Chaos in the relativistic generalization of the standard map

A. A. Chernikov
Space Research Institute, Profsoyuznaya 84/32, Moscow 117810, Union of Soviet Socialist Republics

T. Tél and G. Vattay
Institute for Theoretical Physics, Eötvös University, Puskin utca 5-7, H-1088 Budapest, Hungary

G. M. Zaslavsky
Space Research Institute, Profsoyuznaya 84/32, Moscow 117810, Union of Soviet Socialist Republics
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The relativistic generalization of the standard map is introduced, based on the problem of particle acceleration in the electric field of an electrostatic wave packet. It is shown that the diffusion is, in general, less effective than that for nonrelativistic particles. In the ultrarelativistic case chaotic motion appears in the form of intermittency.

I. INTRODUCTION

Recently the evolution of chaos in area-preserving maps modeling Hamiltonian systems has intensively been studied.[1,2] One of the most frequently examined two-dimensional maps is the standard map (SM) introduced by Chirikov for describing higher-dimensional Hamiltonian systems near resonances.[3] This map can also be interpreted in terms of a simple mechanical system. It connects the momentum and coordinate of a particle kicked by a periodical force, between two kicks, if the force depends on the coordinate in a sinusoidal fashion.[3]

In our paper we introduce the relativistic generalization of the SM. The introduction of the map will be based on a plasma physics problem (although a mechanical interpretation will also be given). We are interested in how relativistic effects modify the properties of particle acceleration and what the qualitative features of the deterministic diffusion are in the relativistic generalization.

We point out the appearance of intermittency in this map. Chaos is very weak in the ultrarelativistic case, in comparison with the regular component of the motion. The time evolution of the system can be represented as a sequence of randomly connected long parts of regular oscillations. The character of the phenomenon described above is general and can be found not only in the theory of dissipative[4] and Hamiltonian[5] systems, but also in many problems of plasma physics[6,7] and mechanics of continuous media. For instance, it is known in the theory of turbulence[8] where the term "intermittency" implies a kind of spatio-temporal chaotic dynamics.

The mechanism of stochastic particle acceleration in conditions of strong intermittency has little in common with the usual diffusion acceleration mechanism.

II. THE STANDARD MAP

In this section we collect some well-known qualitative properties of the SM in order to compare them with those of the relativistic standard map (RSM) in the next sections.

The SM is a map of plane (I,U) defined by the following equations:[1,2]

$$\begin{aligned} I' &= I - K\sin U\ , \\ U' &= U + I'\ (\mathrm{mod}\, 2\pi)\ , \end{aligned} \tag{1}$$

where K is proportional to the amplitude of the kicking force. mod 2π appears because of the periodicity in U.

There are four different types of trajectories in the phase space of the map: periodic orbits, quasiperiodic trajectories associated with invariant tori, stochastic or chaotic trajectories, and cantori. The motion along stochastic trajectories gives rise to the so-called deterministic diffusion,[1] which is restricted by the tori. A critical value of the parameter K can be associated with each torus. If K is greater than a critical value K_c, all tori break up and overall diffusion sets in. The most extended chaotic trajectories encircle the fixed points situated at $(U,I)=(0,2\pi m)$, where m is an integer. For K smaller than the critical value, these ringlike chaotic layers are separated by Kolmogorov-Arnold-Moser (KAM) tori dividing the whole phase space into isolated strips. Each KAM torus between these layers vanishes at the same value of K because the map is periodic in I. This happens at $K=K_c=0.97$.[1] Above K_c an infinitely long chain of chaotic layers is formed. The trajectories starting from this chaotic region can extend to arbitrary distances for sufficiently long time. This phenomenon is the so-called global stochasticity.[1]

III. DERIVATION OF THE RSM

A description of the dynamics of charged particles in the field of wave packets is one of the central problems in the theory of low-density plasma. Such wave packets may be excited in a plasma due to the intrinsic plasma instabilities or as a result of the external excitation of plasma by electromagnetic waves (Tajima and Dawson, Ref. 9). Let $E(x,t)$ denote the electric field of the electrostatic wave packet:

$$E(x,t)\sum_{n=-N}^{+N} E_n \sin(k_n x-\omega_n t) , \tag{2a}$$

where $2N+1$ is the number of harmonics in the wave packet. The basic properties of $E(x,t)$ depend on the structure of the wave packet, which is determined by parameters E_n, k_n, and ω_n. In its turn, this structure reflects the particular physical situation. We make the following simplifying assumptions regarding the structure of the packet:

$$k_n=k_0+n\Delta k,\quad \omega_n=\omega_0+n\Delta\omega,\quad E_n\approx E_0 , \tag{2b}$$

where n is an integer. Expressions (2b) mean that dispersive effects are weak and that the spectral amplitudes E_n are changing slowly with n. Let us neglect also the changing of k_n, i.e., consider $k_n \approx k_0$ and suppose that N is large enough. So such a wide packet approximately represents a periodic sequence of impulses with characteristic period $T=2\pi/\Delta\omega$. A particle moving in this field has the following equations of motion:

$$\dot{x}=c^2 p/\mathcal{E} ,$$
$$\dot{p}=-eE(x,t) , \tag{3}$$

where $\mathcal{E}$, p, and x are the energy, the momentum and the coordinate of the particle, respectively; c denotes the light velocity and e is the unit charge. Introducing the notation $u=k_0x-\omega_0 t$ we obtain

$$\dot{u}=k_0c^2p/\mathcal{E}-\omega_0 ,$$
$$\dot{p}=-eE_0\sin(u)\sum_{n=-\infty}^{+\infty}\cos(n\Delta\omega t) \tag{4}$$
$$=-eE_0T\sin(u)\sum_{n=\infty}^{+\infty}\delta(t-nT) ,$$

where $T=2\pi/\Delta\omega$.

By integrating between "kicks" these equations can be transformed into a map:

$$I_{n+1}=I_n-K\sin(U_n) ,$$
$$U_{n+1}=U_n+\frac{I_{n+1}}{[1+(I_{n+1}/L)^2]^{1/2}}-\omega_0T \pmod{2\pi} , \tag{5}$$

where

$$K=eE_0k_0T^2/m,\quad L=k_0cT,\quad I_n=Tk_0p_n/m .$$

The subscript n refers to values taken at time $n(T-0)$, i.e., just before the nth kick. For simplicity, we consider the case $T\omega_0=2\pi s$, where s is an integer. The map obtained in this way:

$$I_{n+1}=I_n-K\sin(U_n) ,$$
$$U_{n+1}=U_n+\frac{I_{n+1}}{[1+(I_{n+1}/L)^2]^{1/2}} , \tag{6}$$

is the RSM, the relativistic generalization of the SM.

The same recursion could have been derived if one had studied a relativistic particle of mass m, which was kicked by a force depending on the coordinate in a sinusoidal fashion. One can easily see that if I and U are chosen to be proportional to the momentum and coordinate of the particle just before the kicks, map (5) is recovered. In particular,

$$I=p2\pi T/ml,\quad U=2\pi x/l ,$$
$$K=C2\pi T/ml,\quad L=2\pi cT/l,$$

where p and x are the momentum and the coordinate of the particle. The quantities l and T are the periods of the kicking force in space and time, respectively, C is the amplitude of the force, m is the mass of the particle, and c is the light velocity.

It is sufficient to examine $U \bmod 2\pi$ because of the invariance of the map under the translation $U\to U+2\pi$. Note that the dimensionless parameter L is proportional to the light velocity c. If L goes to infinity the classical limit is recovered.

IV. GLOBAL STOCHASTICITY IN THE RSM

First, notice that the RSM is not periodic in I. Consequently, its fixed points are situated not equidistantly and their number is finite. The fixed points P_m which will be of interest for us are lying along the line $U=0$. Their coordinates are given as

$$(U_m,I_m)=\left[0,\frac{2\pi m}{[1-(2\pi m/L)^2]^{1/2}}\right] , \tag{7}$$

where m is an integer. Since the quantity under the root cannot be negative, we find the restriction

$$-L/2\pi<m<+L/2\pi . \tag{7a}$$

Thus the number of such fixed points is $2[L/2\pi]+1$ where [ ] stands for the integer part. The most extended chaotic layers encircle the fixed points similarly as in the SM, but their shape is deformed (Fig. 1).

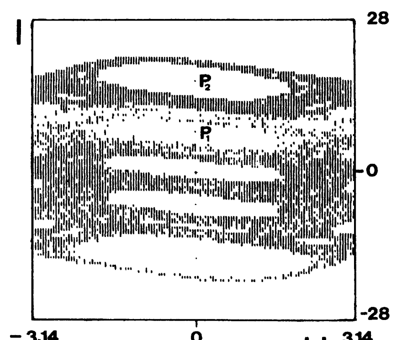

FIG. 1. The phase space of RSM at $K=2$, $L=20$. At this value of L there exist only four fixed points P_0, P_1, P_2, and P_3 for $I\geq 0$. A single trajectory of length 20 000 was used to plot the figure, which could not cross the KAM torus between P_2 and P_3 since $K_c^2=1.9<2<K_c^3=7.4$. We can observe also the deformation of the chaotic layer because of the relativistic effects. The trajectory was started from the point (2.5,0).

