Quasipotential approach to critical scaling in noise-induced chaos

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When a dynamical system exhibits transient chaos and a nonchaotic attractor, as in a periodic window, noise can induce a chaotic attractor. In particular, when the noise amplitude exceeds a critical value, the largest Lyapunov exponent of the attractor of the system starts to increase from zero. While a scaling law for the variation of the Lyapunov exponent with noise was uncovered previously, it is mostly based on numerical evidence and a heuristic analysis. This paper presents a more general approach to the scaling law, one based on the concept of quasipotentials. Besides providing deeper insights into the problem of noise-induced chaos, the quasipotential approach enables previously unresolved issues to be addressed. The fractal properties of noise-induced chaotic attractors and applications to biological systems are also discussed.

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I. INTRODUCTION

In experimental situations, an observed trajectory is always subject to some external perturbations, e.g., of thermal or of technical origin. In the lack of any specific information about their own dynamics, other than their time scale being much shorter than that of the original signal, we can assume that the external perturbations are random. Their inclusion into the dynamics can be modeled via an additive noise term, which converts the purely deterministic equations of motion into a set of stochastic equations [1–6]. Suppose a deterministic system exhibits a nonchaotic attractor, e.g., a periodic attractor. The coupling of a weak noise into the system can cause the attractor to become chaotic. This is the phenomenon of noise-induced chaos [7–9], which is fundamental in nonlinear and statistical physics and has received continuous attention [10–13].

A typical situation where noise-induced chaos can arise is periodic windows. In such a window, there is a periodic attractor and a nonattracting chaotic set that leads to transient chaos. Noise can cause a trajectory to visit both the original attractor and the nonattracting chaotic set, giving rise to an extended noisy chaotic attractor. In a smooth dynamical system that exhibits chaos, in the absence of noise a chaotic attractor is structurally unstable because, periodic windows are dense and occupy open sets in the parameter space. As a result, an arbitrarily small perturbation can place the system in such a window, destroying the chaotic attractor. When noise is present, chaos is enhanced in the sense that noise-induced chaos can even occur in periodic windows and, hence, chaotic attractor can now occur in open sets in the parameter space. Noise-induced chaos thus provides the reason for observing chaotic attractors in realistic systems. This phenomenon is, of course, not restricted to periodic windows. Insofar as a nonchaotic attractor coexists with a nonattracting chaotic set in the phase space, a chaotic attractor can arise due to noise.

A number of previous works have explored the critical behavior associated with noise-induced chaos [13]. A basic issue concerns the scaling of the Lyapunov exponent. In particular, consider a continuous-time dynamical system. When the system is in a periodic window and exhibits a periodic attractor, in the deterministic case the largest Lyapunov exponent λ_1 is zero. As the noise amplitude σ is increased from zero and passes through a critical point σ_c, λ_1 becomes positive. It has been argued and supported by numerical evidence [13] that λ_1 obeys the following scaling law with the variation in the noise amplitude beyond the critical value:

$$\lambda_1 \sim (\sigma - \sigma_c)^{\alpha}, \tag{1}$$

for $\sigma > \sigma_c$, where $\alpha = 1$. In previous works, the arguments leading to Eq. (1) are heuristic as they are based on analyzing the overlaps between the natural measure of the noise-enlarged periodic attractor and that of the stable manifold of the nonattracting chaotic set. To give more credence to the validity of Eq. (1), it is desirable to derive a more solid theoretical approach. The main purpose of this paper is to present such an approach based on the concept of *quasipotentials* [14–21]. We will show that, by utilizing this concept, the equivalent noise-induced escape problem can be treated analytically, leading to the scaling law Eq. (1). It will also be demonstrated that, for maps, the critical noise strength at which points become observable on the noise-induced attractor is not the same as the one above which the largest Lyapunov exponent is positive. To better understand the geometry of the noise-induced chaotic attractors, we will also discuss their fractal properties.

In Sec. II, we review the concept of quasipotentials. In Sec. III, we present a series of arguments based on quasipotentials for the scaling law Eq. (1). In Sec. IV, we discuss the fractal properties of noise-induced chaotic attractors and justify applications to biology. Conclusions are presented in Sec. V.

II. QUASIPOTENTIALS

A. Setting

We consider noisy versions of continuous- and discrete-time dynamical systems which, in dimensionless forms, are written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x},p) + \sigma\boldsymbol{\xi}(t), \tag{2}$$

and

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n,p) + \sigma\boldsymbol{\xi}_n, \tag{3}$$

respectively, where p is a parameter, $\sigma > 0$ represents the noise amplitude, and the $\boldsymbol{\xi}$ terms are independent, identically distributed random processes of zero mean and unit variance. The distribution $P(\boldsymbol{\xi})$ is assumed to be known and to be *independent of time* so that the stochastic process generating the noise is stationary. An example of $P(\boldsymbol{\xi})$ is a Gaussian distribution,

$$P(\boldsymbol{\xi}) \sim \exp(-\boldsymbol{\xi}^2/2). \tag{4}$$

This form implies that, even for small noise strength σ, the random perturbation can be arbitrarily large, but the probabilities for large perturbations are exponentially small.

