Chaotic Systems with Absorption

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Motivated by applications in optics and acoustics we develop a dynamical-system approach to describe absorption in chaotic systems. We introduce an operator formalism from which we obtain (i) a general formula for the escape rate κ in terms of the natural conditionally invariant measure of the system, (ii) an increased multifractality when compared to the spectrum of dimensions D_q obtained without taking absorption and return times into account, and (iii) a generalization of the Kantz-Grassberger formula that expresses D_1 in terms of κ, the positive Lyapunov exponent, the average return time, and a new quantity, the reflection rate. Simulations in the cardioid billiard confirm these results.

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The design of concert halls was probably the first problem in which the importance of the partial absorption of energy along trajectories was fully recognized [1,2]. In Berry's elegant formulation, "confinement is needed to prevent sound from being attenuated by radiating into the open air. But if the confinement were perfect, that is, if the walls of the room were completely reflecting, sounds would reverberate forever. To avoid these extremes, the walls in a real room must be partially absorbing" [3]. Besides acoustics [1–7], chaotic dynamical systems in which trajectories are partially absorbed appear nowadays in an increasing number of different areas [8], ranging from optics (microlasers) [9,10] to environmental sciences (resetting mechanism) [11] and quantum chaos [12]. The analogy of the decay of the sound intensity with the survival probability of transient chaos has early been recognized [7]; here, we add that a sharp distinction between the attenuation of energy (absorption) and the escape of particles (transport) is necessary.

A seemingly unrelated problem is monitoring continuous time in flows represented by discrete-time maps $\vec{x}_{n+1} = f(\vec{x}_n)$ through a proper Poincaré surface of section. Both problems can be handled extending the phase space of map f [13] to include the true physical time t_n and the ray intensity J_n at the nth intersection with the Poincaré section as

$$f_{\text{extended}}:\begin{cases} \vec{x}_{n+1} = f(\vec{x}_n), \\ t_{n+1} = t_n + \tau(\vec{x}_n), \\ J_{n+1} = J_n R(\vec{x}_n), \end{cases} \tag{1}$$

where the return time $\tau(\vec{x}) \geq 0$, chosen as the time between intersections $\vec{x}$ and $\vec{x}' \equiv f(\vec{x})$, and the reflection coefficient $0 < R(\vec{x}) \leq 1$ are functions of the coordinate $\vec{x}$ on the Poincaré section. Probably the most prominent systems incorporating both properties are billiards such as the one in Fig. 1. Concert halls can be modeled as 3D billiards [1]. Trajectory-based simulations (ray tracing [4,9]) in these systems are performed from Eq. (1) by tracking t and J along each trajectory.

In this Letter, we show that absorption and true time lead to surprising modifications of fundamental results of chaotic dynamics. This is done by introducing an operator-based formalism. We use it to derive an expression for the escape rate κ as a function of the natural conditionally invariant measure of the system. As a consequence, we show that κ depends on the entire distributions of $\tau(\vec{x})$ and $R(\vec{x})$ and not only on their averages. In terms of the spectrum of fractal dimensions D_q of the invariant sets, we show that τ and R typically enhance multifractality and that D_1 can be expressed as a function of κ, the average Lyapunov exponent, and a new parameter.

We start with the well-known operator formalism for open maps [13–16]. The escape rate κ of an open (possibly noninvertible) map f is related to the largest eigenvalue $e^{-\kappa}$ of the Perron-Frobenius operator acting on the density of trajectories ϱ,

$$\varrho_{n+1}(\vec{x}') = \sum_{\vec{x} \in f^{-1}(\vec{x}')} e^{\kappa} \frac{\varrho_n(\vec{x})}{|\mathcal{D}_f(\vec{x})|}, \tag{2}$$

where $\mathcal{D}_f(\vec{x})$ is the Jacobian at $\vec{x}$. Equation (2) expresses that the probability in a small region around $\vec{x}$ at step n is the same as the f image of that region at step $n+1$, when compensating for the escape. κ follows from the requirement that the integral of ϱ_n over a fixed phase space region containing the underlying nonattracting chaotic set (a repeller or a saddle) remains finite for $n \to \infty$. In this limit, $\varrho_n \to \varrho_c$ concentrates on the unstable manifold of the chaotic saddle according to the conditionally invariant measure (c measure) [14].