Another new feature is that in the present case the KAM tori do not break up at the same K value. Figure 2 shows the dependence of the critical K values of KAM tori on the parameter L as obtained in a numerical simulation of (6). K_c^i belongs to the KAM torus separating the chaotic layers around the fixed points P_{i-1} and P_i. The curves begin at the points $L=2\pi, 4\pi, 6\pi, \ldots$ because the fixed point P_i exists for $L>2\pi i$ only. The value of K_c^i does not depend on the sign of i due to the fact that the KAM tori separating the stochastic layers around the fixed points (P_i, P_{i+1}) and (P_{-i}, P_{-i-1}) break up simultaneously because of the inversion symmetry. The K_c^i-s tend to the classical value $K_c=0.97$ as L goes to infinity.

The overall picture of the phase space is quite different for $L\neq 2\pi m$ and $L=2\pi m$. In the first case the chaotic motion is restricted to the vicinity of the fixed points. The chaotic layers of the highest and lowest fixed points are bounded by KAM tori if K is arbitrary but finite. The deterministic diffusion can only occur between these tori. Above and below these limiting KAM tori the motion is regular. These areas are filled by KAM tori which are nearly parallel to the limiting tori. Thus, we find that, contrary to the SM, the phase space of the RSM is bounded by KAM tori at any value of K (Fig. 3).

We can determine the shape of the limiting KAM tori, for $|I|\gg L$, which corresponds to the ultrarelativistic case. In this limit the RSM can be approximated by the following map:

$$\begin{aligned} I_{n+1} &= I_n - K\sin(U_n)\ , \\ U_{n+1} &= U_n + L\,\mathrm{sgn}(I_{n+1})\ (\mathrm{mod}\ 2\pi)\ . \end{aligned} \tag{8}$$

It is easy to see that an invariant curve exists then given by the expression

$$I = A + \frac{K}{2\sin(L/2)}\cos(U-L/2)\ , \tag{9}$$

where A is arbitrary and the sign of I must be kept fixed. This proves the regularity of the motion for sufficiently great I. Figure 3 shows a chaotic layer and a KAM tori over it, determined by Eq. (9). For L values being close (but not infinitesimally) to an integer multiple of 2π, the I_m coordinate of the highest fixed point is much larger than L [see (7)]. Along the KAM torus is, therefore, $I\gg L$ and consequently, Eq. (9) provides a good approximation for the form of the outermost KAM torus.

Let us turn now to the resonant case $L=2\pi m$, for which a new mechanism of diffusion can be observed. The last KAM torus between infinity and the highest fixed point is then not bounded but approaches infinity at $U=\pm\pi$. Thus a chaotic channel is opened (Fig. 4). According to our numerical experience the width of the channel and the efficiency of diffusion to infinity increase with the increase of K.

We can also determine approximately the shape of the last KAM torus again. Expanding the RSM into series in $1/I$ we obtain in first order

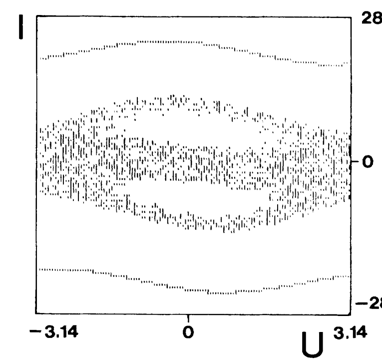

FIG. 3. The phase space of RSM at $K=20$, $L=20$. The KAM torus between infinity and the outmost fixed point P_3 never breaks up at fixed L. Beyond this KAM torus the motion proves to be regular. The continuous curve is an invariant torus in this regular region.

