P H Y S I C A L   R E V I E W   L E T T E R S

Poincaré Recurrences from the Perspective of Transient Chaos

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We obtain a description of the Poincaré recurrences of chaotic systems in terms of the ergodic theory of transient chaos. It is based on the equivalence between the recurrence time distribution and an escape time distribution obtained by leaking the system and taking a special initial ensemble. This ensemble is atypical in terms of the natural measure of the leaked system, the conditionally invariant measure. Accordingly, for general initial ensembles, the average recurrence and escape times are different. However, we show that the decay rate of these distributions is always the same. Our results remain valid for Hamiltonian systems with mixed phase space and validate a split of the chaotic saddle in hyperbolic and nonhyperbolic components.

The idea of recurrence to a certain region of the phase space was introduced by Poincaré in his famous studies of the three body problem as a stability criterion [1]. The wider implications of his results were soon recognized and Poincaré recurrences have played an important role ever since the first debates on the foundation of nonequilibrium processes [2–4]. More recently, recurrences have become a standard tool to investigate low-dimensional *closed* chaotic systems [5–13].

Open systems, on the other hand, are usually investigated as scattering problems: the long-lasting chaotic transients are related to an invariant fractal saddle [14–16] and only short time escapes depend on the initial ensemble. A sharp distinction between open and closed systems has existed since the time of Poincaré [1].

Closed systems can be converted into open ones by defining a finite region of the phase space as *a leak*. Leaking dynamical systems mimics the effect of experimental observations [17–19] and has also been applied as a tool to investigate the dynamics of closed systems [20–22].

In this Letter, we find the complete correspondence between the recurrence and the leak problems. We then apply the transient chaos theory of open systems to describe Poincaré recurrences of closed systems. More precisely, considering the recurrence region to be the leak, we show that for chaotic (both dissipative and Hamiltonian) systems (i) the exponential relaxation rate of the recurrence problem and the escape rate from the leaked system are always identical, (ii) the relaxation rate can be expressed by the conditionally invariant measure [23] of the leaked region, (iii) under a properly chosen initial ensemble of the escape problem, the entire recurrence and escape time distributions coincide, and (iv) the points above remain valid even in Hamiltonian systems with mixed phase space, where for recurrence or leak regions far from Kolmogorov-Arnold-Moser (KAM) islands [15,16], the escape rate and

an associated hyperbolic saddle are shown to be well defined for times shorter than a crossover time.

To be specific, consider a chaotic map M defined on a bounded phase space Γ with an invariant chaotic set: a chaotic attractor or a chaotic sea. The natural ergodic measure on this invariant set is denoted by μ, its density by ρ_μ. The recurrence region is a subset $I \subset \Gamma$. We *do not* restrict ourselves to small $\mu(I)$ [10–12]. The Poincaré recurrence theorem [2–4] ensures that for almost any initial condition in I, there is a first recurrence at some discrete time n_1, a second one at time $n_1 + n_2$, etc. In the long time limit, the set of recurrence times n_i defines the recurrence time distribution $p_r(n)$. For chaotic systems [4,9,11], this distribution is exponential for n larger than some small n_r^*

$$p_r(n) \approx g_r e^{-\gamma_r n}. \qquad (1)$$

Prefactor g_r and the relaxation rate γ_r depend on the choice of I. While the average recurrence time $\langle n \rangle_r$ is given by Kac's lemma [2],

$$\langle n \rangle_r \equiv \sum_{n=1}^{\infty} n p_r(n) = \frac{1}{\mu(I)}, \qquad (2)$$

it has been unknown whether the relaxation rate γ_r can be expressed by means of any measure of I. The equality $\gamma_r = 1/\langle n_r \rangle = \mu(I)$ is proved only in the unrealistic limit $\mu(I) \to 0$ (see [9], and references therein).