We now introduce an operator formalism for the extended map [Eq. (1)]. Imposing a uniform decay of

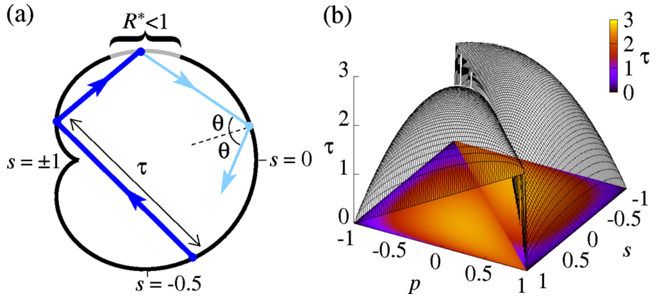

FIG. 1 (color online). Billiards naturally incorporate both partial reflection at the boundary and nontrivial return times between collisions. (a) Cardioid billiard, whose boundary in polar coordinates is $r(\phi) = 1 + \cos(\phi)$ [24]. The intensity J of the rays decays due to $R(\vec{x}) = R^* < 1$ in the gray boundary interval at the top ($R(\vec{x}) = 1$ otherwise). (b) Return time distribution $\tau(\vec{x})$ in the cardioid billiard (velocity modulus is chosen to be unity). Birkhoff coordinates $\vec{x} = (s, p = \sin\theta)$ are used where s is the arc length along the boundary and θ is the collision angle.

trajectories in time t, instead of the number n of iterations, it is natural to replace e^{κ} in Eq. (2) by $e^{\kappa\tau(\vec{x})}$. The reflection coefficient $R(\vec{x})$ corresponds to an immediate loss of intensity and is therefore introduced also on the right hand side of Eq. (2). Altogether, the density function ρ of map [Eq. (1)] evolve as

$$\rho_{n+1}(\vec{x}') = \sum_{\vec{x}\in f^{-1}(\vec{x}')} e^{\kappa\tau(\vec{x})} \frac{R(\vec{x})\rho_n(\vec{x})}{|\mathcal{D}_f(\vec{x})|}. \quad (3)$$

This operator generalizes the true-time formalism of Gaspard [17] and Kaufmann and Lustfeld [18] by introducing reflection in a similar spirit as in Tanner's work on driven acoustic systems [4,5]. Among the different generalized transfer operators [19] and other possible generalizations of Eq. (2), Eq. (3) is the one that remains faithful to the physical picture used in the extension of maps f to extended maps f_{extended} in Eq. (1). Indeed, the operator we recently introduced [8] differs from Eq. (3) precisely because of the different convention of τ (defined as a function of the endpoint $\vec{x}'$) in Eq. (1). Equation (3) is an extension to noninvertible maps of this previously defined operator. For $n \to \infty$, ρ_0 approaches a limit distribution ρ_∞ (of finite integral), which is ρ_c associated to the c measure μ_c of the extended map [Eq. (1)], normalized over the region of interest Ω on the Poincaré map. The support of ρ_c and ϱ_c from Eq. (2) coincide, but the densities are typically different. In open systems there is a region of escape $E \subset \Omega$ in which trajectories escape Ω within one iteration of the Poincaré map f. Because this escape is not due to absorption and happens instantaneously, we choose $R(\vec{x}) = 1$ and $\tau(\vec{x}) = 0$ for $\vec{x} \in E$.

We can now derive a relation for κ as a function of ρ_c. By integrating, for $n \to \infty$, both sides of Eq. (3) over Ω we obtain

$$\begin{aligned} 1 &= \int_\Omega d\vec{x}' e^{\kappa\tau(\vec{x})} \frac{R(\vec{x})\rho_c(\vec{x})}{|\mathcal{D}_f(\vec{x})|}\Big|_{\vec{x}=f^{-1}(\vec{x}')} \\ &= \int_{f^{-1}(\Omega)} d\vec{x} R(\vec{x}) e^{\kappa\tau(\vec{x})} \rho_c(\vec{x}) \\ &= \int_\Omega d\vec{x} R(\vec{x}) e^{\kappa\tau(\vec{x})} \rho_c(\vec{x}) - \int_E d\vec{x} \rho_c(\vec{x}). \end{aligned} \quad (4)$$