B. Basic notions of quasipotentials

For a nonlinear dynamical system under noise, it is often desirable to know the probability distribution over the entire phase space. A normalizable distribution exists only if there are attractors in the system. Based on a well established theory in the weak-noise limit [14–16], we summarize the results for maps of the type of Eq. (3). The steady-state probability distribution $W(\mathbf{x})$ can be written for Gaussian noise with $\sigma \ll 1$ as

$$W(\mathbf{x}) \sim Z(\mathbf{x})e^{-\Phi(\mathbf{x})/\sigma^2}. \tag{5}$$

The exponential factor is of special importance since it is similar to the form describing fluctuations in thermal equilibrium. The function Φ plays a central role in the theory: it is analogous to the free energy. The noise intensity σ^2 plays the role of thermal energy k_BT. A difference from equilibrium thermodynamics is that, here an explicit form of Φ cannot be obtained from the first principles; Φ is therefore called the *quasipotential* (or nonequilibrium potential) of the map. An example of the quasipotential is shown in Fig. 1. Note that neither Φ nor the prefactor Z depends on the noise strength; they depend solely on the underlying deterministic dynamics.

The quasipotential satisfies an extremum principle and it can be constructed based on methods from Hamiltonian mechanics [17–23]. The basic observation is that the system can come to a phase-space point $\mathbf{x}$ via a large number of noise realizations. Since, however, noise is weak, there are rare realizations that are sharply peaked about a single optimal realization, namely, the most probable path that leads to $\mathbf{x}$. In N iterations, the optimal path for noise should *maximize* the probability

$$P(\boldsymbol{\xi}_0)P(\boldsymbol{\xi}_1)\cdots P(\boldsymbol{\xi}_N) \sim \exp\left(-\sum_{n=0}^{N}\frac{(\sigma\boldsymbol{\xi}_n)^2}{2\sigma^2}\right). \tag{6}$$

Equivalently, the path corresponds to the *minimum* of the "noise energy" $\Sigma_{n=0}^{N}(\sigma\boldsymbol{\xi}_n)^2$. The iteration process Eq. (3) plays the role of a constraint that can be taken into account

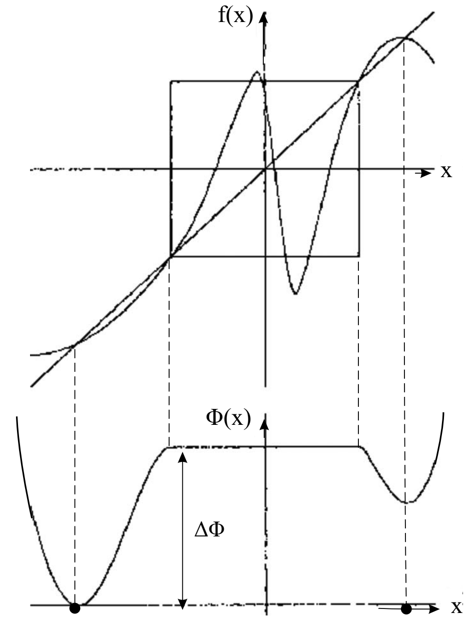

FIG. 1. Schematic diagram of quasipotential Φ for a one-dimensional map f that has two coexisting fixed-point attractors (denoted by dots) and a chaotic repeller. The value of the quasipotential is constant over the interval containing the repeller. The activation energy $\Delta\Phi$ in the valley of the left fixed point is marked.

by means of Lagrangian multipliers $\boldsymbol{\eta}_n$ that are effectively control variables. The task of finding the optimal path, thus, boils down to minimizing the following Lagrangian:

$$L = \sum_{n=0}^{N}\left(\frac{1}{2}(\sigma\boldsymbol{\xi}_n)^2 + \boldsymbol{\eta}_n[\mathbf{x}_{n+1} - \mathbf{f}(\mathbf{x}_n) - \sigma\boldsymbol{\xi}_n]\right). \tag{7}$$

In the presence of the multipliers, the variables $\boldsymbol{\xi}_n$ and $\mathbf{x}_n$ can be regarded as independent. Setting the partial derivatives of L zero yields the following coupled map between $\boldsymbol{\eta}_n$ and $\mathbf{x}_n$ for the optimal path

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n,p) + \boldsymbol{\eta}_n, \quad \boldsymbol{\eta}_{n+1} = \mathbf{J}(\mathbf{x}_{n+1},p)^{-1}\boldsymbol{\eta}_n, \tag{8}$$

where $\mathbf{J}$ denotes the derivative (Jacobian) matrix of map $\mathbf{f}$. A comparison with Eq. (3) indicates that $\boldsymbol{\eta}_n/\sigma$ is nothing but the optimizing noise process. Another feature is that the map Eq. (8) is area preserving as it is independent of the dissipative character of $\mathbf{f}$. It can therefore be regarded as a kind of Hamiltonian extension of the deterministic dynamics $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p)$ through the control variable $\boldsymbol{\eta}_n$. The optimal escape path from an attractor was numerically determined in a number of cases [22–24]. This path passes through a saddle point (or a saddle periodic orbit) lying away from the attractor. The saddle is often part of a nonattracting chaotic set of the noise-free system, which might also be a subset of a fractal basin boundary [22,23].

