

## Poincaré recurrences and transient chaos in systems with leaks

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In order to simulate observational and experimental situations, we consider a leak in the phase space of a chaotic dynamical system. We obtain an expression for the escape rate of the survival probability by applying the theory of transient chaos. This expression improves previous estimates based on the properties of the closed system and explains dependencies on the position and size of the leak and on the initial ensemble. With a subtle choice of the initial ensemble, we obtain an equivalence to the classical problem of Poincaré recurrences in closed systems, which is treated in the same framework. Finally, we show how our results apply to weakly chaotic systems and justify a split of the invariant saddle into hyperbolic and nonhyperbolic components, related, respectively, to the intermediate exponential and asymptotic power-law decays of the survival probability.

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### I. INTRODUCTION

The numerical and theoretical study of dynamical systems can resemble observational and experimental situations if the escape of an initial ensemble of trajectories through a leak placed in the otherwise closed phase space is considered. The idea of introducing a leak in a closed chaotic dynamical system to generate transient chaos was first suggested by Pianigiani and Yorke [1]. Later this problem was discussed in detail in the context of fractal exit boundaries [2], of geometrical acoustics [3], of quantum chaos [4], of controlling chaos [5], of resetting in hydrodynamical flows [6,7], of leaked Hamiltonian systems [8], of astronomy [9] and of cosmology [10]. The subject has recently received renewed attention [11–17]. Part of this renewed interest comes from recent developments of quantum chaos, which started to consider carefully the effect of measurement devices, absorption, and other forms of leaking both theoretically and experimentally [11–14]. It is now clear that leaking dynamical systems provides a way to “peep at chaos” [16], and can be used as a kind of chaotic spectroscopy [4].

From a theoretical point of view, systems with leaks provide an interesting bridge between open and closed dynamical systems. In closed systems the phase space coordinates remain confined, and a rigorous mathematical approach based on attractors and ergodic components can be employed [18]. An important quantity is the distribution  $p_r(T)$  of the first Poincaré recurrence time  $T$  [19–25] to a preselected region of the phase space. As originally proposed by Chirikov and Shepelyansky [24] (see also [25]),  $p_r(T)$  is a useful analyzer of the entire dynamics. In contrast, in open systems trajectories may leave the region of interest (e.g., tend to  $\pm\infty$ ) and the method of transient chaos is usually employed [18,26]. In this case, the dynamics is characterized by the escape time distribution  $p_e(n)$  (the derivative of the survival probability).

Chaotic dynamics usually leads to an asymptotic exponential decay of both  $p_r$  and  $p_e$ ,

$$p_{r,e}(t) \sim e^{-\gamma_{r,e}t}, \quad (1)$$

for large enough  $t=\{T,n\}$ . Exponentials are the signature of *strong* chaotic properties as seen, e.g., in the divergence of nearby initial conditions, in the decay of correlations, and in the convergence to equilibrium distributions. The exponent  $\gamma_{r,e}$  is the key quantifier of system-specific characteristics and will be considered in detail in this paper.

The possibility of comparing a system with leaks to its corresponding closed system shows numerous advantages in comparison to naturally opened systems. For instance, the rate  $\gamma_{r,e}$  in Eq. (1) can be estimated from the probability of escaping or returning. For small leaks, this probability is obtained by computing the closed-system natural measure  $\mu$  of a typical leak region  $I$ , and it follows that [5,23,27]

$$\gamma_r = \gamma_e = \mu(I) = \frac{1}{\langle t \rangle} \quad \text{for } 1 \gg \mu(I) > 0, \quad (2)$$

where  $\langle t \rangle$  denotes the mean recurrence or escape time,  $\langle T \rangle$  or  $\langle n \rangle$ .

Despite its simplicity, relation (2) is of little practical relevance since the limit  $\mu(I) \rightarrow 0$  is never achieved in either numerical or experimental applications [27–29]. Here we do not restrict ourselves to this limit and find that, apart from being unrealistic, it masks different interesting relations. We perform a formal description through the theory of transient chaos, i.e., in terms of a nonattracting chaotic set [18] and the conditionally invariant measure [1,30], and obtain distinct relations for  $\gamma$  and  $1/\langle t \rangle$  as a function of the position and size of the region  $I$  and of the initial ensemble. Relation (2) does not hold, but we show that  $\gamma_r = \gamma_e \equiv \gamma$  remains valid. This is based on a full correspondence with the recurrence problem obtained for a special initial ensemble and setting the *leak to be equal to the recurrence region of the closed system*. We have explored the most striking consequences of this unified treatment from the point of view of Poincaré recurrences in a previous short paper [31]. Here, after describing the relationship between the two problems with additional details, we focus on the escape problem and consider the recurrence problem as a particular, yet important, case.

We also use this connection to apply and adapt a method to determine  $\gamma$  [27] based only on the escape probabilities for *short* times. This is a main advantage over genuine open systems where long-term periodic orbits are needed to determine  $\gamma_e$ .

In systems showing *weak* chaos, such as, e.g., Hamiltonian systems with mixed phase space, the exponential decay law (1) experiences a crossover for longer times toward an asymptotic power-law behavior [25,23,32]. We argue, however, that on intermediate times a decay rate  $\gamma$  can be well defined:

$$p_{r,e}(t) \propto \begin{cases} e^{-\gamma t} & \text{for intermediate } t < t_c, \\ t^{-\alpha} & \text{for } t > t_c. \end{cases} \quad (3)$$

We interpret our results based on an effective splitting of the chaotic saddle into hyperbolic and nonhyperbolic components.

The paper is organized as follows. In Sec. II we review the main properties of Poincaré recurrence times. The corresponding results for escape times appear in Sec. III, together with the dependence of  $\gamma$  and  $\langle t \rangle$  on  $I$ . The equivalence between recurrence and escape problems is carefully discussed in Sec. IV. In Sec. V numerical illustrations of the results are provided for paradigmatic dissipative models. In Sec. VI, the nonhyperbolic case is investigated using an area-preserving map. Relations to periodic orbits and an efficient method to determine  $\gamma$  are described in Sec. VII. Finally, conclusions and discussions appear in Sec. VIII, where the case of higher-order recurrences is also investigated.

## II. POINCARÉ RECURRENCES IN CLOSED SYSTEMS

Poincaré recurrences to a certain region of the phase space have played an important role since the foundation of the kinetic description of nonequilibrium processes [21,23]. We define the problem in the context of low-dimensional chaotic maps that can be dissipative [33] or Hamiltonian [23,24].

Consider a discrete-time chaotic system  $\vec{x}_{n+1} = M(\vec{x}_n)$  defined on a bounded phase space  $\vec{x} \in \Gamma$ . Let the natural ergodic measure on the invariant chaotic set (e.g., strange attractor or chaotic sea) be denoted by  $\mu$  and its density by  $\rho_\mu$ . We define the *recurrence region* as a subset  $I \subset \Gamma$  with  $\mu(I) > 0$ . As stated above, we do *not* restrict ourselves to the unrealistic limit  $\mu(I) \rightarrow 0$  [27–29]. The Poincaré recurrence theorem ensures that, for almost all initial conditions  $\vec{x}_0 \in I$ , there are infinitely many time instants  $n = n_1, n_2, \dots$  such that  $M^n(\vec{x}_0) \in I$ . The first recurrence times are defined as  $T_i = n_i - n_{i-1}$ , with  $i \geq 1$  and  $n_0 \equiv 0$  (if the points remains in  $I$  at the  $i$ th iterate  $T_i = 1$ ). The recurrence time distribution  $p_r(T)$ ,  $T \geq 1$ , is the probability of finding  $T_i = T$  in an infinitely long trajectory. The cumulative version can be written as  $P_r(\tau) = \sum_{T=\tau}^{\infty} p_r(T)$ . The mean recurrence time is defined as

$$\langle T \rangle \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N T_i \equiv \sum_{T=1}^{\infty} T p_r(T).$$

Due to the ergodicity of the measure  $\mu$ , the specific choice of the initial point  $\vec{x}_0 \in I$  is irrelevant for  $p_r(T)$ . Instead of fol-

lowing a single trajectory,  $p_r(T)$  can also be obtained by using an ensemble of trajectories chosen inside  $I$  according to the invariant density  $\rho_\mu$ . A consequence of ergodicity is Kac's lemma [19]:

$$\langle T \rangle = \frac{1}{\mu(I)}, \quad (4)$$

which is valid for any recurrence region  $I$ .