We used $|\mathcal{D}_f(\vec{x})| = |d\vec{x}'|/|d\vec{x}|$, and the fact that $f^{-1}(\Omega) \cap \Omega = \Omega \setminus E$. After rearrangement

$$\langle Re^{\kappa\tau}\rangle_c = 1 + \mu_c(E), \quad (5)$$

where $\langle \ldots \rangle_c \equiv \int_\Omega \ldots d\mu_c = \int_\Omega \ldots \rho_c(\vec{x}) d\vec{x}$. This new implicit formula for κ involves the c measure of map (1) and contains both $\tau(\vec{x})$ and $R(\vec{x})$. It generalizes the Pianigiani-Yorke formula [14] $\kappa = -\ln[1 - \mu_c(E)]$ valid for usual maps, for which τ, $R \equiv 1$ for $\vec{x} \in \Omega \setminus E$ while $\tau = 0$, $R \equiv 1$ for $\vec{x} \in E$. To see this, notice that Eq. (5) can be written as $\langle Re^{\kappa\tau}\rangle_c = e^{\kappa}[1 - \mu_c(E)] + \mu_c(E) = 1 + \mu_c(E)$.

We now explore the implications of Eq. (5). As an approximation of a closed concert hall, consider the case of closed systems ($E = \emptyset$) with homogeneous absorption [$R(\vec{x}) = R < 1$] and nontrivial τ's, in which case (5) becomes $\langle e^{\kappa\tau}\rangle_c = 1/R$. Consider the cumulant expansion $\ln\langle e^{\kappa\tau}\rangle_c = \sum_{r=1}^{\infty} (\kappa)^r C_r(\tau)/r!$, where C_r are the cumulants of τ with respect to the c measure. The $r = 1$ approximant of κ is $\kappa_1 = -\ln R/\langle\tau\rangle_c$. For $R \to 1$, we obtain $\langle\tau\rangle_c \to \langle\tau\rangle$ and $\kappa_1 \approx (1 - R)/\langle\tau\rangle$, which corresponds to Sabine's celebrated formula for the reverberation time [1], where $\langle\tau\rangle$ is the closed billiard average return time. The $r = 2$ approximant is

$$\kappa_2 = \frac{\sqrt{\langle\tau\rangle_c^2 - 2\sigma_c^2 \ln R} - \langle\tau\rangle_c}{\sigma_c^2} \approx \kappa_1\left(1 - \frac{\kappa_1}{2}\frac{\sigma_c^2}{\langle\tau\rangle_c}\right), \quad (6)$$

where the approximation is valid for small variance σ_c of τ and was obtained in different contexts [1,6]. The accuracy of these expressions depends on the rate of convergence of $\kappa_r \to \kappa$; see Supplemental Material [20] for details and general cases.

The importance of our general and exact formula (5) becomes clear in view of Joyce's pessimistic conclusion from 1975: "It is further proven that the functional form of Sabine's expression cannot be modified so as to become correct for large absorption" [2]. While this negative result is an unavoidable consequence of the argumentation being restricted to the properties of closed dynamics, Eq. (5) provides the answer to Joyce's search based on the modern theory of open dynamical systems [14,16].

We now turn to the effect of R and τ on the spectrum of fractal dimensions D_q. In a closed system ($E = \emptyset$) trajectories visit the whole phase space and thus D_0 equals the phase space dimension. We argue below that a nontrivial D_q (multifractality) is obtained even in this case, and that

D_q depends on both R and τ. We illustrate this through four examples (I–IV) with increasing complexity.

I. Consider the tent map $f(x) = ax$ with $a > 2$ for $x < 1/2$, and $f(x) = a(1-x)$ for $1/2 \le x \le 1$. We extend $f(x)$ by adding return times $\tau(x)$ and reflection coefficients $R(x)$ which, for simplicity, are chosen to be constant on the two elements $i = 1, 2$ of the generating partition: (τ_1, R_1) on $I_1 = [0, 1/a]$, and (τ_2, R_2) on $I_2 = [1 - 1/a, 1]$. The escape region is $E = (1/a < y < 1 - 1/a)$ [where $(\tau, R) = (0, 1)$]. Direct substitution into the steady state of Eq. (3) with $|\mathcal{D}_f| \equiv |f'| = a$ shows that $\rho_c = 1$ on $x \in [0, 1]$, and that the relation for κ is

$$P_1 + P_2 = 1, \quad \text{with} \quad P_i \equiv R_i e^{\kappa\tau_i}/a. \tag{7}$$

To see that Eq. (7) is consistent with Eq. (5), notice that there are only three intervals (I_1, I_2, and E) with different $Re^{\kappa\tau}$ and, due to the constancy of ρ_c, their c measure equal their length. It follows that $\mu_c(E) = 1 - 2/a$ and $\langle Re^{\kappa\tau}\rangle_c = R_1 e^{\kappa\tau_1}/a + R_2 e^{\kappa\tau_2}/a + (1 - 2/a) = 2 - 2/a = 1 + \mu_c(E)$, where Eq. (7) was used. P_i in Eq. (7) can be interpreted as the proportion of weighted trajectories, initiated uniformly in I_i, after one iteration of Eq. (3). Analogously, the weights on the preimages of I_i of length $1/a^2$ are P_1^2, P_2P_1, P_2^2, and P_1P_2. As will be clear from example II, the continuation of this procedure provides a multifractal measure, μ different from μ_c, which corresponds to the weights on small intervals covering the never escaping points (chaotic repeller).