Open (transiently) chaotic maps are characterized by the escape time distribution $p_e(n)$, which gives the fraction of trajectories of a certain initial ensemble that leaves the system at time n. This distribution is also exponential [15,16]. We are interested in the class of open systems obtained by leaking a closed system at a region $I \subset \Gamma$, described by map $M^\dagger(\vec{x}) = M(\vec{x})$ for $\vec{x} \in \Gamma \setminus I$ and $M^\dagger(\vec{x}) = $ "exit" for $\vec{x} \in I$. Notice that the escape happens one step *after* entering I, and initial conditions can be in I. In particular, for such a leaked system, we have

$$p_e(n) \approx g_e e^{-\gamma_e n}, \quad (3)$$

for $n \geq n_e^*$, where γ_e is the escape rate. The average escape time $\langle n \rangle_e$, also called the lifetime of chaos, is usually estimated by the reciprocal of γ_e [15,16]. While the full $p_e(n)$, and the average $\langle n \rangle_e$, depend also on the choice of the *initial density* ρ_0 used in generating $p_e(n)$, the escape rate does not (provided it exists). All these quantities depend on I. The theory of transient chaos [15,16] explains these results based on the existence of an invariant chaotic saddle in the complement set of the leak $\Gamma \setminus I$ [24]. The saddle is the union of all trajectories that *never* enter the leak, neither forward nor backward in time. It is a fractal set of measure zero for strongly chaotic systems. Trajectories spending a long time outside I must come close to the saddle along its stable manifold, stay in the vicinity of the saddle, and leave it along its unstable manifold. Therefore, irrespective of ρ_0, the long-term emptying process is governed by the invariant saddle, which is uniquely defined by I.

We now apply this picture to Poincaré recurrences. Long recurrences should correspond to trajectories that fall near to the chaotic saddle's stable manifold right after exiting the recurrence region I. Therefore, we expect the relaxation rate of the recurrence distributions γ_r and the decay rate of the escape distribution γ_e to coincide

$$\gamma_r = \gamma_e \equiv \gamma, \quad (4)$$

for a given recurrence or leak region I. This equality is illustrated for the Hénon map in Fig. 1, where it is apparent that the slopes of $\ln p_r(n)$ vs n and $\ln p_e(n)$ vs n coincide. Equation (4), which will be rigorously justified below,
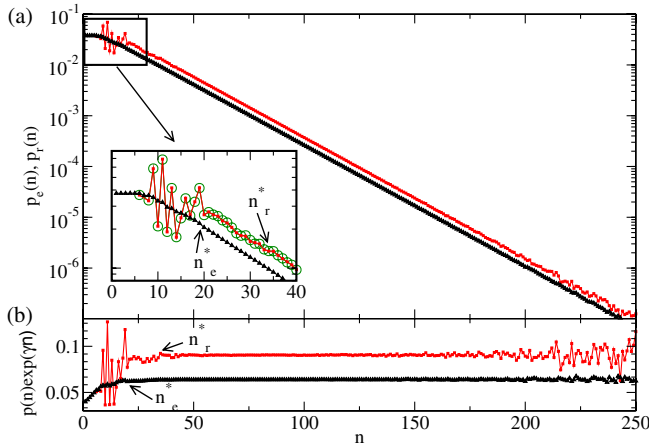

FIG. 1 (color online). (a) Recurrence $p_r(n)$ (■/above) and escape $p_e(n)$ (▲/below) time distributions in the Hénon map shown in Fig. 2. For $p_r(n)$, a single initial condition $(x_0, y_0) = (x_c, y_c)$ was iterated over 10^{11} steps. For $p_e(n)$, $\rho_0 = \rho_\mu$ was built by 7×10^9 trajectories. Inset: short time behavior, where $p_e(n)$ with $\rho_0 = \rho_r$, given by (5), is marked by ○. (b) $p_{e,r} \exp(\gamma n)$, with $\gamma = 0.055$ [$\mu_c(I) = 0.0535 < \mu(I) = 0.0377$]. Numerical limitations appear for large n.

implies that to any Poincaré recurrence problem, there exists a chaotic saddle (that of the corresponding leaked system) that governs the long-term recurrences and determines γ_r. This saddle is shown in Fig. 2(b). Its double fractal character makes evident the improvement of Eq. (4) in comparison to the previous results being limited to $\mu(I) \rightarrow 0$ [7,9].

