}(a_{01} - b_{10})(a_{10} + b_{10})^{-1} \ , \end{aligned} \tag{3.6}$$

under the assumption that $a_{10} + b_{01}$ is different from zero. Based on (3.6), the coefficients of the power-series expansion can be calculated systematically order by order. Generally, we have

$$T\Phi(n) = -f(n) \ , \tag{3.7}$$

with

$$\begin{aligned} T = {} & xG_{11}\left[\frac{\partial}{\partial x}\right] + yG_{12}\left[\frac{\partial}{\partial x}\right] \\ & + xG_{21}\left[\frac{\partial}{\partial y}\right] + yG_{22}\left[\frac{\partial}{\partial y}\right] \ , \end{aligned} \tag{3.8}$$

$$\begin{aligned} G_{11} &= a_{10} + 2\Phi_{20}, \quad G_{12} = a_{01} + \Phi_{11} \ , \\ G_{21} &= b_{10} + \Phi_{11}, \quad G_{22} = b_{01} + 2\Phi_{02} \ , \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} f(n) = {} & \sum_{m=2}^{n-2} [\mathbf{K}(m) + \tfrac{1}{2}\nabla\Phi(m+1)] \cdot \nabla\Phi(n-m+1) \\ & + \mathbf{K}(n-1) \cdot \nabla\Phi(2) \ , \end{aligned} \tag{3.10}$$

where ∇ denotes the gradient operator and

$$\begin{aligned} K^1(m) &= \sum_{\mu+\nu=m} a_{\mu\nu} x^\mu y^\nu \ , \\ K^2(m) &= \sum_{\mu+\nu=m} b_{\mu\nu} x^\mu y^\nu \ , \\ \Phi(m) &= \sum_{\mu+\nu=m} \Phi_{\mu\nu} x^\mu y^\nu \ . \end{aligned} \tag{3.11}$$

A detailed derivation of (3.6)–(3.11) can be found in Ref. 7.

A remarkable feature of the procedure is that the expansion starts with quadratic terms of the potenial since the drift has nonzero linear terms. When we deal with the potential around a singular point of order p with $p > 2$, the situation will be completely different. For instance, in case of second-order singular points (codimension-2 problems)

$$a_{ij} = 0, \quad b_{ij} = 0, \quad i + j \leq 1$$

the power series expansion of Φ must start with cubic terms. Equation (2.13) is then fulfilled if

$$\begin{aligned} & [(a_{20} + \tfrac{3}{2}\Phi_{30})x^2 + (a_{11} + \Phi_{11})xy + (a_{02} + \tfrac{1}{2}\Phi_{12})y^2](\tfrac{3}{2}\Phi_{30}x^2 + \Phi_{21}xy + \tfrac{1}{2}\Phi_{12}y^2) \\ & \quad + [(b_{20} + \tfrac{1}{2}\Phi_{21})x^2 + (b_{11} + \Phi_{12})xy + (b_{02} + \tfrac{3}{2}\Phi_{03})y^2](\tfrac{1}{2}\Phi_{21}x^2 + \Phi_{12}xy + \tfrac{3}{2}\Phi_{03}y^2) = 0 \end{aligned} \tag{3.12}$$

holds, which implies a set of five nonlinear algebraic equations for four variables. It is apparent that they possess no solution other than zero unless the coefficients $a_{nm}, b_{nm}, n + m = 2$ satisfy a special constraint. Constraints for the existence of polynomial expansions at bifurcation points of higher codimension have been derived in Ref. 7 (see also Refs. 6 and 8). In the general case when these constraints are not satisfied the potential at the bifurcation point cannot be of polynomial form. What happens to the potential as it loses its Taylor expandibility at singular points of order p ($p \geq 2$), i.e., at bifurcation points of codimension higher than 1? This question, obviously, cannot be answered by a direct application of the polynomial expansion. However, the question can be answered in cases where the perturbation theory of Sec. II B is applicable. In Sec. IV this will be done for a simple model chosen in such a manner that it still contains all relevant features.