From Eq. (6), the probability for a trajectory to be at point $\mathbf{x} = \mathbf{x}_{N+1}$ after the Nth iteration is

$$P(\boldsymbol{\xi}_0)P(\boldsymbol{\xi}_1)\cdots P(\boldsymbol{\xi}_N) \sim \exp\left(-\frac{1}{\sigma^2}\sum_{n=0}^{N}\frac{1}{2}[\mathbf{x}_{n+1}-\mathbf{f}(\mathbf{x}_n,p)]^2\right). \tag{9}$$

In order to find a time-independent distribution of $\mathbf{x}$, we take the limit $N\to\infty$. The quasipotential defined by Eq. (5) becomes

$$\Phi(\mathbf{x}) = \min\sum_{n=0}^{\infty}\frac{1}{2}[\mathbf{x}_{n+1}-\mathbf{f}(\mathbf{x}_n,p)]^2\bigg|_{\mathbf{x}_\infty\equiv\mathbf{x}} + \text{const}, \tag{10}$$

where the minimum is taken with respect to the value of the control variable at the initial and end points. The quasipotential must be independent of the initial condition $(\mathbf{x}_0,\boldsymbol{\eta}_0)$ within the basin of attraction of an attractor A. This can be realized by letting Eq. (8) evolve according to the deterministic dynamics ($\boldsymbol{\eta}_n=0$, $n=0,1,\cdots$) until the attractor is reached. This initial evolution does not contribute to Φ and, hence, for all practical purposes the initial condition can be taken to be $\mathbf{x}_0\in A$, and $\boldsymbol{\eta}_n\to 0$ for $n\to 0$. A more detailed formulation of the initial value problem was given in [23] where the invariant manifold on which the optimal escape path should start is determined, along with the local form of the quasipotential.

Since, the probabilities of visiting different regions of an attractor cannot differ exponentially, the quasipotential is *constant* on the attractor. For a chaotic attractor, the quasipotential is constant on the entire fractal set. The differences in the probabilities of visiting different regions of the attractor are characterized by the prefactor $Z(\mathbf{x})$ of Eq. (5). Since it is σ independent, the prefactor evaluated on the attractor must coincide with the density ϱ associated with the natural measure. The role of noise becomes thus important outside the attractor where the essential contribution to the dynamics is characterized by $\exp(-\Phi/\sigma^2)$. In particular, the quasipotential increases with the distance from the attractor, and the "Boltzmann factor" $\exp(-\Phi/\sigma^2)$ yields the probability that noise pushes a trajectory point to $\mathbf{x}$, away from the attractor. In the case of a fractal chaotic attractor, the factor is the probability for a trajectory to fall in-between two branches of the fractal manifolds as result of noise.

C. Quasipotential plateaus associated with nonattracting sets

Analogous to the situation with attractors, quasipotentials are constant on nonattracting chaotic sets. Such sets are either repellers (that repel in all directions of the phase space) or saddles (that possess, besides the repelling directions, at least one attracting direction) [25]. For example, in a one-dimensional map, the potential is constant on the interval containing the repeller, as shown in Fig. 1.

For chaotic saddles arising from invertible two-dimensional maps, noise is more likely to push trajectories along than across the unstable manifold. In this case, a quasipotential plateau extends along the unstable manifold of the saddle only. In between branches of the unstable manifold, the potential assumes larger values. As a result, Φ changes in the stable direction outside the saddle but remains constant along the unstable direction in a region containing the entire chaotic saddle [18–21]. The constant value of the quasipotential taken along the unstable manifold branches is the quasipotential plateau.

When a nonattracting chaotic set is present within the basin of attraction of an attractor, its existence can increase the exit rate [24]. To understand this, consider an invertible system with smooth basin boundaries, where the most probable exit path passes through an unstable periodic point $\mathbf{x}_e$ on the boundary. Before reaching the boundary, the path also extends through the chaotic saddle since, from the point of view of energy, motion along the unstable manifold of the saddle does not contribute to the quasipotential. From the quasipotential plateau of the chaotic saddle, a trajectory can reach $\mathbf{x}_e$ on the boundary with a relatively low increase in the quasipotential. Numerical simulations reveal [24] that, although the activation energy $\Phi(\mathbf{x}_e)-\Phi_{\text{attractor}}$ is smaller than that in the absence of the saddle by about 50 percent only, the exit rate can be enhanced by several orders of magnitude. The origin of the reduction in the activation energy can be understood by noting that the exit process actually consists of three stages: reaching the quasipotential plateau of the saddle, moving on the plateau, and leaving the plateau to reach the boundary. The chaotic saddle thus acts as a "shortcut" for minimizing the quasipotential in the exit process.