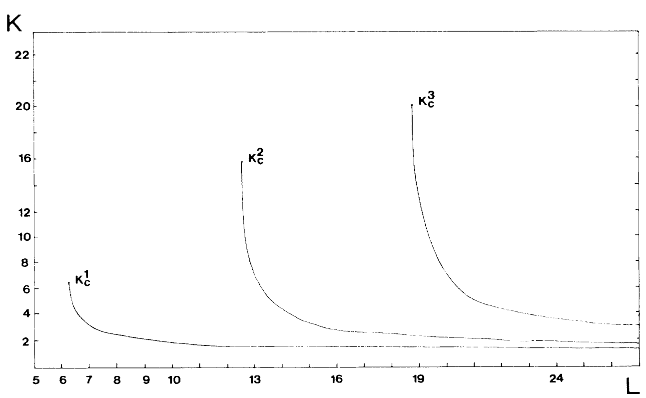

FIG. 2. The critical curves $K_c^i(L)$ characterizing the destruction of the last KAM torus between the fixed points (P_0, P_1) and (P_1, P_2) and (P_2, P_3). The curves stop at 2π, 4π, and 6π with limiting values 6.25, 15.9, and 20.1, respectively. If L tends to infinity the K_c^i approach the critical value $K_c=0.97$ of the SM.

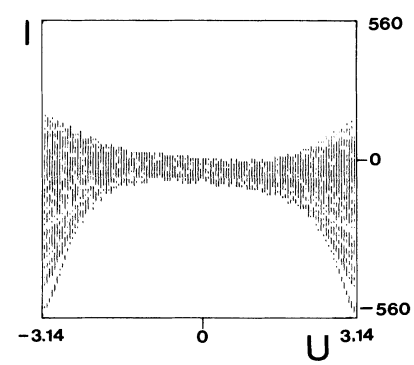

FIG. 4. One single trajectory of length 20 000 for $L=3\times 2\pi$, $K=31>K_c^3=20.1$. A chaotic channel (or web) is opened at $U=\pm\pi$. The diffusion can now spread arbitrarily far away from the origin along the channel. After 20 000 steps the trajectory has reached the distance $I=560$. The trajectory was started from the point (2.5,0).

$$I_{n+1}=I_n-K\sin(U_n)\ ,$$
$$U_{n+1}=U_{n+1}+L\ \mathrm{sgn}(I_{n+1})\left[1-\frac{L^2}{2I_{n+1}^2}\right]\ (\mathrm{mod}\ 2\pi)\ . \tag{10}$$

Because of the resonance condition

$$L\ \mathrm{sgn}(I_{n+1})\ (\mathrm{mod}\ 2\pi)=0\ . \tag{11}$$

So we obtain

$$I_{n+1}=I_n-K\sin(U_n)\ ,$$
$$U_{n+1}=U_n-\frac{(2\pi m)^2}{2I_{n+1}^2}\mathrm{sgn}(I_{n+1})\ (\mathrm{mod}\ 2\pi)\ . \tag{12}$$

If $|I_{n+1}-I_n|\ll I_n$, $|U_{n+1}-U_n|\ll U_n$ we may write approximately

$$I_{n+1}-I_n\approx dI,\quad U_{n+1}-U_n\approx dU\ , \tag{13}$$

and, consequently

$$\frac{dI}{dU}=\frac{2K\sin(U)I^2}{(2\pi m)^3}\ . \tag{14}$$

Looking for a solution of this differential equation with the condition of being singular at $U=\pm\pi$ one finds

$$|I|=\left[\frac{2K}{(2\pi m)^3}[\cos(U)+1]\right]^{-1}\ . \tag{15}$$

The resonance condition $L=2\pi m$ means that the light velocity divided by the quantity $1/T$ is an integer. Along the chaotic channel the particle can then accelerate to arbitrarily high energies with relatively weak kicking force.

Summarizing, we can say that the smaller the parameter L is chosen the less important the diffusion is in the RSM, except for the resonant case when trajectories within a restricted region of the phase space can go arbitrarily far from the starting point.

V. INTERMITTENCY IN THE RSM

If, in nonresonant cases, K is sufficiently large, simulation shows that as long as I is far from zero, the trajectories are regular and seem to lie on an arch of sinusoidal shape (Fig. 5). The reason for the regularity is the fact that the motion is strongly chaotic around the fixed points only. The range of the region where the fixed points are situated is approximately proportional to the I_m coordinate of the highest fixed point P_m and, in this way, proportional to L. A typical value of I for the deterministic part of the motion is determined by K. Therefore for $K\gg L$ the approximation $K\sim I\gg L$ is valid and one can use the approximation (8) except for the vicinity of $I=0$. The regular part of the trajectories can be determined from expressions (8) as

$$U_n=U_0+nL\ ,$$
$$I_n=I_0-K\frac{\sin[(n+1)L/2+U_0]\sin(nL/2)}{\sin(L/2)}\ . \tag{16}$$

The motion is described by these expressions as long as I_n is far from zero. Close to zero, approximation (8) loses its validity. Therefore in the vicinity of $I=0$ we must use the solution of the original equations which is chaotic so the motion, close to the line $I=0$ consists of unpredictable chaotic jumps.