The recurrence time distribution of an uncorrelated random process with fixed return probability  $\mu$  follows a binomial distribution [27]

$$p_r(T) = \mu(1 - \mu)^{T-1} = \frac{\mu}{1 - \mu} e^{\ln(1 - \mu)T}. \quad (5)$$

The relaxation rate  $\gamma_r$  defined in Eq. (1) is thus given for such a random process by

$$\gamma_r^* = -\ln(1 - \mu). \quad (6)$$

For small recurrence probability  $\mu \rightarrow 0$ ,  $\gamma_r^* \rightarrow \mu$ , and the Poisson distribution  $p_r(T) = \mu \exp(-\mu T)$  is obtained. Identifying  $\mu$  with  $\mu(I)$  (the recurrence probability equals the natural measure of the recurrence region), we see that a Poisson process is the simplest process leading to Eq. (2). Indeed, the Poisson distribution is proved to describe deterministic hyperbolic systems (see references in [34]).

We study next the distribution of recurrence times  $p_r(T)$  for generic chaotic systems. In agreement with Eq. (1), we write

$$p_r(T) \approx \begin{cases} \text{irregular} & \text{for } 1 < T < T^*, \\ g_r e^{-\gamma_r T} & \text{for } T \geq T^*. \end{cases} \quad (7)$$

Implicit is the view that the approach toward the exponential distribution for  $T \rightarrow \infty$  does not occur uniformly, but that after some short time  $T^*$  the distribution is practically very close to an exponential [27]. This will be theoretically justified in the next section and numerically illustrated in Sec. V. We are interested in the decay rate  $\gamma_r$ , which typically deviates from the random estimate (6), and also in the short-time irregular fluctuations for  $T < T^*$ , where  $T^*$  is of the order of  $\langle T \rangle$  and will be better interpreted in Sec. IV. These short-time irregular oscillations [denoted as *irregular* in Eq. (7) and equations hereafter] depend strongly on the choice of the recurrence region  $I$ . For instance, if a periodic orbit of (short) period  $p$  is present inside  $I$ , there is a higher probability of recurrence at  $T = p$ . As shown in Ref. [27] and as will be discussed in Sec. VII, normalization and Eq. (4) imply that the short-time fluctuations uniquely determine the deviation of  $\gamma_r$  from  $\gamma_r^*$ . Therefore, for different recurrence regions  $I$  of the same natural measure  $\mu(I)$  [while  $\langle T \rangle$  is fixed and given by (4)], the value of  $\gamma_r$  depends strongly on the position of  $I$  [27].

## III. ESCAPE IN OPEN SYSTEMS

### A. General relations for transiently chaotic systems

By now, transient chaos has a well established theory [18,26]. In this case, the discrete system  $\vec{x}_{n+1} = \bar{M}(x_n)$  is de-

defined in an unbounded phase space  $\tilde{\Gamma}$ . A region of interest  $A \subset \tilde{\Gamma}$  (nontrivial dynamics) is defined in such a way that trajectories may leave  $A$  [ $\tilde{M}(A) \not\subseteq A$ ] but do not return to it [ $\tilde{M}(\tilde{\Gamma} \setminus A) \cap A \equiv \emptyset$ ]. An initial ensemble is started according to a smooth density  $\rho_0(\vec{x})$ , with  $\vec{x} \in A$ . The escape time distribution  $p_e(n)$  is given by the fraction of trajectories that leave  $A$  at time  $n \geq 1$ . For chaotic systems it can be written as

$$p_e(n) = \begin{cases} \text{irregular} & \text{for } 1 < n < n_e^*, \\ g_e e^{-\gamma_e n} & \text{for } n \geq n_e^*, \end{cases} \quad (8)$$

where  $n_e^*$  corresponds to a short convergence time and  $\gamma_e$  is the escape rate. The survival probability inside  $A$  up to time  $n$  is given by  $P_e(n) = \sum_{n'+1}^{\infty} p_e(n')$ , and is also an exponential distribution with the same  $\gamma_e$  as in Eq. (8). The average escape time  $\langle n \rangle_e \equiv \sum_1^{\infty} n p_e(n)$ , also called the lifetime of chaos, is usually estimated by the reciprocal of  $\gamma_e$  [18,26]. Note, however, that the exact average is, in general, different from  $1/\gamma_e$  and depends (just like  $n_e^*$ ) on  $\rho_0$ .

The ergodic theory of transient chaos explains the above properties by the existence of a nonattracting chaotic set. In invertible systems this set is, e.g., a chaotic saddle:<sup>1</sup> the set of orbits that never escape either forward or backward in time [18,20,26]. For systems with strong chaos, the saddle is an invariant fractal set of zero measure. Long-time transients correspond to trajectories that approach it closely, along its stable manifold, and leave it along its unstable manifold. Being an invariant object, the chaotic saddle is, of course, independent of the initial density  $\rho_0$ . Therefore, assuming that the distribution  $\rho_0$  is nonvanishing in at least some region around the stable manifold of the saddle (and that there are no disjoint saddles), the escape rate  $\gamma_e$  is *independent* of the initial density. It can be related to the geometrical and dynamical properties of the saddle by the Kantz-Grassberger formulas [35], which for two-dimensional invertible maps are

$$\gamma_e = \lambda(1 - d_u), \quad \lambda d_u = -\lambda' d_s, \quad (9)$$

where  $\lambda$  ( $\lambda'$ ) is the positive (negative) Lyapunov exponent on the saddle and  $d_u$  ( $d_s$ ) is the information dimension of the saddle along its unstable (stable) manifold [18,26]. The time  $n_e^*$ , below which orbits escape without approaching the saddle, can be estimated by the inverse of the negative Lyapunov exponent  $\lambda'$ .

A rigorous description of open systems is possible in terms of the *conditionally invariant measure* [1,30,36,37], hereafter called the *c measure*. A measure  $\mu_c$  is said to be conditionally invariant if

$$\frac{\mu_c(\tilde{M}^{-1}(E))}{\mu_c(\tilde{M}^{-1}(A))} = \mu_c(E), \quad (10)$$

for any  $E \subset A$ . In words, the *c measure* is not invariant under the map,  $\mu_c(\tilde{M}^{-1}(E)) \neq \mu_c(E)$ , but is preserved under the incorporation of the compensation factor in the denominator of

(10). The compensation factor  $\mu_c(\tilde{M}^{-1}(A)) < 1$  corresponds to the *c measure* of the set remaining inside  $A$ , i.e., the trajectories that do not escape in one iteration. The *c measure* of the region of interest is  $\mu_c(A) = 1$ . The *c measure* has a density  $\rho_c$ , which is unique and the only attractor for smooth initial densities  $\rho_0$ . Taking into account the exponential escape expressed in (8), one sees that convergence is possible only if [1]

$$\gamma_e = -\ln[\mu_c(\tilde{M}^{-1}(A))]. \quad (11)$$

Considering the evolution of  $\rho_0$  through an evolution operator, the Perron-Frobenius operator [20,30],  $\exp(\gamma_e)$  is the largest eigenvalue and  $\rho_c$  is the corresponding (right) eigenvector. The *c measure* describes the escape process of the system and therefore  $\rho_c$  is concentrated on the saddle and along its unstable manifold [30].

Numerically,  $\rho_c$  is obtained by multiplying  $\rho_0$  at each iteration by a constant factor which, after many iterations, is necessarily equal to  $\exp(\gamma_e)$ . In practice, we iterate a large number of trajectories and we obtain  $\rho_c$  by renormalizing the surviving trajectories at a long time.  $\mu_c(\tilde{M}^{-1}(A))$  in (11) [or  $1 - \mu_c(I)$  in (14) below] is obtained as the fraction of all surviving trajectories that escape in the next time step or, equivalently, by fitting  $\gamma_e$  to  $p_e(n)$  and inverting (11).

## B. Systems with leaks

Open systems  $\tilde{M}$  can be obtained from the closed system  $M$  defined on  $\Gamma$ , described in Sec. II, by introducing a leak in a region  $I \subset \Gamma$

$$\vec{x}_{n+1} = \tilde{M}(\vec{x}_n) = \begin{cases} M(\vec{x}_n) & \text{if } \vec{x}_n \notin I, \\ \text{escape} & \text{if } \vec{x}_n \in I. \end{cases} \quad (12)$$

Notice that, since the escape happens one step *after* entering  $I$ , the map  $\tilde{M}$  is defined in  $I$  and initial conditions can be in  $I$ . This procedure defines an open system as that of Sec. III A, with  $A \equiv \Gamma$ .