The role of chaotic saddle in enhancing the exit rate suggests that, when a dynamical system undergoes a basin boundary metamorphosis [26] by which a smooth boundary becomes fractal so that a nonattracting chaotic set arises on the boundary, the rate of exiting the basin due to noise can be enhanced significantly. This has indeed been observed [27].

The average lifetime τ_P of particles in the potential well of the periodic attractor around which a non-attracting chaotic set exists on the basin boundary is determined, in the weak noise limit, by the Arrhenius factor [28] $\exp(\Delta\Phi/\sigma^2)$, where $\Delta\Phi$ is the activation energy, i.e., the difference between the quasipotential value of the plateau and of the periodic attractor (see Fig. 1). We can thus write τ_P as

$$\tau_P = \tau_0 e^{\Delta\Phi/\sigma^2}, \tag{11}$$

where τ_0 is a constant.

III. SCALING LAW FOR NOISE-INDUCED CHAOS

A heuristic analysis and extensive numerical support for the scaling law Eq. (1) has been provided in Ref. [13]. Here, we provide a more rigorous analysis based on quasipotentials, valid both for maps and for time-continuous flows.

A noise-induced chaotic attractor is an extended attracting invariant set of a noisy system, the deterministic counterpart of which contains regular attractors only, coexisting with a non-attracting chaotic set. As described in Sec. I (Introduction), a noise-induced chaotic attractor contains the union of the deterministic regular attractor and the non-attracting chaotic set. Since noise-induced chaotic attractors typically appear in the weak-noise case, the theory of quasipotentials is applicable. We emphasize, however, that a noise-induced attractor is not a substitute for "long-lasting transient." It is a permanently attracting set appearing in the presence of noise.

A. Critical noise strength for noise-induced chaos

The concept of quasipotential provides a convenient way for estimating the critical noise strength required for noise-induced chaos [21]. The first observation is that the periodic attractor appears to be fuzzy in the presence of noise. We can define a noisy attractor as the region in which the probability distribution takes on large values. Note that any practical observation of the stationary distribution relies on the existence of a finite threshold resolution, χ (say 10^{-5} of the maximum of the probability density W). In leading order, we can define, depending on the threshold, a noisy attractor as the set of phase-space points that satisfy $Z\exp[-(\Phi(\mathbf{x})-\Phi_{\text{attractor}})/\sigma^2]\geq\chi$ where Z is a constant. For small σ, the distribution is strongly localized and the extension of the noisy attractor beyond the deterministic attractor is small but increases with the noise strength. A noise-induced chaotic attractor becomes observable at a critical noise strength σ_c where the noisy attractor touches the edge of the quasipotential plateau. Then, the difference $\Phi(\mathbf{x})-\Phi_{\text{attractor}}$ takes on the value of the activation energy $\Delta\Phi$, and the condition for the critical noise-strength is $Z\exp(-\Delta\Phi/\sigma_c^2)=\chi$. We thus obtain

$$\sigma_c = \sqrt{\Delta\Phi/\ln(Z/\chi)} \sim \Delta\Phi^{1/2}. \tag{12}$$

A sudden spreading of the support of distribution W has indeed been observed [21].

For noise strength slightly above σ_c, the probability distribution observed with resolution χ extends over the unstable-manifold branches of the quasipotential plateau. The noise-induced chaotic attractor contains these branches of the unstable manifold of the nonattracting chaotic set, but the probability about the original periodic attractor is much larger than that of being further away. The mean first-exit time from the corresponding potential well is equal to the average lifetime in the noisy system about the original attractor.

To illustrate the usefulness of the quasipotential concept, we mention that close to certain bifurcations, a scaling law of the critical noise strength can be obtained from Eq. (12) [21]. For example, close to the saddle-node bifurcation that initiates a period-m window, the deterministic dynamical system can effectively be reduced to a normal form that is one-dimensional [29]. The quasipotential about the fixed-point attractor, the node, increases quadratically with the distance from the attractor with a coefficient proportional to $(p-p_b)^{1/2}$ [20,30]:

$$\Delta\Phi(\Delta x) \sim (p-p_b)^{1/2}\Delta x^2,$$

where p is a system parameter and p_b denotes the bifurcation point. For a smooth one-dimensional map, the phase-space distance between the saddle and the node about the bifurcation point is proportional to $(p-p_b)^{1/2}$. Since the saddle is part of the non-attracting set, the activation energy is

$$\Delta\Phi = c(p-p_b)^{3/2}, \tag{13}$$

where c is a constant. From Eq. (12), at a fixed resolution, the critical noise strength scales with $(p-p_b)$ as

$$\sigma_c \sim (p-p_b)^{3/4}, \tag{14}$$

which has been verified numerically [30,31].