II. Consider general noninvertible expanding maps $f(x)$ defined on $x \in [0, 1]$ with general $\tau(x)$ and $R(x)$. In the most typical single humped family, the nth preimages, of number 2^n, of the unit interval are the so-called cylinders $I_i^{(n)}$ [16]. In general, ρ_c is not constant, but is continuous and covers $x \in [0, 1]$. A fractal measure μ can be found by considering the analogues of the weights P_1 and P_2 for cylinders $P(I_i^{(n)})$. For shrinking cylinders, P approximates the measure μ of the repeller $\mu_i^{(n)} \equiv \mu(I_i^{(n)}) \approx P(I_i^{(n)})$. To compute this, consider the n fold iterated map f^n and the corresponding Eq. (3). For large n the logarithm of the slope of f^n at x is approximately constant in a cylinder and therefore $P(I_i^{(n)}) = e^{n\kappa\tau_i^{(n)}} e^{n \ln R_i^{(n)}} \epsilon_i^{(n)}$, where $\epsilon_i^{(n)} = 1/|(f^n)'(x_0)| \equiv e^{-\lambda_i^{(n)} n}$ with $x_0 \in I_i^{(n)}$. In turn, $\tau_i^{(n)}$ and $\ln R_i^{(n)}$ are sums of $\tau_{1,2}$ and $\ln R_{1,2}$, respectively, over a typical trajectory $(x_0, \ldots, x_j, \ldots, x_{n-1})$ of length n divided by n. Here x_{n-1} is arbitrary, but fixed, and $x_0 = f^{-n}(x_{n-1}) \in I_i^{(n)}$ for all cylinders. Altogether,

$$\mu_i^{(n)} \sim e^{n(\kappa\tau_i^{(n)} + \ln R_i^{(n)} - \lambda_i^{(n)})} \sim \Pi_{j=0}^{n-1} \frac{e^{\kappa\tau(x_j)} R(x_j)}{|f'(x_j)|}, \tag{8}$$

which hardly depends on x_{n-1} (because of the shrinking cylinders) and differs from $\mu_c(I_i^{(n)})$ (which is proportional to $\epsilon_i^{(n)}$).

Averages of an observable A over the repeller measure μ are obtained as $\bar{A} \equiv \lim_{n\to\infty} \sum_{i=1}^{2^n} A_i^{(n)} \mu_i^{(n)}$ (e.g., the average Lyapunov exponent is $\bar{\lambda}_+ = \lim_{n\to\infty} \sum_i \lambda_i^{(n)} \mu_i^{(n)}$). The information dimension of the repeller D_1 follows from the general relation $D_1 = \lim_{n\to\infty} \sum_i \mu_i^{(n)} \ln \mu_i^{(n)} / \sum_i \mu_i^{(n)} \ln \epsilon_i^{(n)}$ [16,21]. Substituting Eq. (8), we find

$$D_1 = 1 - \frac{\kappa\bar{\tau} + \overline{\ln R}}{\bar{\lambda}_+}. \tag{9}$$

Due to the reflection rate $\overline{\ln R}$, this is a generalization of the Kantz-Grassberger relation ($D_1 = 1 - \kappa/\bar{\lambda}_+$) [22] to any chaotic 1D map with absorption. For the tent map of example I, $\lambda_+ = \ln a$, $\bar{\tau} = P_1\tau_1 + P_2\tau_2$, $\overline{\ln R} = P_1 \ln R_1 + P_2 \ln R_2$, and the order-$q$ dimension can be calculated (from $\sum_i \mu_i(\epsilon)^q \sim \epsilon^{(q-1)D_q}$ [21]) as