An estimate  $\gamma_e^*$  of the escape rate  $\gamma_e$  in systems with leaks is usually given in terms of the Frobenius-Perron relation [5,20], which expresses that the number of particles not escaping within one time step is proportional to the natural measure outside of the leak,  $\exp(-\gamma_e^*) = 1 - \mu(I)$ , i.e.,

$$\gamma_e^* = -\ln[1 - \mu(I)] \quad [\approx \mu(I) \quad \text{for } \mu(I) \rightarrow 0], \quad (13)$$

where  $\mu(I)$  is the natural measure of leak  $I$ . This relation is equivalent to the binomial estimate for recurrence times (6) and we call it hereafter the naive estimate of  $\gamma_e$ . Deviations of  $\gamma_e$  from  $\gamma_e^*$  were reported in Refs. [5,7] and associated with the existence of short-time periodic orbits of the closed system inside  $I$ .

The results of Sec. III A on open systems can be applied to systems with leaks. Since  $\tilde{M}^{-1}(A) \equiv A \setminus I$ , it follows that  $\mu_c(\tilde{M}^{-1}(A)) = \mu_c(A) - \mu_c(I) = 1 - \mu_c(I)$ , and the following simple relation is obtained from (11):

<sup>1</sup>In noninvertible systems the set is a chaotic repeller.

$$\gamma_e = -\ln[1 - \mu_c(I)]. \quad (14)$$

This exact expression, which was previously obtained in Ref. [5], corresponds to the naive estimate (13) replacing  $\mu(I)$  by  $\mu_c(I)$ . They become equivalent in the limit of the small-leak region  $\mu(I) \rightarrow 0$  because  $\mu_c$  tends to  $\mu$  [38,39]. It is the  $c$  measure of the *leak* which determines the escape rate. Even though numerically such a measure is easily calculated. Proving the existence of such conditionally invariant measure may be a very involving mathematical task [1,36] (see, e.g., Ref. [37] for a class of maps with leaks). Here we employ a pragmatic approach and implicitly assume that such a measure exists in the large class of systems with leaks where numerical simulations indicate exponential decay of  $p_e(n)$ .

### C. Dependence of the mean escape time on the initial density

While the escape rate  $\gamma_e$  is independent of the initial density  $\rho_0$ , as expressed by Eq. (14), the mean escape time  $\langle n \rangle_e$  and the full distribution  $p_e(n)$  change considerably with  $\rho_0$ . It is thus instructive to discuss the following special initial conditions.

*Conditionally invariant density  $\rho_0 = \rho_c$ :* From a formal mathematical perspective the natural choice of initial conditions is according to the density of the  $c$  measure  $\rho_c(\vec{x})$ , which can be considered as the natural measure of the open system. This corresponds to setting the initial density in agreement with the escape process. Therefore, no oscillations are observed in Eq. (8), i.e.,  $n^* = 1$ . Since  $\gamma_e$  is given by (14) and the distribution is normalizable,  $p_e(n)$  follows a binomial distribution

$$p_e(n) = \mu_c(1 - \mu_c)^{n-1} = \frac{\mu_c}{1 - \mu_c} e^{\ln(1 - \mu_c)n},$$

with  $\mu_c = \mu_c(I)$ , in analogy with Eq. (5). The mean escape time is in this case

$$\langle n \rangle_c = \frac{1}{\mu_c(I)} = \frac{1}{1 - e^{-\gamma_e}} \quad \text{with } \rho_0 = \rho_c. \quad (15)$$

*Natural density  $\rho_0 = \rho_\mu$ :* Coherent with the idea of systems with leaks, the initial density is chosen according to the natural measure of the closed system over the full  $\Gamma$ . It corresponds to leaking the system after equilibrium has been achieved in the closed system. We show in Sec. V that

$$\langle n \rangle_\mu \approx \frac{1}{\mu_c(I)} = \frac{1}{1 - e^{-\gamma_e}} \quad \text{with } \rho_0 = \rho_\mu. \quad (16)$$

Similar results are obtained for densities proportional to  $\rho_\mu$  (equal to  $\rho_\mu$  apart from normalization in a subspace of  $\Gamma$ ).

*Smooth density  $\rho_0 = \rho_s$ :* In some systems, and for numerical simulations, it is natural to define the initial density to be a smooth function over  $\Gamma$  or a part of  $\Gamma$ . A particular case is that of a uniform density. Estimate (16) remains valid for such initial densities as well, as shown in Sec. V.

## IV. RELATION BETWEEN RECURRENCE AND ESCAPE

In this section we show that there is an initial density  $\rho_0 = \rho_r$  for which  $p_r(T) = p_e(n)$  for  $T = n$ . We use this special

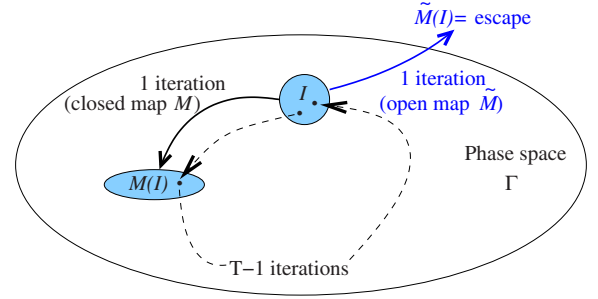


FIG. 1. (Color online) Illustration of the equivalence between recurrence and escape times.

initial condition to establish a relation between recurrence and escape.

*Density equivalent to recurrence  $\rho_0 = \rho_r$ :* Consider that trajectories are injected from outside through the leak. More precisely, consider the distribution  $\rho_r$  obtained as the first iterate through map  $M(\vec{x})$  of the points  $\vec{x} \in I$  distributed according to the natural density  $\rho_\mu$ . This can be obtained applying the Perron-Frobenius operator [20] as

$$\rho_r(\vec{x}) = \frac{\rho_\mu(M^{-1}(\vec{x}) \cap I)}{J(M^{-1}(\vec{x}) \cap I)\mu(I)} \quad \text{for } \vec{x} \in M(I), \quad (17)$$

where  $M^{-1}(x) \cap I$  denotes the points that come from  $I$ ,  $J$  is the Jacobian of the map, and  $\mu(I)$  ensures normalization.<sup>2</sup> In the case of invertible maps,  $M^{-1}$  is unique and the density (17) is equivalent to

$$\rho_r(\vec{x}) = \begin{cases} \frac{\rho_\mu(M^{-1}(\vec{x}))}{J(M^{-1}(\vec{x}))\mu(I)} & \text{if } \vec{x} \in M(I), \\ 0 & \text{else.} \end{cases} \quad (18)$$

Consider now an infinitely long trajectory used to calculate the recurrence times  $T$  described in Sec. II. In view of the ergodic theorem (time average equals ensemble average), we see that *one iteration after* returning to  $I$  the points of this trajectory are distributed precisely as  $\rho_r$ . As illustrated in Fig. 1, due to the one-step time shift inserted in definition (12), all escape times  $n$  of  $\rho_r$  correspond to a recurrence time  $T$ .<sup>3</sup> Since this is valid for all  $n$ ,  $p_e(n) \equiv p_r(T)$  for  $\rho_0 = \rho_r$ . In particular,  $\gamma_r = \gamma_e$ , and since  $\gamma_e$  is independent of the initial density, this implies that for a fixed recurrence and escape region  $I$  the decay rate of the Poincaré recurrences and the escape rate of the corresponding leaked system coincide:

$$\gamma_r = \gamma_e \equiv \gamma. \quad (19)$$

The mean recurrence time is given, however, according to Kac's lemma (4) as

<sup>2</sup>This extends the validity of Eq. (5) of Ref. [31] to maps with nonconstant  $J$ .

<sup>3</sup>Escapes at  $n=1$  occur when  $I \cap M(I) \neq \emptyset$  and  $p_e(1) = \mu(I \cap M(I))$ .