It should be emphasized that the quasipotential approach is applicable if the deterministic influence dominates the stochastic influence. This implies that, given a fixed finite value of σ, the results presented here are valid only if $p-p_b$ exceeds some minimum value because, noise is dominant for parameter values quite close to the bifurcation point.

B. Scaling of positive Lyapunov exponent

A standard approach to defining an attractor under noise to be chaotic is the sensitive dependence on initial conditions, as characterized by the existence of at least one positive Lyapunov exponent [5,6]. This is because the Lyapunov exponents are the time-averaged stretching or contracting rates of infinitesimal vectors along a typical trajectory in the phase space, which can be defined for both deterministic and stochastic dynamical systems. In particular, in the absence of noise, since the attractor is not chaotic, the largest Lyapunov exponent of the asymptotic attractor is a negative number for maps (zero for flows). As noise is turned on and its strength becomes sufficiently large, there is a nonzero probability that a trajectory originally on the attracting set escapes it and wanders near the coexisting nonattracting chaotic set. In this case, the largest Lyapunov exponent λ_1 becomes positive, indicating that the asymptotic attractor of the system is chaotic for trajectories starting from random initial conditions.

Since the problem is basically a two-state problem with a periodic and a chaotic state, the probability of being in state $i=P$ (periodic) or $i=C$ (chaotic) can be expressed as a ratio of the average lifetimes:

$$f_i(\sigma) = \frac{\tau_i(\sigma)}{\tau_P(\sigma)+\tau_C(\sigma)}. \tag{15}$$

The lifetime τ_P about the periodic attractor is given by the Arrhenius factor Eq. (11). The lifetime about the chaotic set depends nonexponentially on the noise strength and can be considered to be constant. It is practically the average lifetime τ_C of transient chaos in the deterministic case. This lifetime can be approximated as the reciprocal of the escape rate κ [25] of the nonattracting chaotic set, i.e., $\tau_C\approx 1/\kappa$. Taking this into account in the weak noise limit $\Delta\Phi/\sigma^2\gg 1$, we have

$$f_P(\sigma)\approx 1, \quad f_C(\sigma)\approx\frac{\tau_C}{\tau_0}e^{-\Delta\Phi/\sigma^2}. \tag{16}$$

If the value of $f_C(\sigma)$ falls below the threshold χ, mentioned in Sec. III A, $f_C(\sigma)$ is unmeasurable in practice. To express this, we rewrite $f_C(\sigma)$ as

$$f_C(\sigma) = \frac{\tau_C}{\tau_0}e^{-\Delta\Phi/\sigma^2} - \chi$$

$$\text{for} \quad \frac{\tau_C}{\tau_0}e^{-\Delta\Phi/\sigma^2} > \chi. \tag{17}$$

By identifying normalization constant Z in Eq. (12)) with τ_C/τ_0, we see that $f_C(\sigma)>0$ for $\sigma>\sigma_c$. Deterministic time-continuous flows always have a zero Lyapunov exponent, the case of stochastic flows and maps should therefore be considered separately.

1. Flows

Consider, for example, a three-dimensional flow described by Eq. (2). Let $\lambda_3^P \le \lambda_2^P < \lambda_1^P = 0$ and $\lambda_3^C < \lambda_2^C = 0 < \lambda_1^C$ be the Lyapunov spectra of the periodic attractor and of the chaotic saddle, respectively, in the absence of noise. Let $\lambda_3 < \lambda_2 < \lambda_1$ be the Lyapunov spectrum of the noisy system.

In the two-state approximation, the Lyapunov spectrum can be written as

$$\lambda_1(\sigma) = f_P(\sigma)\lambda_1^P + f_C(\sigma)\lambda_1^C = f_C(\sigma)\lambda_1^C,$$

$$\lambda_2(\sigma) = f_P(\sigma)\lambda_2^P + f_C(\sigma)\lambda_2^C = f_P(\sigma)\lambda_2^P \approx \lambda_2^P < 0,$$

$$\lambda_3(\sigma) = f_P(\sigma)\lambda_3^P + f_C(\sigma)\lambda_3^C \approx \lambda_3^P + f_C(\sigma)\lambda_3^C < 0. \quad (18)$$

Because of the averaging effect of noise, we expect the dependence on noise of the largest Lyapunov exponent λ_1^C of the original chaotic set to be weak. Thus the main dependence of λ_1 on noise comes from $f_C(\sigma)$, the frequency of visit to the originally non-attracting chaotic set.

For $\sigma < \sigma_c$, the noisy attractor is only a flattened version of the original periodic attractor and we have $\lambda_i = \lambda_i^P$ $(i=1,2,3)$ since $f_C(\sigma)=0$. In particular, there is still a null Lyapunov exponent $\lambda_1 = 0$, despite the presence of noise, indicating that the topology of the flow is preserved. The critical noise strength σ_c is set by the condition that an intermittent hopping of the trajectory between regions that contain the original periodic attractor and the chaotic saddle becomes observable.