$$D_q = \frac{\ln(P_1^q + P_2^q)}{(1-q)\ln a}. \tag{10}$$

III. We now apply our operator formalism (3) to an invertible 2D map, the analytically solvable baker map; see Fig. 2 [16,21]. Consider initially $\rho_0 \equiv 1$. In the next step, ρ_0 is multiplied by $R_i e^{(\kappa\tau_i)}/(ab)$, $i = 1$ or 2, leading to two columns of width b parallel to the $x = 0$ and $x = 1$ axes with measures P_1 and P_2, respectively, as given in Eq. (7). The construction goes on in a self-similar way. Prescribing that the c measure corresponds to a case when the full measure $(P_1 + P_2)^n$ after $n \gg 1$ steps remains unity, Eq. (7) is recovered (the dynamics along unstable manifolds of 2D maps is faithfully represented by 1D maps). In addition, P_i's are the c measures of the columns of width b and of unit height.

Concerning the dimensions of the c measure $D_{q,c}$, we concentrate on the partial dimensions $D_q^{(2)}$ along the stable (x) direction because μ_c is constant along y and therefore $D_{q,c} = 1 + D_q^{(2)}$. After n steps, the boxes in the x direction are of length b^n and therefore $D_q^{(2)} = D_q\lambda_+/\lambda_-$, where $\lambda_- = -\ln b$ is the modulus of the contracting Lyapunov exponent and D_q is given by Eq. (10). Although Eq. (10) was obtained as D_q of the tent map repeller, an analogous

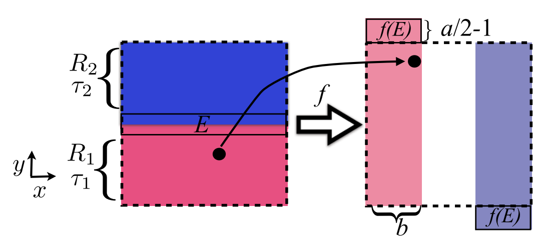

FIG. 2 (color online). Open baker map with absorption and return times. Intensity J decays due to $R < 1$. Trajectories leave the system (unit square) when $(x, y) \in E$. The extended map [Eq. (1)] is $(x', y') = (bx, ay)$ for $y < 1/2$, $(x', y') = 1 - b(1-x), 1 - a(1-y)$ for $y \ge 1/2$, $(\tau, R) = (\tau_1, R_1)$ if $y < 1/a$ and (τ_2, R_2) if $y > 1 - 1/a$. $b \le 1/2$, $a \ge 2$; $\mathcal{D}_f = ab$.

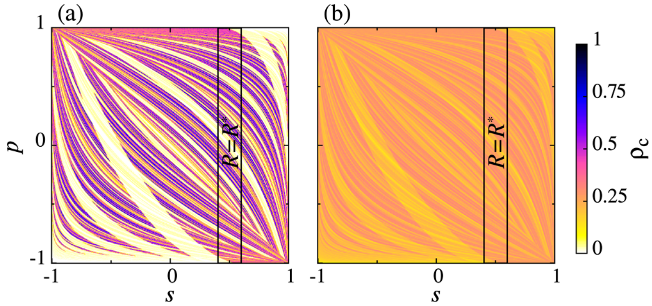

FIG. 3 (color online). Conditionally invariant density ρ_c for the cardioid billiard with absorption. As shown in Fig. 1, $R = 1$ everywhere except in $\{s \in [0.4, 0.6], p \in [-1, 1]\}$ where $R = R^* < 1$. (a) $R^* = 0.1$; and (b) $R^* = 0.75$. Structures in (b) amount to 0.2% difference between $D_{0,c}$ and $D_{1,c}$; see Table I.

procedure applied to the horizontal bands of height $(1/a)^n$ yields that the order-q dimension of the baker saddle's stable manifold is $1 + D_q$ [23]. D_q of Eq. (10) can therefore be considered to be the partial dimension $D_q^{(1)}$ along the unstable (y) direction of the baker map $D_q^{(1)} = D_q$. The dimension of the saddle is $D_q^{(1)} + D_q^{(2)}$.