TABLE I. The escape and recurrence rate  $\gamma = \gamma_r = \gamma_e$  and the mean escape time (recurrence time for initial density  $\rho_r$ )  $\langle n \rangle$  expressed in terms of the natural invariant measure  $\mu(I)$  and the conditionally invariant measure  $\mu_c(I)$  of the leak and recurrence region  $I$ , for large and small leaks.

	Large leaks			Small leaks
Measures	Finite $\mu(I) \neq \mu_c(I)$			$\mu(I) = \mu_c(I) \rightarrow 0$
Escape rate	$\gamma = -\ln[1 - \mu_c(I)] \neq -\ln[1 - \mu(I)] = \gamma^*$			$\gamma = \mu_c(I) = \mu(I)$
Mean time	$\langle n \rangle_r = \frac{1}{\mu(I)} \neq \frac{1}{1 - e^{-\gamma}}$	$\langle n \rangle_c = \frac{1}{\mu_c(I)} = \frac{1}{1 - e^{-\gamma}}$	$\langle n \rangle_{\mu,s} \approx \frac{1}{\mu_c(I)} = \frac{1}{1 - e^{-\gamma}}$	$\langle n \rangle = \frac{1}{\mu(I)} = \frac{1}{\gamma}$
$\rho_0$	recurrence $\rho_r$	$c$ measure $\rho_c$	natural, smooth $\rho_{\mu,s}$	$\rho_{r,c,\mu,s}$

$$\langle n \rangle_r = \langle T \rangle = \frac{1}{\mu(I)} \quad \text{with } \rho_0 = \rho_r. \quad (20)$$

The mean recurrence time is thus determined by the natural measure  $\mu(I)$  and deviates considerably from the typical mean escape time given by (16). The reason is that  $\rho_r$  in (17) is very atypical from the point of view of the  $c$  measure. In fact,  $\rho_c$  is concentrated along the saddle's unstable manifold, i.e., points  $\vec{x}$  such that  $\tilde{M}^{-i}(\vec{x})$  never escape. On the other hand, all points on  $\rho_r$  came from the leak and therefore the support of  $\rho_r$  does not overlap with the unstable manifold of the saddle (but it does with the stable manifold).

In view of the results above, we consider hereafter the recurrence problem as an escape problem with  $\rho_0 = \rho_r$ .<sup>4</sup> Coherently, we omit the indices  $r$  and  $e$ , and  $T$  will also be denoted by  $n$ . The values of the escape rate  $\gamma$  and of the mean escape time  $\langle n \rangle$  for the different initial densities  $\rho$  are summarized in Table I.

It is now possible to compare the results of Ref. [5] for escapes with those of Ref. [27] for recurrence. In both cases the fully chaotic logistic map was carefully investigated and deviations of  $\gamma$  from the naive estimate  $\gamma^*$  were reported. The deviations were associated with the presence of periodic orbits inside  $I$  and interpreted in terms of the overlap of the preimages of  $I$  (in Ref. [5]) and of the short time oscillations (in Ref. [27]). In our unified perspective, the results of Refs. [27,5] correspond to choosing initial densities  $\rho_r$  and  $\rho_\mu$  or  $\rho_s$ , respectively (see Table I). Since the escape rate is independent of this choice, we see that both references are giving intuitive interpretations for the deviation of the actual  $\gamma$  from the naive estimate  $\gamma^*$ . Analogous observations in Hamiltonian systems have been reported in [22] (recurrence) and in [7] (escape). In a different approach,  $\gamma$  has been related to the diffusion coefficient in both recurrence [23] and escape [40] problems.

Formally the problem can be described in terms of the  $c$  measure and the convergence to it. The difference between  $\gamma$  and  $\gamma^*$  corresponds to the difference between  $\mu(I)$  and  $\mu_c(I)$ , as given by Eqs. (13) and (14). As emphasized in our previous publication [31], this is surprising since a measure of

open systems is employed to describe properties of closed systems (recurrence). In practice, initial densities are most often taken proportionally to the natural density  $\rho_\mu$  or to a smooth density  $\rho_s$ . In such cases we expect a fast convergence to  $\rho_c$ , and this is why  $1/\mu_c(I)$  [Eq. (15)] is a better estimate of  $\langle n \rangle$  [in Eq. (16)] than  $1/\mu(I)$  [Eq. (4), which is obtained by the atypical  $\rho_0 = \rho_r$ ]. The escape rate is related to  $\mu_c(I)$  which is itself especially sensitive to the choice of the location, shape, and size of the finite leak  $I$ . It is natural to expect that  $\mu_c(I) < \mu(I)$  since the saddle exists only in  $A \setminus I$  and therefore  $\mu_c$  is concentrated only along its unstable manifold. This implies that the mean recurrence time  $\langle T \rangle$ , i.e.,  $\langle n \rangle$  with  $\rho_0 = \rho_r$ , should be shorter than the mean escape time  $\langle n \rangle$  for a typical  $\rho_0$ . This statement is rigorously proved in Ref. [41] for  $\rho_0 = \rho_\mu$ .

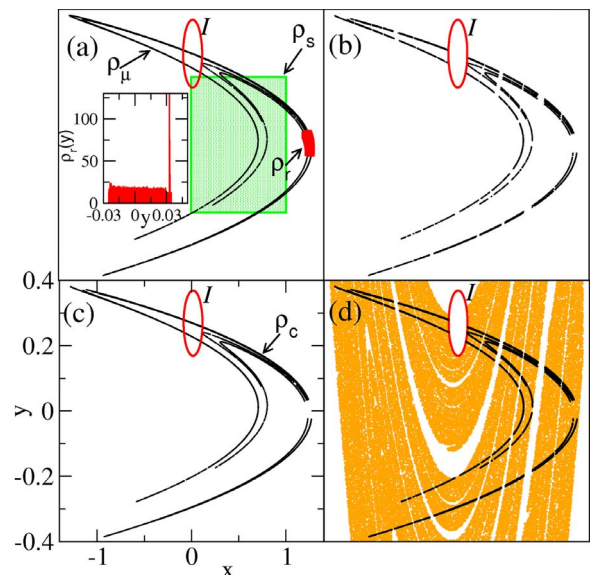


FIG. 2. (Color online) Hénon map (21) with a leak at  $(x_c, y_c) = (1.0804, 0.0916)$  of radius  $\delta = 0.1$ , depicted as a red circle. (a) Different initial densities:  $\rho_\mu$  of the natural measure on the Hénon attractor,  $\rho_s = \text{const}$  on  $0 < x < 1, -0.2 < y < 0.2$ , and  $\rho_r$ , equivalent to recurrence. The inset shows the nontrivial dependence of the projection  $\rho_r(y)$  of  $\rho_r$  on the  $y$  axis. (b) Invariant saddle composed of the points that do not enter  $I$  for forward and backward iterations. (c) Unstable manifold of the invariant saddle ( $\rho_c$ ) and (d) stable manifold in gray (orange) of the invariant saddle.

<sup>4</sup>This is not valid if one is interested in the full sequence of recurrence times  $\{T_1, T_2, \dots, T_N\}$ .

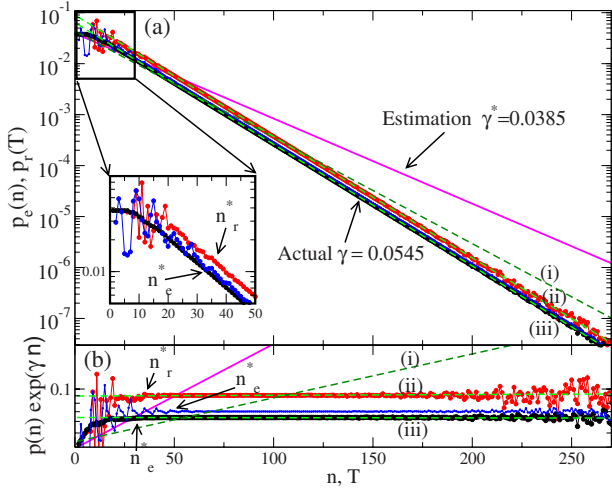


FIG. 3. (Color online) (a) Escape time distributions for the Hénon map (21) with a leak of radius  $\delta=0.05$  at  $(x_c, y_c) = (1.0804, 0.0916)$ . Results for initial densities  $\rho_r$ ,  $\rho_\mu$ , and  $\rho_s$  (shown in Fig. 2) are presented from top to bottom, respectively. The thin solid line corresponds to  $\gamma^*=0.0385$  given by (6) and strongly differs from the actual exponent  $\gamma=0.0545$ . The inset shows the short-time behavior. (b) The same as in (a) multiplied by  $\exp(\gamma n)$ . Three different procedures were employed to fit  $\gamma$  using  $p(n)$  for  $n < n^*$  (see Sec. VII) and are indicated by the dashed lines (i)–(iii). In all cases  $n^*=35$  and (i) corresponds to the use of relation (33), (ii) follows from relations (31) and (32) with the numerical value of  $\langle n \rangle$  for  $\rho_0=\rho_r$ , and (iii) from relations (31) and (32) with the numerical value of  $\langle n \rangle$  for  $\rho_0=\rho_s$ .