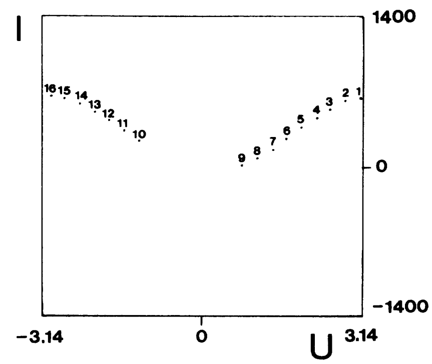

FIG. 5. The first 16 steps of a trajectory started at $U=3.14$, $I=600$ ($K=100$, $L=6$). The first nine points follow the invariant curve (9) with high accuracy. The motion is regular here. A jump occurs as the $I=0$ line is approached. After the jump the motion is regular again.

The phenomenon when regular motion is interrupted by chaotic jumps is called intermittency.[4] Figures 6 and 7 show that the sinusoidal, regular parts of the trajectories are mixed by chaotic jumps. In this way, the intermittent trajectories fill the chaotic layer. The latter is bounded by a sinusoidal KAM torus, which is described by expression (9).

Intermittency is a general phenomenon in area-preserving maps.[5] In our map the reason for intermittency is the relativistic feature of the motion. In the ultrarelativistic case the particle moves uniformly at speed close to c so one can determine the momentum of the particle. The quantity

$$I-\frac{K}{2\sin(L/2)}\cos(U-L/2) \tag{17}$$

is an adiabatic invariant for the motion. The condition for this limit holds in a broad region of the phase space

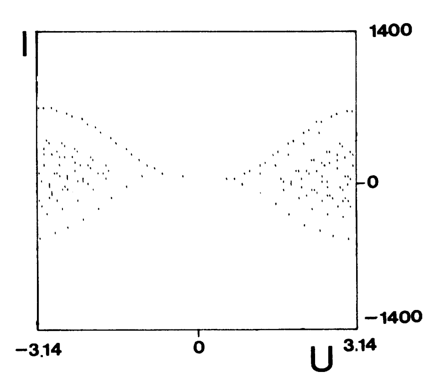

FIG. 6. The first 170 steps of the trajectory described in Fig. 5. This chaotic trajectory contains long regular pieces.

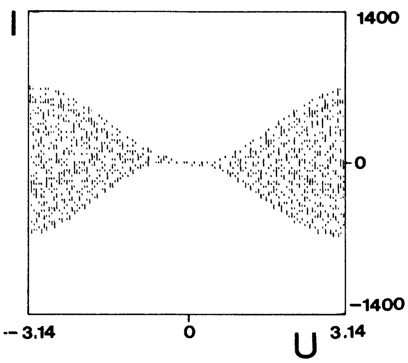

FIG. 7. 1465 steps of the intermittent trajectory described in Fig. 5. The boundary of the chaotic region itself is well approximated by a curve of type (9).

but not everywhere. In regions where the motion is nonultrarelativistic, chaos shows up. The wandering between chaotic and regular regions appears as intermittency.

VI. SUMMARY

The investigation of the relativistic generalization of the standard map presented in this paper clarified that the character of the interaction between relativistic particle and an electrostatic wave packet differs qualitatively from the dynamics of nonrelativistic particles. In the nonresonant case, i.e., when the phase velocities of the waves in the packet differ significantly from the speed of light, energy of the accelerated particle is bounded from above. The situation is quite different if the phase velocity of one of the waves is close to the speed of light. In this case the way for an unlimited stochastic particle acceleration is open and there is an analogy with the case of interaction of relativistic particles with electromagnetic waves propagating across a given magnetic field.[10] The dynamics of charged particles undergoing stochastic acceleration is very interesting because intermittency takes place. In other words, long periods of the quasiregular motion are intermitted with short periods of chaotic motion. The phenomenon of intermittency can be found in almost any nonintegrable system so its investigation provides us also with a more profound knowledge of many general properties of chaotic systems.

ACKNOWLEDGMENTS

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