As an example, consider the closed area preserving map $(a = 1/b = 2)$ with weak absorption $1 - R_i \ll 1$, for which κ is small. Assuming κ to be of the same order as $1 - R_i$, in leading order, $P_i = [1 - (1 - R_i) + \kappa\tau_i]/2$ and, from Eq. (7), $\kappa = (1 - R_1 + 1 - R_2)/(\tau_1 + \tau_2)$. Inserting $P_1 = (1 - \Delta)/2$ and $P_2 = (1 + \Delta)/2$ into Eq. (10), we obtain

$$D_q^{(1)} = 1 - q\frac{\Delta^2}{2\ln 2} \quad \text{and} \quad D_{q,c} = 2 - q\frac{\Delta^2}{2\ln 2}, \tag{11}$$

valid for $0 \le q < 1/\Delta^2$, where $\Delta = [(1 - R_1)\tau_2 - (1 - R_2)\tau_1]/(\tau_1 + \tau_2)$, and $D_q^{(2)} = D_q^{(1)}$ because $\lambda_+ = \lambda_-$. This illustrates that both inhomogeneous absorption $(R_1 \neq R_2)$ and return time $(\tau_1 \neq \tau_2)$ distributions lead to $D_q^{(i)} \neq D_0^{(i)} = 1$. Multifractality becomes stronger with increasing absorption. In contrast, for the usual closed area preserving baker map $\kappa = 0$ and $D_q^{(1,2)} = 1$, illustrating how the results from the traditional operator (2) and the generalized operator (3) can differ even for the same map f.

IV. Our final example is the fully chaotic cardioid billiard [24] with an absorbing segment of the boundary where $R = R^* < 1$; see Fig. 1. Figure 3 shows ρ_c for two values of R^*, computed using ray simulations (1) [8]. $D_{q,c}$ for $q = 0$, 1, and 10 are reported in Table I and exhibit R-dependent multifractality like in the baker map. A comparison with the $R^* = 0$ case (trajectories escape) shows that the slightest nonzero R^* without trajectory escape leads to a space-filling unstable manifold $(D_{0,c} = 2)$ whose $D_{1,c}$ is close to $D_{0,c}$ of the $R^* = 0$ case. The difference between $D_{0,c}$ and $D_{1,c}$ quantifies the enhancement in multifractality due to absorption.

TABLE I. Escape rate κ and order-q dimensions $D_{q,c}$ of the c measure of the cardioid billiard described in Figs. 1 and 3 for different R^*. $D_{q,c}$ is measured from the c measure of partitions of the phase space (Fig. 3) and $D_{1,c}^{\text{Eq.}(12)}$ from the saddle's measure and Eq. (12) (we found that $\bar{\lambda}_+ = \bar{\lambda}_- \approx 0.35\bar{\tau}$ for all R^*; see the Supplemental Material, Fig. S1 [20]). κ is measured by fitting the survival probability and κ_r' are approximants obtained directly from ρ_c; see the Supplemental Material (text and Figs. S2 and S3) [20].

R^*	0.01	0.05	0.25	0.5	0.75	0
$D_{0,c}$	2.00	2.00	2.00	2.00	2.00	1.84
$D_{1,c}$	1.84	1.87	1.94	1.981	1.996	1.82
$D_{10,c}$	1.79	1.80	1.86	1.923	1.975	1.75
$D_{1,c}^{\text{Eq.}(12)}$	1.83	1.86	1.94	1.980	1.996	1.81
κ	0.06470	0.06155	0.04663	0.02954	0.01410	0.06559
κ_1'	0.06434	0.06131	0.04641	0.02945	0.01408	0.06520
κ_2'	0.06468	0.06163	0.04660	0.02953	0.01410	0.06557

In summary, we argued that chaotic systems with absorption should be considered as a class of dynamical systems on its own. Absorption converts the closed dynamics of trajectories into an open dynamics of weighted rays which we have shown to have fundamentally different chaos characteristics when compared to those of traditional open systems (in which trajectories escape). Among such properties are the new Perron-Frobenius operator [Eq. (3)], an implicit formula [Eq. (5)] for the escape rate, a generalized Kantz-Grassberger relationship [Eq. (9)] for 1D maps, and an enhanced multifractality of invariant measures. We anticipate that absorption also has important consequences in other operator approaches based on Markov partitions, which received renewed interest with the concept of almost invariant sets [25] and Ulam's method [26]. Furthermore, we conjecture that, provided the direct product structure seen in the baker example holds, for invertible chaotic 2D maps with absorption

$$D_{1,c} = 1 + \left(1 - \frac{\kappa\bar{\tau} + \overline{\ln R}}{\bar{\lambda}_+}\right)\frac{\bar{\lambda}_+}{\bar{\lambda}_-} \tag{12}$$

(see Table I for a numerical test). Our results apply to any chaotic system with absorption or partial reflection, provide new relations that have been looked after for decades [2], have direct implications for wave-chaotic systems [10,12,27], and are directly accessible to experiments (e.g., measuring the spatial distribution of decaying states in optical and acoustic systems).

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