## V. NUMERICAL RESULTS

We illustrate the previous results through numerical simulations of the dissipative Hénon map

$$x_{n+1} = 1 - 1.4x_n^2 + y_n, \quad y_{n+1} = 0.3x_n. \quad (21)$$

Initial conditions inside a large basin of attraction close to the origin converge to the well-studied Hénon attractor, shown in Fig. 2(a). We let trajectories leak the system through the leak region  $I$ : a circle of radius  $\delta$  centered at some  $(x_c, y_c)$  on the attractor. The Sinai-Ruelle-Bowen (SRB) measure is the natural measure  $\mu$  describing the closed system. All the initial densities discussed in Sec. III C are marked in Fig. 2. Figures 2(b)–2(d) show the invariant saddle and its manifolds. A new feature in comparison with saddles of naturally open systems is that the unstable manifold is not a single fractal curve but consists of disjoint pieces cut by the leak and its images.

In Fig. 3 we show the escape time distribution for the initial densities depicted in Fig. 2. It is apparent that all initial densities lead to *different* short time behavior but to the *same* escape rate  $\gamma$ , which is significantly different from the naive estimate (13).

We investigate now the dependence of  $\gamma$  and  $\mu(I)$  on the radius  $\delta$  of the leak. Since the natural and the  $c$  measure have different fractal properties, for small (but not infinitesimally small)  $\delta$  one expects the scaling relation

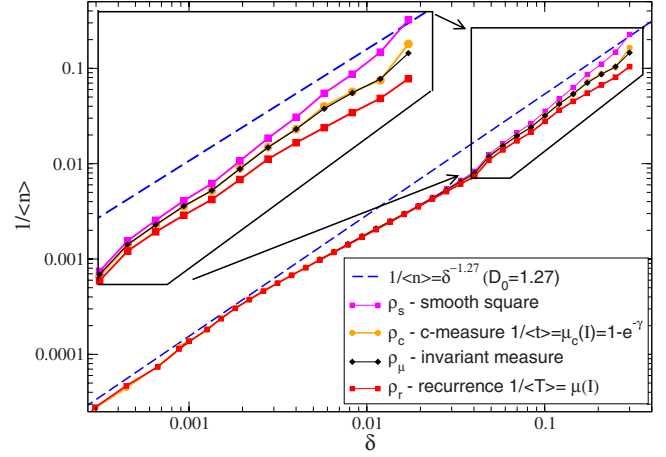


FIG. 4. (Color online) Dependence of the inverse mean escape time  $1/\langle n \rangle$  on the radius  $\delta$  of the leak for fixed  $(x_c, y_c) = (0.015714, 0.268892)$ . Initial densities equivalent to those shown in Fig. 2 are taken. The straight line corresponds to  $\delta^{-D_0}$ , where  $D_0=1.27$  is the fractal dimension of the attractor [53]. The inset shows a magnification of the main graph for large  $\delta$ .

$$\frac{1}{\langle T \rangle} = \mu(I) \sim \delta^{D_1},$$

$$\gamma \approx \mu_c(I) \sim \delta^{1+d_s} = \delta^{1+(\lambda-\gamma)/|\lambda'|}, \quad (22)$$

where  $D_1$  is the information dimension of the chaotic attractor,  $d_s$  is the partial information dimension of the saddle (system with leak) along the stable direction (the  $c$  measure is smooth along the unstable one), and (9) has been used. The numerical comparison can be seen in Fig. 4 where the inverse mean escape time is plotted for different initial densities. For the  $c$  measure and  $\delta \ll 1$ ,  $1/\langle n \rangle$  corresponds to  $\mu_c(I)$  through relation (14), while  $1/\langle n \rangle = \mu(I)$  with  $\rho = \rho_r$ . The deviation between these two curves is a measure of the error of the naive estimate (13). The general dependence, for  $\delta \rightarrow 0$ , is in agreement with the expected  $\delta^{-D_1}$  relation, by taking into account that  $D_1 \lesssim D_0$ , where  $D_0$  is the fractal dimension of the attractor. From the inset it is apparent that initial conditions  $\rho_\mu$  and  $\rho_s$  according to the natural measure and to the homogeneous distribution, respectively, are closer to the results obtained with the density  $\rho_c$  of the  $c$  measure than that obtained with initial density  $\rho_r$  of the recurrence problem. In other words, the mean escape time for typical smooth densities is better approximated by Eq. (16) than by  $1/\mu(I)$  [Eq. (4)].

A systematic investigation of the dependence of  $\gamma$  on the *position* of the leak in the Hénon map would be very involved. Apart from the difficulty of having two dimensions of the phase space, the natural (SRB) measure of the system is not known *a priori*. Therefore, in order to fix the natural measure of the leak  $\mu(I)$ , a different value of  $\delta$  should be chosen for each new  $(x_c, y_c)$ . We investigate therefore this issue in the fully chaotic logistic map

$$x_{n+1} = 4x_n(1-x_n). \quad (23)$$

In this case the density  $\rho_\mu$  of the natural measure of the

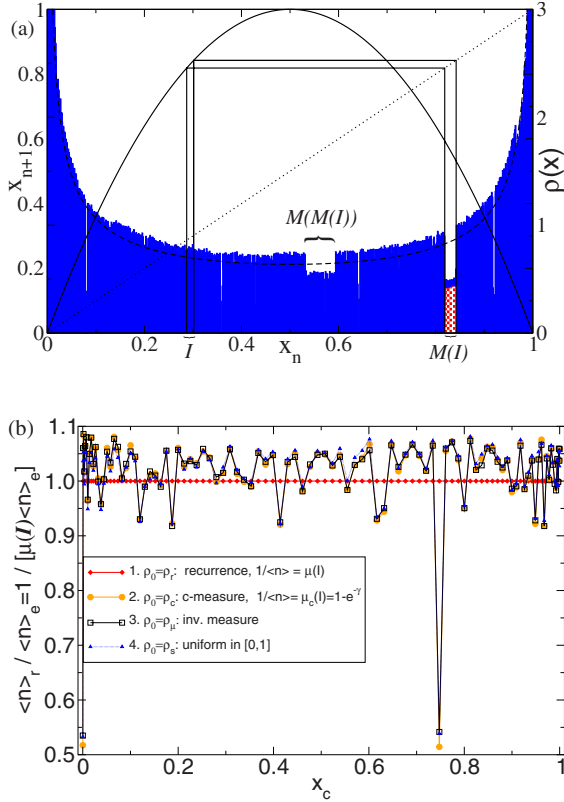


FIG. 5. (Color online) (a) Graphical representation of the logistic map (23) with a leak  $I$  at  $x_c = 0.29427$  of half-width  $\delta = 0.00716$  so that  $\mu(I) = 0.01$ . The dashed line corresponds to  $\rho_\mu(x)$ , Eq. (24), and the (blue) histogram to a numerical distribution of the  $c$  measure of the chaotic repeller obtained by leaking the system at  $I$ . The dotted (red) histogram corresponds to  $\rho_r(x)$  given by (17) multiplied by  $\mu(M(I))$ . (b) Dependence of the inverse mean escape time, multiplied by  $\langle n \rangle_r$ , on the leak position  $x_c$  in the logistic map (23) at an interval length such that  $\mu(I) = 0.01$ . The initial densities used are (see Table I)  $\rho_r$ , equivalent to recurrence,  $\rho_\mu$ , natural density Eq. (24), and  $\rho_s$ , chosen uniform in  $0 < x < 1$ .

system is given by [18]

$$\rho_\mu(x) = \pi^{-1}[x(1-x)]^{-1/2}. \quad (24)$$

The results are shown in Fig. 5(b) and confirm the strong dependence of  $\gamma$  on  $I$ , as well as the conclusion drawn from Fig. 4 that for general smooth initial densities  $\gamma$  is better approximated by (16) than by  $1/\mu(I)$ , reflecting once more the fast convergence of  $\rho_0$  to  $\rho_c$ . Moreover, it is clear that for most  $x_c$  one has  $\langle n \rangle_e > \langle n \rangle_r$ , i.e.,  $\mu(I) > \mu_c(I)$ , in agreement with the discussion at the end of Sec. IV. The remarkable exceptions are related to the existence of short-time periodic orbits inside the leak, as previously described in Refs. [5,17,27]. An explanation for these cases can be given as follows (see Sec. VII for details): if a periodic orbit of period  $n_p$  is inside  $I$  we expect a high recurrence probability  $P(T = n_p)$ ; since  $\langle T \rangle$  is fixed by (4), we predict that  $\gamma < \gamma^*$  and therefore  $\langle n_r \rangle / \langle n_e \rangle < 1$ , as observed in Fig. 5(b).