IV. THE MODEL

A. Statement and perturbative analysis

In order to illustrate the connection between the nonequilibrium potential and its polynomial approximant we consider a simple two-variable problem. Let

$$x \equiv q_1, \quad y \equiv q_2 ,$$
$$K^1(x,y) = ax - x^3 , \tag{4.1}$$
$$K^2(x,y) = ay - y^3 - \epsilon x^2 y$$

be the drift and $Q^{\nu\mu} = \delta^{\nu\mu}$ the diffusion matrix. For $a < 0$ the deterministic system possesses a point attractor

$$A = (0,0) . \tag{4.2}$$

For $a > 0$ a pair of attracting points

$$A = (\pm a^{1/2}, \pm a^{1/2}(1-\epsilon)^{1/2}) \tag{4.3}$$

appears and the origin itself becomes a repeller

$$R = (0.0) . \tag{4.4}$$

As the deterministic system $\dot{q}^\nu = K^\nu(q)$ is invariant under the transformation $q \to -q$ and as the diffusion matrix is constant, the nonequilibrium potential will be a function of x^2 and y^2 only. Taking into account this symmetry property the two attractors appearing for $a > 0$ can be considered as being identical.

The unperturbed system ($\epsilon = 0$) is characterized by the nonequilibrium potential

$$\Phi_0(x,y) = -a(x^2+y^2) + \tfrac{1}{2}(x^4+y^4) . \tag{4.5}$$

The Hamiltonian trajectory (2.10) is then specified by the equations

$$\dot{x} = -ax + x^3, \quad \dot{y} = -ay + y^3 . \tag{4.6}$$

Note that these are just the time-reversed deterministic equations of the unperturbed problem. Their solution is easily found to be

$$x(t) = x \left[\frac{x^2}{a} + \left(1 - \frac{x^2}{a} \right) e^{2at} \right]^{-1/2} ,$$
$$y(t) = y \left[\frac{y^2}{a} + \left(1 - \frac{y^2}{a} \right) e^{2at} \right]^{-1/2} . \tag{4.7}$$

This trajectory ends at x,y at $t=0$ and starts on the attractor. Really, for $t = t_0 \to -\infty$ we have $x(t_0) = y(t_0) = 0$, as long as $a < 0$, and $x^2(t_0) = y^2(t_0) = a$ for $a > 0$, which is the equation for the attractor in zeroth order. In the latter case end points x,y with $x=0$ or $y=0$ cannot be reached from the attractor up to $t=0$, while points with x or y arbitrarily small but finite can be reached. Therefore, end points with $x=0$ or $y=0$ must be treated in this case by taking the limit $x \to 0$ or $y \to 0$ after the limit $t_0 \to -\infty$ has been taken.

By using the general result (2.12) and introducing a new integration variable, we find

$$\Phi_1(x,y) = \min \int_{t_0 \to -\infty}^{t=0} 2[-a + y^2(t)] x^2(t) y^2(t) dt$$
$$= \min \lim_{t_0 \to -\infty} a^2(-a+y^2) x^2 y^2 \tag{4.8}$$
$$\times \int_{y^2+(a-y^2)e^{2at_0}}^{a} \frac{du}{u^2[u(a-x^2)+a(x^2-y^2)]} . \tag{4.9}$$

Performing the integral explicitly, one obtains

$$\Phi_1(x,y) = \lim_{t_0 \to -\infty} x^2 y^2 \frac{a-y^2}{x^2-y^2} \left[\frac{a}{u} - \frac{a-x^2}{x^2-y^2} \ln \left(1 + \frac{(x^2-y^2)a}{(a-x^2)u} \right) \right] \Bigg|_{u=y^2+(a-y^2)e^{2at_0}}^{u=a} . \tag{4.10}$$

Here the min has been dropped since the integral is single valued for $x \neq 0$, $y \neq 0$. We have to consider, however, the cases $a < 0$ and $a > 0$ separately, since the lower limits are drastically different for these cases.