For $\sigma > \sigma_c$, we have $\lambda_1(\sigma) \approx f_C(\sigma)\lambda_1^C > 0$. We see that, immediately after the noise strength exceeds the critical value σ_c, the noisy attractor is chaotic in the sense that its largest Lyapunov exponent becomes positive. For $\sigma > \sigma_c$, the periodic attractor and the chaotic saddle are dynamically connected but, for σ slightly above σ_c, a trajectory visits the chaotic saddle only occasionally. Under this circumstance the sets can be regarded as distinct but only in an approximate sense. That is, Eq. (18) is valid only for σ slightly above σ_c.

Since σ_c is defined as the critical noise amplitude for which $\tau_C/\tau_0 \exp(-\Delta\Phi/\sigma_c^2) = \chi$ [Eq. (12)] holds, we can write

$$\lambda_1(\sigma) = \lambda_1^C f_C(\sigma) = \lambda_1^C \chi (e^{-\Delta\Phi(\sigma^{-2}-\sigma_c^{-2})} - 1). \quad (19)$$

For σ slightly above σ_c the exponent is $\Delta\Phi(\sigma^2 - \sigma_c^2)/(\sigma^2\sigma_c^2) \approx 2\Delta\Phi(\sigma-\sigma_c)\sigma_c^{-3}$, and the exponential function can be expanded to yield

$$f_C(\sigma) \sim \Delta\Phi \frac{\sigma - \sigma_c}{\sigma_c}, \quad (20)$$

which leads to

$$\alpha = 1, \quad (21)$$

independent of any system details.

2. Maps

For map Eq. (3), we have a negative largest Lyapunov exponent on the periodic attractor. Let $\lambda_1^P < 0$ and $\lambda_1^C > 0$ denote the largest Lyapunov exponent of the periodic attractor and of the nonattracting chaotic set, respectively, in the absence of noise. The largest Lyapunov exponent of the noisy system is denoted by λ_1. We shall see that noise-induced chaos sets in for weak noise only if $|\lambda_1^P|/\lambda_1^C \ll 1$ is fulfilled.

For $\sigma < \sigma_c$, we have $\lambda_1 = \lambda_1^P$. For $\sigma > \sigma_c$, there is an intermittent hopping of the trajectory between regions that contain the original periodic attractor and the nonattracting chaotic set. In the two-state approximation, we have

$$\lambda_1(\sigma) = f_P(\sigma)\lambda_1^P + f_C(\sigma)\lambda_1^C \approx \lambda_1^P + f_C(\sigma)\lambda_1^C. \quad (22)$$

For $\sigma < \sigma_c$, $f_C(\sigma)$ is negligible and we have $\lambda_1 = \lambda_1^P$. For σ slightly above σ_c, the second term in Eq. (22) has a positive contribution but the Lyapunov exponent is still negative. It is at a somewhat larger $\sigma = \sigma_c'$ where the Lyapunov exponent changes sign. We thus have $f_C(\sigma_c')\lambda_1^C = |\lambda_1^P|$.

The critical noise strength at which the largest Lyapunov exponent vanishes in maps thus fulfills, according to Eqs. (17) and (22), the following relation:

$$e^{-\Delta\Phi/\sigma_c'^2} = \frac{\tau_0}{\tau_C}\left(\frac{|\lambda_1^P|}{\lambda_1^C} + \chi\right). \quad (23)$$

For σ close to σ_c' we obtain

$$\lambda_1(\sigma) \approx (|\lambda_1^P| + \lambda_1^C\chi)(e^{-\Delta\Phi(\sigma^{-2}-\sigma_c'^{-2})} - 1). \quad (24)$$

Note that if $\chi \ll |\lambda_1^P|/\lambda_1^C \ll 1$, the role of the observational threshold χ becomes negligible. Equation (24) has the same σ-dependence as in Eq. (19) and leads to $\lambda_1(\sigma) \sim (\sigma - \sigma_c')$, i.e., again to the exponent $\alpha = 1$. We see that the critical noise strength σ_c at which points become observable on the noise-induced attractor is not the same as the one (σ_c') above which the largest Lyapunov exponent is positive. This situation is somewhat different from flows, where the two critical noise values are approximately the same. This is due to the fact that in a Poincaré map, noise acts rarely on the system dynamics, roughly once every oscillation cycle of the flow. The two critical values are, however, of the same order of magnitude.

3. Remarks

In fact, the argument presented above applies to any physical quantity A which takes on values A^P and A^C on the original periodic attractor and on the nonattracting chaotic set, respectively. The noise dependence of the average value $A(\sigma)$ of A is then

$$A(\sigma) \approx f_P(\sigma)A^P + f_C(\sigma)A^C. \quad (25)$$

An application of this rule leads to a surprising result in a model of Brownian motion in a symmetric periodic potential in the presence of bias and periodic diving [32]. In particular, the effect of driving pushes the system out of thermal equilibrium even in the presence of temperature fluctuations. For the noiseless system at positive bias, there is a periodic attractor leading to a negative average velocity $v^P<0$ of particles. At the same parameters, there is a coexisting chaotic saddle to which a positive average velocity $v^C>0$ belongs. This, however, is not observable with typical initial conditions. The presence of noise connects the two original invariant sets and leads to noise-induced chaos. In the context of transport, the main interest is, however, in the average velocity $v(\sigma)$, which follows from Eq. (25) with $A^P=v^P$ and $A^C=v^C$. It has the property that $v(\sigma)$ changes sign at a critical value σ_c^v. The behavior of the average velocity about the critical point is linear: $v(\sigma)\sim\sigma-\sigma_c^v$, analogous to the largest average Lyapunov exponent.

