Power-law decaying oscillations of neutrally buoyant spheres in continuously stratified fluid

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The free sinking motion of small spheres is experimentally investigated in a laboratory tank filled up with continuously stratified salt solutions. Special attention is paid to the late stage oscillations around the gravitational equilibrium level. Such motion is generated by slowly decaying weak internal waves originating from the interaction of the solid sphere with the surrounding fluid. The temporal decay of oscillation amplitudes and velocities can be best described by a power law of exponent value close to $-3/2$. Up to our best knowledge, none of the existing theories can predict a similar algebraic decay. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2927461]

One of the most fundamental phenomena in three-dimensional stably stratified fluids is the spontaneous generation of layered structures from virtually any transient distortions of isopycnal surfaces, including turbulent events.[1–5] The late stage of time evolution is characterized by a very slowly decaying mode of wavy motion where the vertical velocity and density fluctuations are in phase, and they attenuate as a result of advection and molecular diffusion.[1,6,7] The appearance of almost horizontal layers can be easily visualized by shadowgraph[8] or schlieren[9] methods. Here we show that macroscopic floating bodies can also be advected by the weak decaying waves, and quantitative information can be extracted by following the motion for sufficiently long times.

Initial transient disturbances in our study are generated by the free fall and settling of small spheres in stably stratified salt solutions. The free fall of solid particles is a classical problem in hydrodynamics with a long research history. Important milestones are the concept of added mass introduced by Bessel *et al.*[10] the Boussinesq–Basset history term,[11] or the Maxey–Riley equation of motion.[12] While most of the early studies considered homogeneous fluids, the extension to continuously stratified media[13–18] or the settling through sharp density interfaces[19,20] attracted also an interest. A general theoretical description is complicated by the fact that entrainment, wake formation, eddies, and waves are all different when the medium is density stratified. It is no wonder that the various models can approximate the early stages of motion through a stratified fluid but predictions generally fail for the long-time behavior, as we demonstrate in this work.

Experiments were carried out in a glass tank of size of $75.0 \times 38.0 \times 50.0$(length $\times$ width $\times$ height) cm^3. Salt density stratification of typical depths of 38–39 cm was produced by a standard double-bucket equipment.[21] Density profiles $\varrho_f(z)$ were obtained by measuring the conductivity and the temperature of the salt solution as a function of height z (measured from the bottom). The local Brunt–Väisälä (BV) or buoyancy frequency $N(z)=\sqrt{-g/\varrho_0(d\varrho_f/dz)}$ was deduced from the local slope of the density profile at z. The settling motion of five different plastic spheres of diameter $d=14.6$ mm and density $\varrho_p \in [1.010;1.048]$ g/cm^3 was investigated in the tank. The density adjustment of the balls was prepared by implanting small metal pieces close to the surface, which helped to suppress rotations. With a given ball, usually four repeated experiments were carried out in the same density profile from the same initial height z_0 slightly below the fluid surface. The balls were initially kept fixed at the end of a rubber tube connected to a vacuum pump, and the motion was initiated by gradually extinguishing the vacuum. The dynamics was monitored by digital cameras (Sony DCR-PC 115E PAL, PCO Pixelfly). The location of the ball was determined as the center of mass for the black pixels representing the ball on a digitalized image. With this method, the position of the body could be determined with a subpixel resolution of 0.01 mm.

Figure 1 illustrates the initial stage of a typical experiment. The observations are in agreement with earlier studies in similar setups,[22–25] which are the following. During the rapid downward falling of the balls, strong density inhomogeneities appear in the hydrodynamical wake generating irregular internal waves. The characteristic separation length-scale is approximately $\lambda \approx 5$ cm in both the horizontal and the vertical directions [Figs. 1(c) and 1(d)]. This gives a Strouhal number[26] $\mathrm{St} \equiv d/\lambda \approx 0.29$, which significantly differs from the constant value $\mathrm{St}=0.175$ characterizing the velocity independent near-wake spiral shedding mode behind a sphere in a homogeneous fluid.[26] The details of the patterns in subsequent runs are very different, thus vertical damping, the structure, and the peeling of the dragged boundary layer [Figs. 1(e) and 1(f)] can also be very different. The consequence of a temporally irregular (chaotic) wake formation is that the reproducibility of repeated experiments with the same ball in the same stratification is somewhat limited, as demonstrated in Fig. 2.