Despite having the same exponential decay (4), $p_r(n)$ and $p_e(n)$ are different. This holds for practically any initial density ρ_0. We show now that there is, nevertheless, a special $\rho_0 = \rho_r$ [9,10] of the leaked system for which $p_e(n) = p_r(n)$. Consider inserting trajectories into the system through the leak. More precisely, consider the density $\rho_r(\vec{x})$ obtained as the first iterate through map M of the points $\vec{x} \in I$ distributed according to the natural density of the closed system ρ_μ:

$$\rho_r(\vec{x}) = \rho_\mu[M^{-1}(\vec{x}) \cap I]/\mu[M(I)] \quad \text{for } \vec{x} \in M(I), \quad (5)$$

where $M^{-1}(\vec{x}) \cap I$ denotes the points that come from I and $\mu[M(I)]$ ensures normalization. This distribution is shown in Fig. 2(a). In view of the ergodic theorem, the distribution of all positions of a single infinitely long recurrent trajectory, one iteration after returning to I, is precisely given by ρ_r. Since the escape times for ρ_r are exactly the recurrence times, it follows that

$$p_r(n) = p_e(n) \quad \text{with } \rho_0 = \rho_r,$$

for all times n [see the inset of Fig. 1(a)]. Of course, $\gamma_r = \gamma_e$ and, because γ_e is independent of ρ_0, this implies (4).

Having shown how recurrences can be viewed as escapes from a leaked system, we now perform a formal description in terms of the ergodic theory of transient chaos. A central concept in this theory is the so-called conditionally invariant measure, c measure in brief [23,25,26]. Smooth initial densities iterated via the action of map $M^\dagger$ do not converge to any finite measure due to the
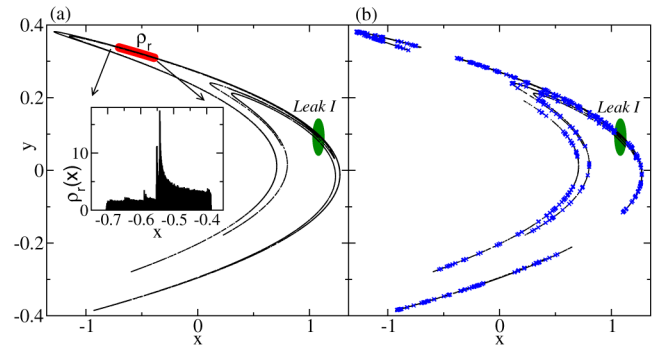

FIG. 2 (color online). Phase space of the Hénon map $(x_{n+1}, y_{n+1}) = (1 - 1.4x_n^2 + y_n, 0.3x_n)$ with I as a circle of radius 0.05 centered at $(x_c, y_c) = (1.0804, 0.0916)$. (a) Hénon attractor: support of ρ_μ (thin line) and of ρ_r (●) at $M(I)$, see (5). Inset: $\rho_r(x, y)$ projected onto x. (b) Chaotic saddle (×) and the support of ρ_c (thin line).

permanent escape. If, however, one applies a compensation in the form of multiplying the density in each step by $\exp(\gamma) > 1$, it converges to a finite measure, the c measure μ_c. Because of the permanent contraction along the stable direction, the density ρ_c of the c measure is nonzero along the unstable manifold of the saddle [25]. Qualitatively speaking, this measure characterizes the escaping process from the saddle and is thus completely different from the natural measure of the closed system. With the normalization $\mu_c(\Gamma) = 1$, the compensation factor $\exp(\gamma)$ is the c measure of the region not escaping within one time step [23]. In the leaked system, this is $\mu_c(\Gamma \setminus I) = \mu_c(\Gamma) - \mu_c(I) = 1 - \mu_c(I)$ and therefore

$$\gamma = -\ln[1 - \mu_c(I)] \; [\approx \mu_c(I) \quad \text{for small } \mu(I)]. \quad (6)$$