## VI. NONHYPERBOLIC HAMILTONIAN SYSTEMS WITH POWER-LAW TAILS

### A. Theory

Our discussion has covered so far the case of strongly chaotic systems where, asymptotically in time, the distribution of escape and recurrence times is exponential (for certain classes of systems this has been formally proved [42]). We concentrate in this section on weakly chaotic systems where deviations from exponential decay are observed for long times. The simplest examples of weakly chaotic systems are one-dimensional intermittent maps with a marginal fixed point, where an asymptotic power-law decay is proved [43,44]. The most prominent examples are, however, Hamiltonian systems with mixed phase space, where asymptotically a power-law decay is observed. This occurs in systems possessing isolated marginally unstable orbits [45], sharply divided phase space [46], hierarchical Kolmogorov-Arnold-Moser (KAM) islands coexisting in a chaotic sea [21,24,47,48], and also in higher dimensions [49]. The power-law behavior is due to the nonhyperbolicity of the dynamics and it is present for both recurrences and escapes. The power-law exponent is independent of the choice of  $I$  and  $\rho_0$ , provided the latter is concentrated away from the nonhyperbolic regions. Below we show that the same holds for the intermediate time exponential decay of the recurrence and escape time distributions, for a given  $I$ .

The distribution of escape and recurrence times in Hamiltonian systems with mixed phase space can thus be given as

$$p(n) \approx \begin{cases} \text{irregular} & \text{for } n < n^*, \\ \gamma a e^{-\gamma n} & \text{for } n^* < n < n_\alpha, \\ \gamma [a e^{-\gamma n} + b(\gamma n)^{-\alpha}] & \text{for } n > n_\alpha, \end{cases} \quad (25)$$

where  $a e^{-\gamma n_\alpha} \gg b(\gamma n_\alpha)^{-\alpha}$ . The factor  $\gamma$  is written out on the right-hand side in order to facilitate the connection to the continuous-time limit. Contrary to what has previously been claimed [50],  $\gamma(a+b) \neq 1$  in Eq. (25).

The question whether the asymptotic regime has a well-defined (universal) power law is still under investigation for the case of area-preserving maps (see Ref. [48] for the latest result which indicates that  $\alpha \approx 2.57$ ). Here, for simplicity, we write the asymptotic decay as  $n^{-\alpha}$ , but it is meant to describe the power-law-like behavior usually observed.

We define the crossover time  $n_c$  between exponential and power-law decay as

$$a e^{-\gamma n_c} = b(\gamma n_c)^{-\alpha} \Rightarrow p(n_c) = 2\gamma a e^{-\gamma n_c}. \quad (26)$$

In the framework of *recurrences*, the different times in Eqs. (25) and (26) can be interpreted as follows.  $n^*$  is the short-time memory, within which fluctuations due to short-time periodic orbits appear. After this time the recurrences lose correlation and a random approximation for the return is valid (Poisson process).  $n_\alpha$  is the minimum time a trajectory inside  $I$  takes to approach the nonhyperbolic region and return to  $I$ .  $n_c$  is the time when returning without touching the nonhyperbolic region becomes as probable as returning after touching. If  $I$  is small and away from any nonhyperbolic region we expect  $n^* < n_\alpha < n_c$ .

In terms of *escapes*, the times in Eq. (25) are interpreted as follows.  $n^*$  is the convergence time, which is proportional to  $1/|\lambda'|$ , where  $\lambda'$  is the negative Lyapunov exponent of the saddle (the time to relax to the hyperbolic component of the saddle along its stable manifold).  $n_\alpha$  is the time needed to approach the nonhyperbolic part of the saddle.  $n_c$  is the time when the nonhyperbolic part of the saddle becomes as important as the hyperbolic one. In this case, the key assumption  $n^* < n_\alpha$  is related to the initial density  $\rho_0$  and leak  $I$  being away from the sticky regions. Being invariant, the division of the saddle is of course independent of the initial density. However, for initial conditions touching the KAM island the importance of the exponential decay can be made arbitrarily small (see Fig. 8 below) and the power-law exponent in (25) is modified to  $\alpha' = \alpha - 1$ ; see Refs. [46,51] (and also Ref. [42]).

The above interpretation of  $p(n)$  given by (25) expresses the view, first suggested in [32], that the nonhyperbolic component of the saddle is approached through the hyperbolic part [32,50,52,54,55], i.e., the fraction of trajectories that reach the nonhyperbolic part is proportional to those that reach the hyperbolic one. When the size of the leak  $I$  decreases, the hyperbolic component increases while the nonhyperbolic one remains the same, meaning that the above picture is even more accurate. In this case we can expect that the fraction of trajectories that arrive at the nonhyperbolic component is proportional to the fraction of trajectories that approach the hyperbolic component and that therefore the ratio  $b/a$  [see Eq. (25)] has a weak dependence on  $\gamma$ . Introducing this assumption in the definition of the crossover time  $n_c$  [Eq. (26)], we obtain

$$n_c \sim 1/\gamma \quad [\approx 1/\mu(I) \quad \text{for small } \mu(I)], \quad (27)$$

meaning that  $n_c$  is proportional to the reciprocal of the escape rate  $\gamma$ . The proportionality constant depends on the size of the nonhyperbolic regions. We have verified relation (27) for different maps, which emphasizes once more that for small  $\gamma$  a well-defined exponential decay exists for intermediate times  $n^* < n < n_c$ .

### B. Numerical results for the standard map

We illustrate the previous results in the standard map [23,24]

$$y_{n+1} = y_n - 0.52 \sin(2\pi x_n), \quad x_{n+1} = x_n + y_{n+1}. \quad (28)$$

The phase space of this map is shown in Fig. 6. Chaotic trajectories stick for an algebraic long time to the hierarchical border of the KAM island, shown in the center of Fig. 6(a), which constitutes the nonhyperbolic component of the saddle. The exponential decay is governed by the hyperbolic component of the saddle, whose unstable manifold is shown in Fig. 6(b) for a specific leak  $I$ . This unstable manifold corresponds to the support of the conditionally invariant density  $\rho_c$ . Notice its apparent filamentary character and that it is present inside  $I$ , but does not enter  $M(I)$ .