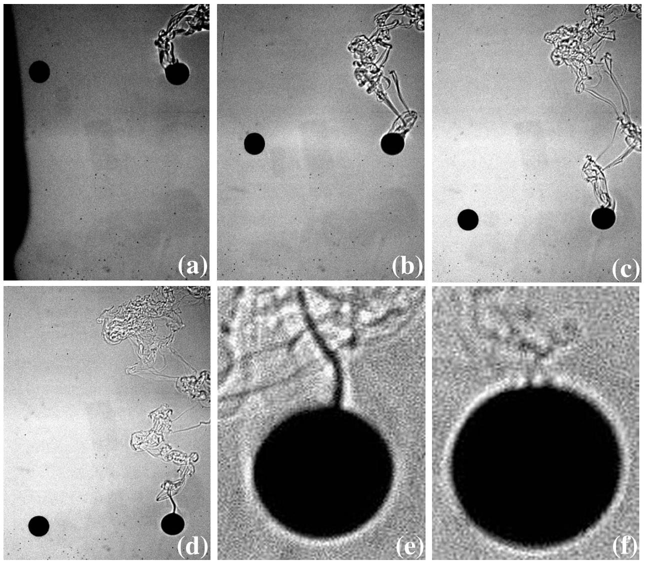

FIG. 1. Shadowgraph images over the first period of motion of a sphere [diameter $d=14.6$ mm, ball density $\varrho_p=1.025(8)$ g/cm^3, $N=0.86\pm0.01$ 1/s, $(z_0-z_{eq})/d=22.0$, where the initial height was $z_0=38.0$ cm]. The times after release are (a) 4 s, (b) 6.4 s, (c) 10 s, (d) 16 s, (e) 22 s, and (f) 54 s. The direct image of the ball appears on the left from the shadow in (a)–(d).

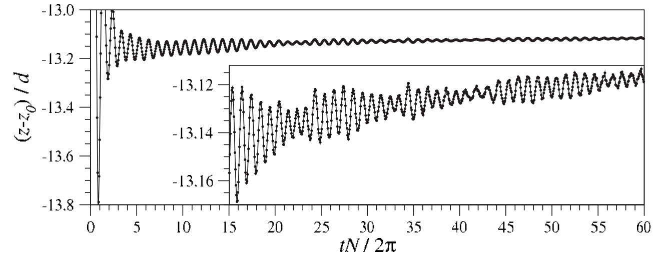

FIG. 3. Normalized vertical displacement $(z-z_0)/d$ as a function of dimensionless time $tN/2\pi$; the inset has a magnified vertical scale after 15 buoyancy periods [$z_0=38.0$ cm, $d=14.6$ mm, $\varrho_p=1.016(5)$ g/cm^3, $N=0.86\pm0.01$ 1/s].

The main characteristics of the vertical oscillations for longer times are illustrated in Fig. 3. The strong initial damping during the first few periods does not quickly lead to a static gravitational equilibrium, weak oscillations survive for remarkably long periods. The inset in Fig. 3 clearly shows that the amplitude of late stage oscillations does not decrease monotonously. The appearance of intervals characterized by large amplitudes suggests repeated transits of internal wave packets. We observed that the apparent gradual upward shift is also a generic feature in our setup, however, its (extremely slow) speed is very unpredictable. We attribute this rising to the nucleation and accumulation of tiny gas bubbles on the surface of the plastic spheres. The gas bubbles became visible after several hours of waiting, over which we were able to trace the continuous ascent of balls. Repeated attempts to minimize this side effect by washing the beads in different chemicals remained unsuccessful. Note that standard deaeration procedures do not work in our experiments because the filling of tanks by the double bucket method unavoidably dissolves some air in the salt solution.

Irrespectively of the imposed stratification and the gravitational equilibrium height z_{eq} of the spheres, the empirical frequency of oscillations ω_0 obtained by Fourier analysis are found to be very close to the local BV frequency N. The summary of measurements is shown in Fig. 4. The average relative frequency anomaly $(\omega_0-N)/N$ is approximately -2%, and we could not resolve any tendency as a function of potential control parameters. This observation agrees again with the reported behavior for larger bodies in similarly stratified fluids.[25] An important corollary of this result is that the added mass effect is apparently negligible because it vanishes when $\omega_0=N$ for the solid body oscillations.[27,28] As for the role of the history force term, we do not expect it to be significant at the late stage of the motion. This is because the average particle Reynolds number (see below) is not infinitesimal throughout the observations, therefore a fast decay of history forces is anticipated.[29–31] Nevertheless, we checked numerically that the incorporation of a Boussinesq–Basset history term in the equation of motion always precluded to reproduce even the basic oscillation frequencies.