IV. GEOMETRY OF NOISE-INDUCED CHAOTIC ATTRACTORS AND APPLICATIONS TO BIOLOGY

A. Fractal properties of noise-induced chaotic attractors

A noise-induced chaotic attractor lies in the union of the periodic attractor and the unstable manifold of the nonattracting chaotic set. Since a periodic attractor is a zero-dimensional object on a Poincaré plane, the box-counting and information dimensions D_0 and D_1 of the noise-induced chaotic attractor are the same as those of the unstable manifold of the non-attracting chaotic set in the absence of noise. For example, for a two-dimensional invertible map, the information dimension of the noise-induced attractor is

$$D_1 = 1 + \frac{\lambda_1 - \kappa}{|\lambda_2|}, \tag{26}$$

where κ is the escape rate of the nonattracting chaotic set. Note that the dimension is independent of the noise strength σ, a valid property in the weak-noise limit. In fact, the information dimension of the noise-induced chaotic attractor is determined uniquely by the parameters of the nonattracting chaotic set in the underlying deterministic system.

It is the fractal property of the attractor which can be used as a condition to assess whether noise is weak. To illustrate this, we consider the map [33]:

$$\theta_{n+1} = \theta_n + 1.32 \sin 2\theta_n - 0.9 \sin 4\theta_n - y_n \sin \theta_n + \sigma\xi_n^{(1)},$$

$$y_{n+1} = -0.9 \cos \theta_n + \sigma\xi_n^{(2)}, \tag{27}$$

where $x\equiv\theta/(2\pi)$ and $\xi_n^{(i)}$ are random Gaussian variables. The system has two coexisting attracting fixed points at (0, −0.9) and (0.5, 0.9), respectively, which are separated by a chaotic saddle whose unstable manifold consists of S-shaped curves, foliations of which are approximately orthogonal to those of the basin boundary, as shown in Fig. 2. A noise-induced chaotic attractor, which extends along the unstable manifold of the chaotic saddle, is shown in Fig. 2(b). As can be seen from Fig. 3, which shows the results of the box-counting algorithm, noise makes the dynamics space filling on small phase-space scales, less than $\varepsilon_c\approx e^{-4}=0.018$ for $\sigma=0.01$. For weak noise, there is always a scaling region, although short, with slope given by the noise-free fractal dimension, which is $D_0=1.5$ in this case. When this scaling region disappears, it is no longer possible to identify the fractality of the noise-induced chaotic attractor, even on larger phase-space scales, as is the case for $\sigma=0.03$. In fact, in this case, noise smears out the dynamics into large, finite bands of the phase space. This indicates that noise begins to dominate the dynamics and, when this happens, noise can be considered as strong.

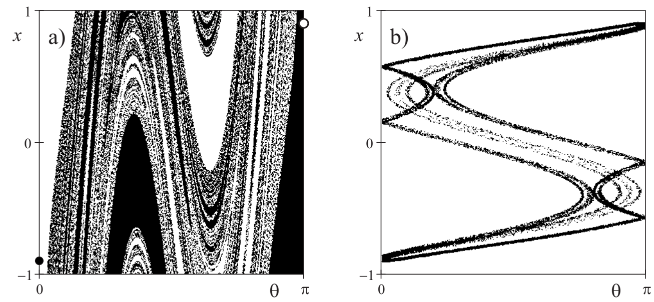

FIG. 2. Noise-induced chaos in the map system Eq. (27), (a) deterministic case, two fixed-point attractors (denoted by large black and blank dots) and their basins of attraction, and (b) noise-induced chaotic attractor.

B. Applications to biology

The concept of noise-induced chaos can play an important role in the dynamical evolution of biological systems since random environmental influences are always present [34–36]. Here we give a few examples:

(1) *Epidemiology*, the controversy between the unpredictability observed in records of chickenpox data and the non-