B. Nonequilibrium potential before bifurcation

For $a < 0$ the lower limit is $u \to -\infty$ and, consequently,

$$\Phi_1(x,y) = x^2 y^2 \frac{a-y^2}{x^2-y^2} \left[1 - \frac{a-x^2}{x^2-y^2} \ln \frac{a-y^2}{a-x^2} \right] . \tag{4.11}$$

Let us now discuss the result. First, we note that, although this form seems to be rather singular for $x = y$, it is not, as is best seen from the integral form (4.9), or by expanding (4.11) around $x = y$. In fact, for any $a < 0$,

$$\Phi_1(x=y) = \tfrac{1}{2} x^4 , \tag{4.12}$$

and $\Phi_1(x,y)$ is a smooth function of its variables. The correction $\Phi_1(x,y)$ can be expanded in a power series around the attractor $A = (0,0)$. As (4.11) suggests, the radius of convergence is given by the condition

$$x^2/|a|, \quad y^2/|a| < 1 . \tag{4.13}$$

By expanding the logarithm up to third order terms we find

$$\Phi_1(x,y) = \tfrac{1}{2} x^2 y^2 + \frac{1}{6a} x^2 y^2 (x^2 - y^2) \tag{4.14}$$

as a polynomial approximant.

Applying the method of Sec. III a direct calculation of a polynomial approximant at any value of ϵ yields up to sixth order:

$$\Phi(x,y) = -a(x^2+y^2) + \tfrac{1}{4}(x^4+y^4) + \frac{\epsilon}{2} x^2 y^2$$
$$+ \left[\frac{\epsilon}{6a} - \frac{\epsilon^2}{12a} \right] x^2 y^2 (x^2 - y^2) . \tag{4.15}$$

By keeping leading order terms in ϵ only we find again (4.14) showing that the polynomial approximation and the perturbation expansion are consistent for $a<0$.

C. Nonequilibrium potential at the bifurcation point

From (4.11) we obtain at $a=0$

$$\Phi_1(x,y)=-\frac{x^2y^4}{x^2-y^2}\left[1+\frac{x^2}{x^2-y^2}\ln\frac{y^2}{x^2}\right] . \tag{4.16}$$

This form is not analytic in the sense that it cannot be expanded in a Taylor series around the origin due to the presence of logarithmic terms, which is in harmony with the comment made after (3.12). The breakdown of a polynomial approximation at the bifurcation point $a=0$ is reflected also in the fact that certain coefficients of expression (4.15) diverge for $a\to 0$, and the radius of convergence of the power series expansion is vanishing at $a=0$. Nevertheless, as expected, the potential is smooth.

D. Nonequilibrium potential after bifurcation

For $a>0$ the lower limit of the integral (4.9) is $u=y^2$. Thus, we find

$$\Phi_1(x,y)=x^2y^2\frac{a-y^2}{x^2-y^2}\left[\frac{y^2-a}{y^2}+\frac{a-x^2}{x^2-y^2}\ln\frac{x^2}{y^2}\right] . \tag{4.17}$$

We recall that this form is obtained for all points $x\neq 0, y\neq 0$ and that Φ_1 in the origin is defined by taking the limit $x\to 0$ or $y\to 0$ in (4.17). In the vicinity of the origin, more precisely in the region $x^2,y^2<<a$, along any line $y=(\tan\alpha)x$ of the x,y plane one obtains

$$\Phi_1=-a^2\left[\frac{1}{1-\tan^2\alpha}+\frac{\tan^2\alpha\ln\tan^2\alpha}{(1-\tan^2\alpha)^2}\right] . \tag{4.18}$$

This means that the correction Φ_1 to the potential is constant along any line $y=(\tan\alpha)x$ around the origin but its value is direction dependent. It is worth emphasizing that (4.17) has, therefore, no power series expansion around the origin which is now a repeller. Consequently, no solution of the Hamilton-Jacobi equation in the form of a power series around the origin can be an approximant to the nonequilibrium potential for $a>0$, even if such a particular solution exists (cf. below).