In order to describe the decay of oscillations, we cut the time series $z(t)$ into pieces of a few buoyancy periods and removed the creeping background trends by local fitting of

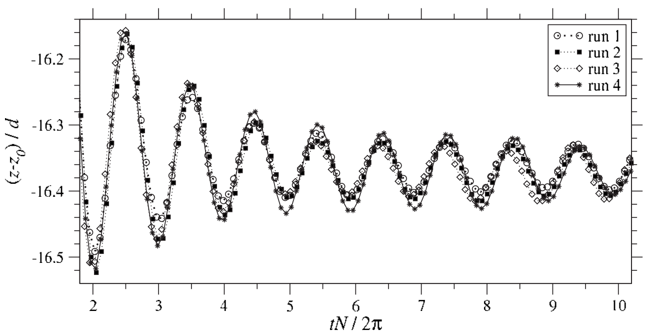

FIG. 2. Reproducibility test with a given sphere in the same stratification. The normalized vertical displacement $(z-z_0)/d$ is plotted as a function of dimensionless time $tN/2\pi$. The different curves are shifted horizontally to achieve the best fit in the period $2\leq tN/2\pi\leq3$. The parameters are $z_0=37.5$ cm, $d=14.6$ mm, $\varrho_p=1.047(2)$ g/cm^3 and $N=1.23\pm0.02$ 1/s.

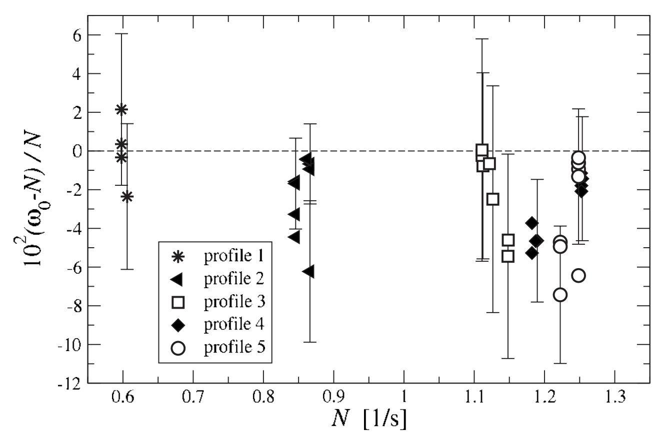

FIG. 4. Relative percentage frequency anomaly as a function of local BV frequency N for the various measurements. Error bars are estimated from the width of spectral peaks. The average BV frequency values for the profiles are (1) $\bar{N}=0.59$, (2) 0.86, (3) 1.12, (4) 1.21, and (5) 1.23 1/s, respectively.

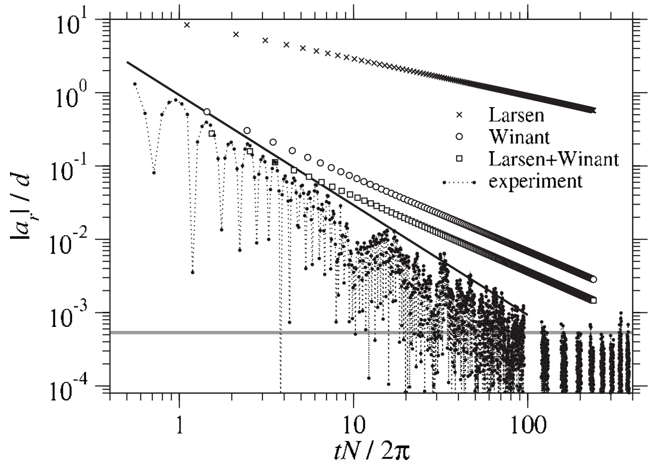

FIG. 5. Temporal decay of the normalized absolute residual amplitudes $|a_r|/d$, see text. The gray line is the estimated noise level, black line illustrates a power law with exponent of $-3/2$. (z_0=38.0 cm, $(z_0-z_{eq})/d$=11.7, ϱ_p=1.025(8) g/cm^3, N=1.248 1/s). Symbols show theoretical predictions by the Larsen (Ref. 13) (crosses), Winant (Ref. 14) (circles), and the combined (Ref. 14) (squares) equations, only successive maxima are plotted.

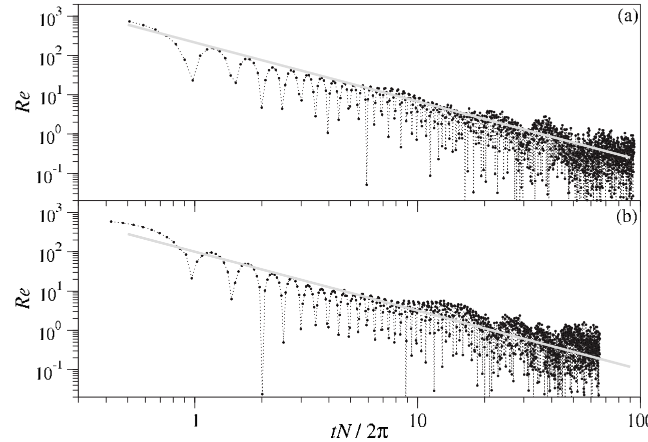

FIG. 6. Decay of the estimated particle Reynolds number $\mathrm{Re}=|\Delta z|/\Delta t d/\nu$ for two experiments. The gray lines (not fitted) illustrate power law with exponent of $-3/2$. (a) z_0=38.0 cm, $(z_0-z_{eq})/d$=16.8, ϱ_p=1.038(0) g/cm^3, N=1.23±0.02 1/s. (b) z_0=37.5 cm, $(z_0-z_{eq})/d$=12.9, ϱ_p=1.016(5) g/cm^3, N=0.86±0.01 1/s.