The distribution of escape and recurrence times for the system presented in Fig. 6 is shown in Fig. 7. As in the case of the Hénon map (Fig. 3), short-time oscillations and an

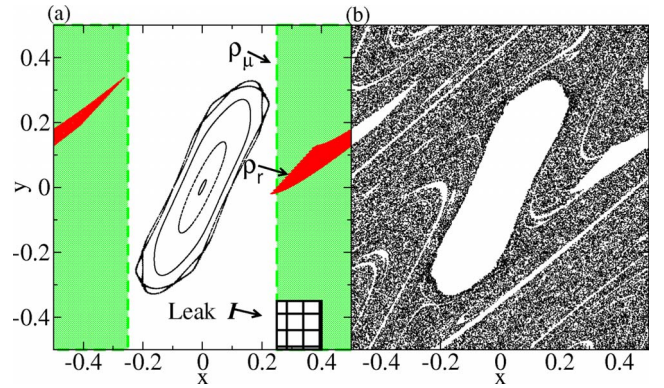


FIG. 6. (Color online) (a) Phase space of the standard map (28) with  $I = [0.25 < x < 0.25 + \varepsilon, -0.5 < y < -0.5 + \varepsilon]$  of size  $\varepsilon = 0.15$ , indicated by a checkered box. The KAM island (nonhyperbolic part of the saddle) is in the center of the figure. The support of  $\rho_r$  at  $M(I)$  is marked as a dark (red) region. (b) The unstable manifold of the hyperbolic part of the saddle (black dots), obtained by using the method of [26], choosing an escape time  $n_s = 80 < n_c = 307$ . The support of  $\rho_r$  is fully outside the unstable manifold.

exponential decay are clearly visible. However, in this case a slower decay is observed for times  $n > n_c \approx 307$ . In the upper inset of Fig. 7(a), a log-log plot of the main graph shows that the long-time decay is roughly a power law with  $\alpha \approx 2.5$ . We have also obtained numerically the value of the crossover time  $n_c$  for leak and recurrence regions  $I$  of different sizes by changing  $\varepsilon$  defined in Fig. 6. The results are shown in the lower inset of Fig. 7(a), and support the theoretical scaling (27). Remarkably, the value of  $n_c$  is almost identical for recurrence and escape times, the same being valid also for other initial distributions  $\rho_0$  away from the KAM island. This provides additional support to the picture that  $\rho_0$  quickly converges to the hyperbolic component of the saddle, and that the nonhyperbolic component is approached in a univer-

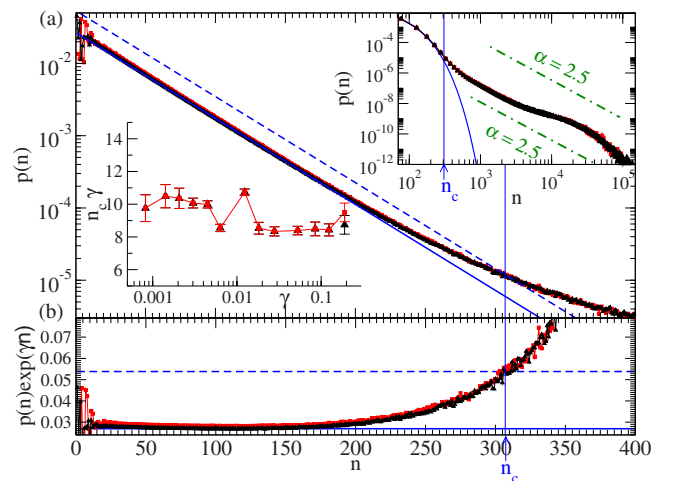


FIG. 7. (Color online) (a) Escape time distributions in the standard map with  $\varepsilon = 0.15$ , as shown in Fig. 6. Results are shown for  $\rho_0 = \rho_r$  (squares below) and for  $\rho_0 = \rho_\mu$  in  $|x| > 0.25$ . Lower inset: scaling of  $n_c \gamma$  with  $\gamma$  obtained by changing  $\varepsilon$  in  $0.03 \leq \varepsilon \leq 0.4$ . Upper inset: log-log plot of the main graph over a long period of time. (b)  $p_{e,r}$  multiplied by  $\exp(\gamma n)$ , where  $\gamma = 0.027403$ .



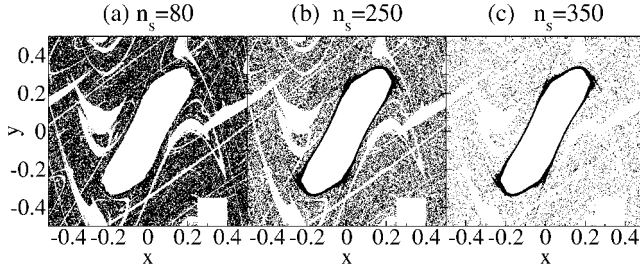


FIG. 8. The saddle characterizing the dynamics at different times,  $n_s$ , is given on top of the panels, obtained by the method of Ref. [26]. For  $n_s < n_c = 307$  in (a), the hyperbolic component obtained is clearly disjoint from the region of the KAM island. By increasing  $n_s$  as in (b) and (c) the nonhyperbolic component appears with an increasing weight.

sal way through the hyperbolic component. The transition between the hyperbolic and nonhyperbolic components of the saddle is illustrated in Fig. 8.

## VII. FITTING PROCEDURE TO OBTAIN THE ESCAPE RATE

In the previous sections we have seen that the escape rate  $\gamma$  can be formally written as a function of the  $c$  measure  $\mu_c(I)$  of the leak region  $I$  through relation (14). However, this relation is of little practical use if one wants to determine  $\gamma$ . The reason is that usually only the Lebesgue measure is known *a priori* and the  $c$  measure  $\mu_c$  may change dramatically depending on the position of the leak. Its numerical calculation is typically much more involved than the determination of  $\gamma$  itself. In this section we search for an efficient method to determine  $\gamma$  that goes beyond the rough naive estimate  $-\ln[1 - \mu(I)]$ , given by relations (13) and (6).

In general open hyperbolic systems a dense set of periodic orbits exists embedded into the nonattracting chaotic set. This can be used to effectively obtain  $\gamma$  as [18,20]

$$e^{-n\gamma} = \sum_{\Gamma_{i,n}} \prod |\Lambda(\Gamma_{i,n})|, \quad (29)$$

where the product is over the expanding eigenvalues  $\Lambda$  of each periodic point  $\Gamma_{i,n}$  belonging to the same periodic orbit of length  $n$ . The sum runs over all periodic orbits of period  $n$  that remain inside the system, i.e., outside the leak region  $I$ .

We want to take further advantage here of the possibility of closing the leak of the system. Note that in the closed system the decay exponent is  $\gamma=0$  and the sum in Eq. (29) runs over all periodic orbits (inside and outside the leak region). We can therefore give  $\gamma$  in terms of the periodic orbits inside the leak region as

$$1 - e^{-n\gamma} = \sum_{\Gamma_{i,n}(\text{inside})} \prod |\Lambda(\Gamma_{i,n})|. \quad (30)$$

From this expression, it is clear that short-time periodic orbits have a strong influence on  $\gamma$  since not only the primitive orbits, but also all their multiples, appear in the sum. This dependence is apparent in Fig. 5(b) and was previously reported in Refs. [5,17,27]. The application of (30) requires,

however, again the identification of unstable periodic orbits of high periods. Since the deviation of  $\gamma$  from the estimate  $\gamma^* = -\ln[1 - \mu(I)]$  has been related to short-time periodic orbits we can expect to determine  $\gamma$  from these orbits. A method for this purpose was described in Ref. [27] in the context of recurrences. We adopt it here in the context of escape.

The distribution of escape times is written as in Eq. (8). The oscillatory part for times  $n < n^*$  will be denoted by  $p_0(n)$  and the remaining exponential part by  $p_{\text{exp}}(n) \equiv g \exp(-\gamma n)$ . The two parameters  $g$  and  $\gamma$  of the asymptotic exponential decay can be obtained then by imposing the following two conditions: normalization

$$\sum_{n=1}^{n^*-1} p_0(n) + \sum_{n=n^*}^{\infty} p_{\text{exp}}(n) = 1 \quad (31)$$

and the value of the mean

$$\langle n \rangle = \sum_{n=1}^{n^*-1} n p_0(n) + \sum_{n=n^*}^{\infty} n p_{\text{exp}}(n) = \frac{1}{\mu_i(I)}. \quad (32)$$

For recurrences, or  $\rho_0 = \rho_r$ , we can use Kac's lemma (4) and thus  $\mu_i(I) = \mu(I)$ , the natural measure of the leak. On the other hand, when initial densities are proportional to the natural measure in the escape problem, the best procedure is to replace (4) by (16) and therefore use  $\mu_i(I) = \mu_c(I) = 1 - e^{-\gamma}$  on the right-hand side of Eq. (32). In the last case, we can solve Eqs. (31) and (32) in terms of  $\gamma$ , which is expressed as the solution of

$$(1 - S_1) \frac{n^* e^\gamma + 1 - n^*}{e^\gamma - 1} = \frac{1}{1 - e^{-\gamma}} - S_2, \quad (33)$$

where  $S_1 = \sum_{n=1}^{n^*-1} p_0(n)$  and  $S_2 = \sum_{n=1}^{n^*-1} n p_0(n)$ . Both fitting procedures, the one given by Eqs. (31) and (32) and the one using Eq. (33), were employed for the case of the Hénon map and are marked as dotted lines in Fig. 3. Notice that the fitting obtained using Eq. (33) alone is worse than the one obtained using as additional information the numerical value of  $\langle n_e \rangle$  in Eqs (31) and (32), but better than the random estimate (6).