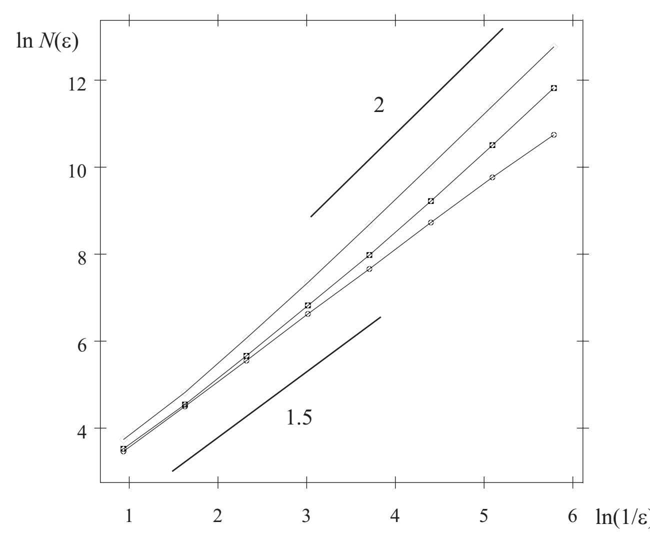

FIG. 3. Results of the box-counting algorithm for the deterministic unstable manifold of Fig. 2(a) (dots) and for the noisy attractor: filled squares, $\sigma=0.01$, as in Fig. 2(b), and filled diamonds, $\sigma=0.03$. The slopes of the thick solid lines represent the fractal dimension $D_0=1.5$ of the unstable manifold of the chaotic saddle in the deterministic system and the phase-space dimension $D=2$. The threshold scale beyond which fractality holds is $\varepsilon_c\approx0.018$ for $\sigma=0.01$. For $\sigma=0.03$, such a threshold value does not exist, indicating that noise dominates the dynamics. Similar plot can be obtained for the scaling of the information dimension of the unstable manifold as determined by Eq. (26).

chaotic nature of the attractor from the mathematical models for realistic parameter values was first resolved by Rand and Wilson [37] who pointed out that weak intrinsic or external noise can convert a chaotic saddle of the model into a noisy chaotic attractor. Noise-induced chaos has proven to be an ubiquitous source of unpredictability in epidemics since then [12,38].

(2) *Physiology*, it has been suggested that pathological destruction of chaotic behavior may induce some types of brain seizures [39] and heart failures [40]. In vital physiological systems chaotic dynamics can in fact be considered as "normal" [41]. Bifurcations to periodic behavior are viewed as a physiological loss of the range of adaptive possibilities [42]. In these situations the presence of noise can be advantageous as it can help induce or restore chaos.

(3) *Ecology*, population dynamical models sometimes also predict regular long-time behavior although the observations find irregular dynamics. Here we present the model of Ellner and Turchin [35] to describe the population dynamics of fennoscandian voles. The time-continuous equations of motions for the scaled prey (vole) density, n, and predator (weasel) density, p, are

$$\frac{dn}{dt} = 4.5n[1 - \sin(2\pi t) - n] - \frac{gn^2}{n^2 + 0.01} - \frac{8np}{n + 0.04},$$

$$\frac{dp}{dt} = 1.25p\left[1 - \sin(2\pi t) - \frac{p}{n}\right], \qquad (28)$$

where the parameters are taken from [43]. The seasonal variation has the period of $t=1$ year. A stroboscopic section is taken with a sampling of once per year (at $t=1,2,\cdots$), generating an invertible two-dimensional map. The attractor of the deterministic problem for $g=0.12$ is a 13-cycle [35]. Figure 4 demonstrates a chaotic saddle coexisting with the 13-cycle [44]. It is the chaotic saddle which is responsible for the appearance of noise-induced chaos described earlier [35].

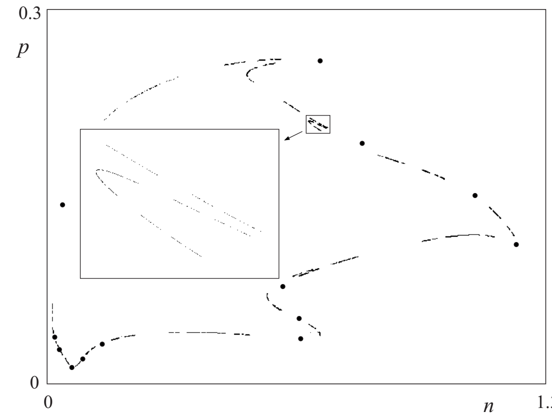

FIG. 4. A chaotic saddle from the ecological model described by Eq. (28) for $g=0.12$. It is obtained by evolving $N_0=5\times10^5$ points uniformly distributed on the rectangle: $0.001<n<1.3$ and $0<p<0.3$. The lifetime of the saddle is $\tau_C=56$ years. Trajectories not entering a circle of size 0.0005 around any of the attractor points (shown by black dots) up to $n_0=100$ years are kept and their points taken at year $n=25$ provide a good approximation to the saddle. The inset shows a magnification of part of the saddle, which exhibits fractal features.

V. CONCLUSIONS

In conclusion, we have used the tool of quasipotentials to explore critical behaviors associated with noise-induced chaos. While our analysis leads to the same scaling law as obtained previously, the use of the quasipotential concept places the law in a firmer setting. We have also explored fractal properties of noise-induced chaotic attractors. Noise-induced chaos is a generic phenomenon in realistic dynamical systems and we have pointed out a number of applications in biological sciences.

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