exponential or third order polynomials (the choice of the functional form proven to be quite indifferent). The normalized absolute value of the residual amplitudes $|a_r|/d$ is shown in Fig. 5 (symbols with dotted line) as a function of nondimensional time $tN/2\pi$, together with a few theoretical predictions (see below). In this experiment, we recorded the position of the sphere for almost 400 buoyancy periods (with breaks of 2 min at the latest stage) which provided a tool to a direct estimate of the noise level (horizontal gray line in Fig. 5). Note that the Fourier analysis could not resolve clear oscillations over ~200 buoyancy periods. The envelope of the decaying amplitudes can be approximated by a power law of exponent of $-3/2$ over two orders of magnitude (black line in Fig. 5).

An alternative standard method for removing slow trends is based on numerical derivation. This is known to produce large noise, nevertheless, we used it for consistency tests. Figure 6 shows two examples, where the time evolution of the particle Reynolds number $\mathrm{Re}=|\dot{z}|d/\nu$ is plotted ($\nu\approx 1$ mm^2/s is the kinematic viscosity and the absolute value of the instantaneous vertical velocity $|\dot{z}|$ was estimated by numerical derivation). The approximate power-law decay is apparent again.

Figures 5 and 6 illustrate that the decay is not a pure power law, smaller or larger deviations are apparent. The evident wobbling of the maxima is most probably due to passing internal waves reflected from the boundaries of the container and from the fluid surface. Future experiments are intended to reveal feasible important effects of the side walls. Fits for the data of other experimental runs resulted in exponent values in the interval [1.3–1.8] centered around 1.5, nevertheless, we cannot formulate any solid statement about a possible universal value. We found, however, that other fits by, e.g., a set of exponential functions with changing time constants, or stretched exponentials, etc., provide a much lower quality description for the decay of the envelope for the oscillation amplitudes and velocities.

In the search toward a possible explanation of the power-law decay with an exponent of $-3/2$, we first point out that the long time decay of Larsen's equation[13,15] follows a power law, but with an exponent of $-1/2$ (Fig. 5, crosses), similarly to recent alternative formulations.[17,18] This behavior is due to the radiation of linear internal waves in a medium of infinite extension. The long-term decay in Winant's nonradiative approach[14] is inversely proportional with time (Fig. 5, circles). Note that the combination of the two theories—wave radiation and velocity-square drag—cannot result in a steeper power law,[14] the decay remains $\sim t^{-1}$ (Fig. 5, squares). When we substitute the drag factor c_D=0.72 (assumed to be constant by Winant[14] owing to large Reynolds numbers, decreasing from 1400 to 300 in the course of his experiments[14]) with the usual form[32] $c_D(\mathrm{Re})=24/\mathrm{Re}$ appropriate for the smaller Reynolds number in the present experiments (see Fig. 6), the damping will be exponential (not shown in Fig. 5).

An interesting fact is that power laws can be very often found in the context of turbulence, but in our case, the motion is an approximately regular oscillation of a macroscopic body. Similar attenuation dynamics was observed in much more demanding experiments by Praud *et al.*,[4] where initial perturbations were generated by towing a vertical grid with various (constant) velocities. Irrespectively of the strength of initial turbulence, the final stage consisted of a quasihorizontal motion of interacting vortices grown up to the width of the tank. They obtained a power-law decay for the average *horizontal* velocity with an exponent of −0.65. *Vertical* velocities were estimated by the continuity equation from horizontal particle image velocimetry (PIV) data.[4] The decay was found to be much faster, although the exponent has not been determined. An evaluation of this exponent for their data appears, however, to be consistent with our estimates (see Fig. 5 in Ref. 4). We also observed the slow horizontal drift of spheres in random final directions, however, the magnitudes were so small (0.01–0.1 mm/s) that we were not able to extract meaningful information about the attenuation dynamics. In several cases, the horizontal drift velocity remained constant during the recorded time interval (usually $\sim$100 buoyancy periods). Dye visualization confirmed that this motion was always the consequence of pancake vortices gradually stretching to the width of the tank.

The results coherently suggest that the late stage motion of our spheres is a pure advection by the long lasting pancake vortices perturbed by passing internal waves. The horizontal motion has attracted more interest in the literature[4,5,26,33] because the decay rate is much faster in the vertical direction.[4] Our findings show that such a simple setup provides a valuable information about the intermediate range of decaying stratified turbulence, and it can thus contribute to a better understanding of a fundamental phenomenon.

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