Note that the c measure itself depends on the form, size, and location of I. In view of (4), it is remarkable that the relaxation rate γ_r of the Poincaré recurrences of the closed system is given by the c measure of the recurrence region, when viewed as a leaked system. Furthermore, $\rho_0 = \rho_c$ is the only initial density for which the exponential decay starts from the very first iterate ($n_e^* = 1$). As a consequence,

$$\langle n \rangle_e = 1/\mu_c(I) \quad \text{with } \rho_0 = \rho_c.$$

The reason for the strong difference between this relation and Kac's lemma (2) (valid for recurrences and for $\rho_0 = \rho_r$) is that ρ_r in (5) is atypical from the point of view of the c measure. In fact, ρ_c is concentrated along the saddle's unstable manifold while all points on ρ_r have come from the leak. Typical initial densities ρ_0 quickly converge to ρ_c, explaining why typical $\langle n \rangle_e$-s, but not $\langle n \rangle_r$, are well estimated by $1/\gamma \approx 1/\mu_c(I)$ [27].

We illustrate now through two examples the new perspectives opened by our unified approach. (i) Recently, it has been surprisingly observed that different leaks I with equal $\mu(I)$ lead to different γ_e's [21,22]. Location dependence of γ_r has also been reported [6,11,12]. In view of (4), these works refer to the same phenomenon. Furthermore, Eq. (6) explains the observations by stating that it is the c measure and not the natural measure that determines γ. (ii) A basic problem in the mathematical approach to recurrences is the convergence of $p_r(n)$ to a Poissonian [$\gamma = \mu(I)$] for $\mu(I) \to 0$ [9]. A fundamental property of the c measure is its converges to the natural measure [23,26]. Equation (6) implies that both convergences occur precisely in the same way.

We turn now to Hamiltonian systems with mixed phase space, where deviations from the exponential decay appear for long times. The trajectories that approach the KAM islands (or other nonhyperbolic structures) *stick* to them for a time that is roughly algebraically distributed [4,5,13]. This phenomenon is usually investigated using recurrences [4,5,13] but appears also in escape problems [16,28]. The asymptotic power-law like decay of $p_{r,e}(n)$ is the same and

independent of the choice of initial densities (away from islands [28]), because it is related to the nonhyperbolic structures. For shorter times, an exponential decay of $p_{r,e}(n)$ is observed [6], what has deserved little attention from the recurrence perspective so far.

Accordingly, for I outside KAM islands and for $n \gtrsim n^*$

$$p_{r,e}(n) \approx \begin{cases} ae^{-\gamma n} & \text{for } n^* < n < n_\alpha \\ ae^{-\gamma n} + b(\gamma n)^{-\alpha} & \text{for } n > n_\alpha, \end{cases} \quad (7)$$

where $ae^{-\gamma n_\alpha} \gg b(\gamma n_\alpha)^{-\alpha}$. The recurrence and escape problems can be again related using ρ_r in (5). However, the analysis based on the chaotic saddle has to be reconsidered in the presence of KAM islands. Numerical results for the standard map in Fig. 3 indicate that, for *intermediate* times, $\gamma_{r,e}$ are well defined and equal, as in (4). These results remain valid [27] for a wide range of I's and of smooth ρ_0's, away from islands (see [28] for ρ_0's touching the island).

To understand these findings, we propose here to effectively split the saddle in a hyperbolic (outside islands) and a nonhyperbolic (nearly space filling [14]) component. To justify this splitting, it is worth defining a crossover time $n_c > n_\alpha$ as $ae^{-\gamma n_c} = b(\gamma n_c)^{-\alpha}$, starting from which the nonhyperbolic component dominates. In the upper inset of Fig. 3, we present evidence for

$$n_c \sim 1/\gamma \; [\approx 1/\mu(I) \quad \text{for small } \mu(I)]. \quad (8)$$