We note that the singularity at $x=y\neq 0$ is apparent again since

$$\Phi_1(x=y)=\tfrac{1}{2}(x^4-a^2) \tag{4.19}$$

for any $a>0$.

Outside the origin $\Phi_1(x,y)$ possesses a power-series expansion. In particular, consider the points $x^2=y^2=a$ which corresponds to the attractor in zeroth order. Let now

$$\xi\equiv x^2-a, \quad \eta\equiv y^2-a , \tag{4.20}$$

and let us assume that

$$\xi/a, \quad \eta/a<1 . \tag{4.21}$$

Expanding Φ_1 in powers of ξ/a, η/a we find

$$\Phi_1(\xi,\eta)=a\left[\eta+\frac{\xi\eta}{2a}-\frac{1}{6a^2}\xi\eta(\xi-\eta)\right] . \tag{4.22}$$

Formally, this expression also diverges as $a\to 0$ but, of course, the radius of convergence then shrinks to zero. It is easy to check that $\Phi_0+\epsilon\Phi_1$ is really minimal at $x^2=a, y^2=a(1-\epsilon)$, i.e., at the attractor specified to first order in ϵ.

The form (4.22) can, of course also be obtained by the method of Sec. III requiring (2.14) and keeping leading order terms in ϵ only.

Finally, we note that the particular solution $\Phi_1(q,t_0)$ of Eq. (2.16) for $a>0$ and $t_0\to+\infty$ yields again the expression (4.11) but now for $a>0$. In fact, like (4.11), $\Phi_1(q,t_0\to\infty)$ can be expanded in a power series around the origin [cf. Eq. (4.14)]. However, as was noted after Eq. (2.17), $\Phi_1(q,t_0\to\infty)$ is *not* the nonequilibrium potential, because it does not satisfy the correct boundary conditions (2.14).

V. CONCLUSIONS

We have shown, by using general arguments and an illustrative example, that nonequilibirum potentials are smoothly differentiable around their minima even if they are not expandable in a power series in their state variables there.

In the following we list a few shortcomings of a direct search for the polynomial approximant to the nonequilibrium potential. They can all be observed in the example of Sec. IV.

(i) A low-order polynomial obtained in the region before or after the bifurcation need not reflect the breakdown of the power-series expansion at bifurcation points of codimension higher than one since the coefficients of such a polynomial might stay constant as the bifurcation point is approached. An example is the result (4.14) if only powers up to the fourth order are kept.

(ii) The polynomial approximant valid before the bifurcation can be formally continued through the bifurcation point and remains a solution of the Hamilton-Jacobi equation also beyond the bifurcation; however, it is not an approximant to the potential there. No warning of this danger is given, e.g. via the nonexistence of a power series solution.

(iii) Beyond the bifurcation a polynomial approximant of a completely different form exists which can be obtained by making an expansion around the new attractor(s). In other words, a polynomial approximation must always be applied *together* with the boundary condition that the potential is minimal in the attractor(s). The radius of convergence cannot be guessed.

Therefore, we conclude that a direct search for a polynomial approximant to the nonequilibrium potential should always be supplemented (at least in a restricted region of the parameter space) by other methods. One such method is the perturbation expansion in powers of a parameter, which is much less restrictive than the Taylor-series expansion, as the latter amounts to a simultaneous expansion in the n variables $q^1,\ldots,q^n$. The expansion in a single parameter is well suited also for getting insight into different kinds of nonanalytic properties of the nonequilibrium potential and estimating the convergence regions for power-series expansions, if they exist.

ACKNOWLEDGEMENTS

Support of this work by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 237 Unordnung und grosse Fluktuationen is gratefully acknowledged. One of us (G.H.) wishes to thank Professor H. Haken for his hospitality at the Institute fuer Theoretische Physik in Stuttgart.

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