The main advantage of this procedure is that, even if theoretically a large  $n^*$  in Eqs. (31) and (33) would render better results, in practice the value  $n^*$  is fairly small, as numerical experience shows ( $n^* \lesssim \langle n \rangle$ ). The values of  $p(n)$  for  $n < n^*$  are thus obtained without much computational effort already with a high precision but allow the computation of the asymptotic decay  $\gamma$ . Moreover, this shows that important information about the long-time dynamics is already contained in the short-time dynamics.

In the nonhyperbolic case one has to take into account also the contribution of the power-law tail. In the continuous limit, the contributions

$$S_3 = \sum_{n=n_\alpha}^{\infty} \gamma b(\gamma n)^{-\alpha} = b(\gamma n_\alpha)^{1-\alpha} / (\alpha - 1),$$

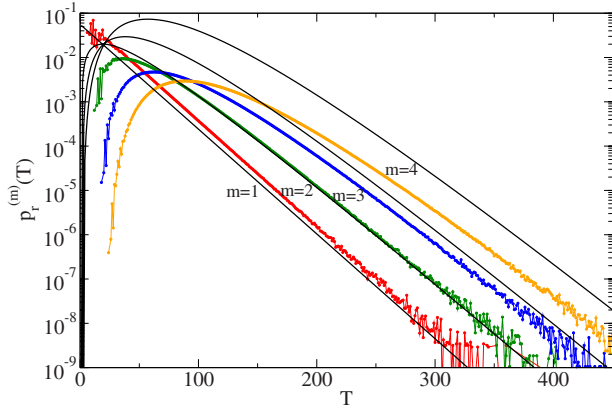


FIG. 9. (Color online) Distribution of the  $m$ th recurrence times ( $m=1, \dots, 4$ , from bottom to top) in the Hénon map with parameters as in Fig. 3. The solid lines correspond to the binomial distribution (35) using  $\mu = \mu_c(I)$ . For long times, the same decay rate  $\gamma = \ln[1 - \mu_c(I)]$  is observed in all curves. The displacement between the distribution and the simulation, as in the  $m=1$  case discussed previously, can be related to the short-time oscillations which modify the prefactors of the distribution.

$$S_4 = \sum_{n=n_\alpha}^{\infty} \gamma m b (\gamma n)^{-\alpha} = b \gamma^{1-\alpha} n_\alpha^{2-\alpha} / (\alpha - 2) \quad (34)$$

are to be added as new terms to the left-hand side of Eqs. (31) and (32), respectively. Since  $n_\alpha \sim n_c \sim 1/\mu(I)$  and  $\alpha > 2$ , one sees that these terms become negligible for small recurrence and leak regions.

## VIII. CONCLUSIONS

Experimental and observational measurements often occur through holes or leaks that naturally exist or are deliberately introduced in an otherwise closed dynamical system [1–10]. The relevant observable quantity in such systems with leaks is the distribution of escape times from inside the system. In strongly chaotic systems, this distribution decays exponentially. Applying the ergodic theory of transient chaos [1] to chaotic systems with leaks an expression for the escape rate  $\gamma_e$  [Eq. (14)] is obtained [5] in terms of the  $c$  measure (associated with the invariant chaotic saddle) of the leak  $I$ . We have provided a theoretical framework to understand the strong dependence of  $\gamma$  and  $\langle n_e \rangle$  on the leak size (Fig. 4) and position (Fig. 5). In the (unrealistic) limit of a small leak, the expression of the escape rate converges to a previously known relation [Eqs. (2)], based on the invariant measure of the closed system. Altogether, our results help to understand previous observations [5,7,27,29] and set a theoretical framework for future applications.

We have shown that the classical problem of Poincaré recurrences in closed systems can be described as a problem of escape from a system with a leak, once the recurrence region is identified with the leak  $I$  and the initial density is chosen properly. This allows us to treat both problems in a unified framework and compare previous similar results that were reported independently in both fields (compare Refs. [5,7,27]), and to adapt (in Sec. VII) a previous method to

efficiently obtain  $\gamma_e$ . More surprisingly, we show that the relaxation rate of the distribution of Poincaré recurrences  $\gamma_r$  is equal to  $\gamma_e$  [Eq. (19)] but that the mean recurrence time  $\langle n_r \rangle$  given by Kac's lemma (4) differs from the typical mean escape time [the analog of Kac's lemma for leaked systems is (15)]. These results are summarized in Table I and provide a more detailed account of the results published in Ref. [31].

In weakly chaotic systems the asymptotic exponential decay is replaced by a power-law decay. We have shown, however, that if the leak region and the initial distribution are away from nonhyperbolic regions, an exponential decay is still well defined for intermediate times. We have obtained the scaling of the crossover time  $n_c$  between the two regimes [Eq. (27)] which confirms the importance of the exponential decay for small leaks. Altogether, these results justify an effective splitting of the chaotic saddle in a nonhyperbolic component, related to the asymptotic power-law decay, and a hyperbolic component, related to the intermediate exponential decay for which our previous results apply. This splitting is particularly important in the case of Hamiltonian systems (found, e.g., in fluid dynamics or optical applications) where the generic phase space shows a mixture of regions of regular and chaotic motion.

Finally, it is worth considering briefly two extensions of the previous results that have potentially interesting applications. First, consider the case when the leak consists of two or more disjoint components (of relevance, e.g., for the problem of resetting in hydrodynamical flows [6]). All our results apply to such cases as well, with the remark that the escape rate is not the mere sum of the escape rates characterizing the components, due to the overlap among preimages, as pointed out by Buljan and Paar [5] and Bunimovich and Dettmann [16]. A second extension is to consider the distribution of the second, third, ...,  $m$ th recurrence times. In the notation of Sec. II, the  $m$ th recurrence time is defined as  $T_i^{(m)} \equiv n_i - n_{i-m}$ . In view of the analogy to escape presented in Sec. IV, this corresponds to the distribution of escape times taking place only after entering the leak  $I$  for the  $m$ th time. The naive (binomial) approximation (5) can be extended to this case as [27]

$$P_r^{(m)}(T) = \frac{(T-1)!}{(T-m)!(m-1)!} \mu(I)^m (1 - \mu(I))^{T-m}, \quad (35)$$

and implies that the asymptotic decay rate  $\gamma^{*(m)} = \ln[1 - \mu(I)]$  [see (6)] remains valid for any  $m$ . In Fig. 9 we show numerical results for the Hénon map. For  $m > 1$  the distributions take, of course, the value zero for  $n=1$ , start growing, and then after a maximum is reached, an exponential decay sets in. The decay rate has been found again to be different from the binomial approximation, but apparently also independent of  $m$ , i.e., to have the value

$$\gamma^{(m)} = \gamma = \ln[1 - \mu_c(I)],$$

discussed in the context of the first recurrence times [see Eq. (13)]. To understand this result, let us first compare the invariant saddles for  $m=1$  ( $S_1$ ) and  $m=2$  ( $S_2$ ). It is easy to see that  $S_1 \subset S_2$ . Trajectories that belong to  $S_2$  but not to  $S_1$  must have a point  $P$  inside  $I$  that never returns to  $I$  again. Hence, these trajectories necessarily approach  $S_1$  for both  $t \rightarrow \pm \infty$ .

To be outside  $I$  for  $t \rightarrow \infty$ ,  $P$  has to belong to the unstable manifold of  $S_1$ . To be outside  $I$  for  $t \rightarrow -\infty$ , the image point  $M(P)$  taken with respect to the closed system's map has to be on the stable manifold of  $S_1$ . Therefore, points  $P$  belong to the intersection of the unstable manifold of  $S_1$  and the pre-image (using map  $M$ ) of the stable manifold of  $S_1$  and also approach  $S_1$  asymptotically. Accordingly, trajectories escaping for long times in the case  $m=2$  appear still to be governed by  $S_1$  having therefore the escape rate  $\gamma$ . This argument can be extended straightforwardly to any finite  $m$  and

can also be important for the case of partial escape in optical systems [12].

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