This scaling can be obtained from the definition of n_c by assuming a weak dependence of b/a on γ. Equation (8) implies that the exponential decay is always dominant for small I and that hyperbolic saddles and c measures can be
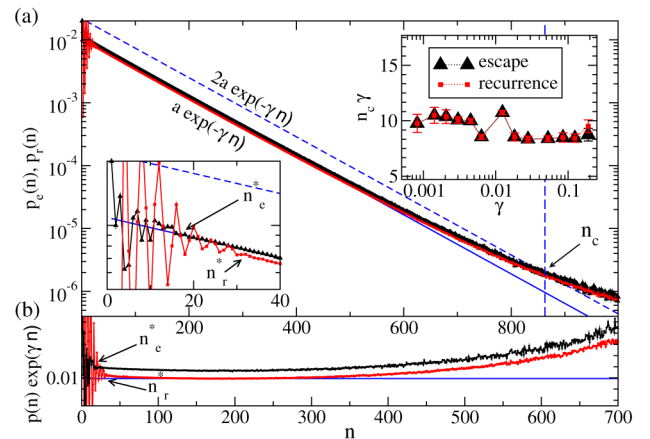

FIG. 3 (color online). (a) Recurrence $p_r(n)$ (■/below) and escape $p_e(n)$ (▲/above) time distributions in the standard map shown in Fig. 4 with $\varepsilon = 0.1$. For $p_r(n)$, a single initial condition was iterated over 10^{11} steps. For $p_e(n)$, $\rho_0 = \rho_\mu$ at $|x| > 0.25$ was built by 2×10^8 trajectories. Lower inset: short time behavior. Upper inset: scaling of $n_c\gamma$ with γ, obtained by changing ε in $0.03 \le \varepsilon \le 0.4$. (b) $p_{e,r}\exp(\gamma n)$, with $\gamma = 0.0108$ [$\mu_c(I) = 0.0107 < \mu(I) = 0.0113$].
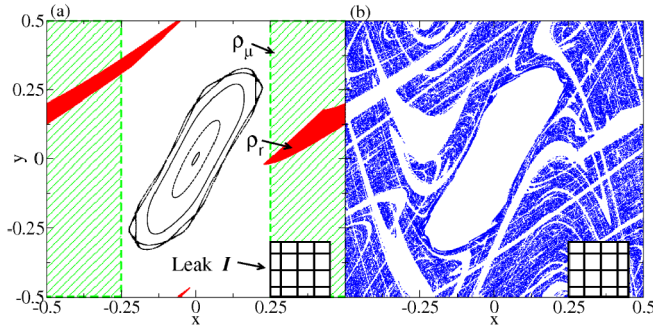
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FIG. 4 (color online). (a) Phase space of the standard map $(x_{n+1}, y_{n+1}) = [x_n + y_{n+1}, y_n - 0.52 \sin(2\pi x_n)]$, periodic in $|x, y| < 0.5$. $I = [0.25 < x < 0.25 + \varepsilon, -0.5 < y < -0.5 + \varepsilon]$, $\varepsilon = 0.2$ (⊞). The KAM island (nonhyperbolic component of the saddle) is at the center. The support of ρ_r is at $M(I)$ (dark/red region). (b) The hyperbolic component of the saddle generated by the method of [16], using $n = 60 < n_c = 160$.

effectively defined for intermediate times $n^* < n < n_c$ [see Fig. 4(b)]. An important difference between the manifolds of the hyperbolic and nonhyperbolic components is that they attract or repel exponentially and subexponentially, respectively. The picture that emerges is that ρ_0 relaxes to the nonhyperbolic component only after, and mainly via, the hyperbolic one. Therefore, b/a hardly depends on ρ_0 and, consequently, neither does n_c. This explains scaling (8) and also why the absolute value of n_c is approximately the same for escape and recurrence (see the upper inset of Fig. 3).

In summary, we have presented a new interpretation of recurrences based on the theory of transient chaos. In particular, we have expressed the exponential relaxation of the recurrence time distribution in terms of properties of chaotic saddles. In addition, in Hamiltonian systems with mixed phase space, we could properly order the exponential and power-law decays, respectively, to hyperbolic and nonhyperbolic components of the saddle. Our results hold in continuous time and in any dimension, being thus valid, e.g., in classical many particle systems.

So far, we have emphasized that Poincaré recurrences can be viewed as an escape problem. However, in view of the correspondence between the two problems presented here, also the results on recurrences [8–12] apply to leaked systems. Furthermore, the split of the chaotic saddle in Hamiltonian systems with mixed phase space provides a theoretical description with practical applications in problems like resetting in hydrodynamics [20] and emission from optical cavities [19].

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