Towards the Harnessing of Chaos
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Controlling Transient Chaos in Dynamical Systems

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We present an efficient algorithm for stabilizing unstable periodic orbits for dynamical systems that exhibit transient chaos based on the controlling chaos idea of Ref. [1]. Dynamically, transient chaos is due to the existence of a nonattracting chaotic saddle in the phase space. By constructing a long reference chaotic orbit on the chaotic saddle which comes arbitrarily close to the desired unstable periodic orbit, we propose to stabilize a trajectory around the reference chaotic orbit and then switch to stabilize it around the target periodic orbit. By doing so, the probability that a trajectory can be stabilized around the target periodic orbit increases significantly as compared with the conventional method of controlling transient chaos in which trajectories are stabilized directly around the periodic orbit.

1 Introduction

Transient chaos is ubiquitous in nonlinear dynamical systems [2, 3]. Dynamically, transient chaos is due to the existence of a non-attracting chaotic saddle (chaotic repeller) containing an infinite number of unstable periodic orbits [2, 3]. Both the stable and unstable foliations associated with the chaotic saddle possess a Cantor-like fractal structure. Physically, a manifestation of transient chaos is that trajectories often evolve chaotically for a finite amount of time before settling into a final state, (for example, an attractor at infinity). This can be seen as follows. A trajectory starting from some initial condition near the chaotic saddle is "pulled" towards the chaotic saddle along the stable manifold under the dynamics. It will then stay close to the chaotic saddle behaving chaotically for a finite amount of time and eventually exit along the unstable manifold. As such, trajectories starting with slightly different initial conditions may stay near the chaotic saddle for quite different amount of time, and they may leave the chaotic saddle with different values of their dynamical variables. In dissipative systems, transient chaos can be induced by boundary crises [2] in which a chaotic attractor is suddenly destroyed after colliding with its own basin boundary, or it can occur in systems that possess fractal basin boundaries [4]. In Hamiltonian systems, transient chaos usually leads to the phenomenon of chaotic scattering [5, 3].

As an example of transient chaos, Fig. 1(a) shows a chaotic saddle for the Hénon map: $(x_{n+1}, y_{n+1}) = (a - x_n^2 + 0.3y_n, x_n)$ together with an unstable period-8 orbit embedded in the chaotic saddle, where $a = 1.5$. The image of the chaotic saddle shown in the Fig. 1(a) is actually a single trajectory with 10000 points obtained by a technique known as the Proper Interior Maximum Triple (PIM-triple) method [6] which is specially designed for tracing continuous trajectories on non-attracting chaotic saddles. Almost all the initial conditions except a set of Lebesgue measure zero in the square region of Fig. 1(a) will escape to infinity after a finite number of iterations. Figure 1(b) shows a time series for x_n at $a = 1.5$ resulting from the initial condition $(x_0, y_0) = (-1.711, 0.0)$. The lifetime of the transient chaos is 288 iterations in this case. At time step $n = 289$, the trajectory escapes the chaotic saddle in a catastrophic way and maps to an attractor at the negative infinity in subsequent iterations.

This paper addresses the issue of stabilizing unstable periodic orbits embedded in the chaotic saddle by applying small perturbations to an accessible parameter of the system. Randomly pick some initial conditions near the chaotic saddle, our goal is to stabilize one of the trajectories resulting from these initial conditions in the neighborhood of a desired unstable periodic orbit before the trajectories escape to infinity.

Control of chaos using unstable periodic orbits embedded in a chaotic attractor was proposed in Ref. [1]. The basic idea is as follows. First one chooses an unstable periodic orbit embedded in the attractor, the one which yields the best system performance according to some criteria. Second, one defines a small region around the desired periodic orbit. Due to ergodicity of the chaotic attractor, a trajectory eventually falls into this small region. When this occurs, small judiciously chosen temporal parameter perturbations are applied to force the trajectory to approach the unstable periodic orbit. This method is extremely flexible because it allows for the stabilization of different periodic orbits, depending on one's needs, for the same set of nominal values of the parameter. It should be mentioned that control of nonlinear dynamical systems without using feedback has also been proposed [7].

A major difference between stabilizing unstable periodic orbits embedded in a chaotic attractor and a chaotic saddle is that for the chaotic attractor, the probability that a chaotic trajectory enters the neighborhood of the desired unstable periodic orbit is one [1]. Hence, trajectories originating from almost any initial condition in the basin of the chaotic attractor can eventually be stabilized. While for the case of transient chaos, only a small set of initial conditions can be controlled. The reason is that most trajectories will have already left the chaotic saddle before entering the neighborhood of the target periodic orbit. A possible remedy is to simultaneously launch an ensemble of initial conditions and to control any one of the trajectories that come close to the target periodic orbit. This strategy has been successfully demonstrated [8]. Clearly, for transient chaos we can only talk about the probability that a randomly chosen initial condition can be controlled. This probability is usually very small [8]. The main objective of this paper is to devise a scheme to maximize this probability.

The key observation is that there exists a dense chaotic orbit on the chaotic saddle *that comes arbitrarily close to the target unstable periodic orbit* [6]. By using the PIM-triple method, one can compute such a long *reference orbit* on the chaotic saddle [6]. The probability that a trajectory approaches this reference orbit is greater than the probability that this same trajectory enters the neighborhood of the target unstable periodic orbit before it escapes, if the length of the reference orbit is long enough. By *stabilizing a trajectory around the reference orbit first, and then switching to stabilize it around the target periodic orbit after the trajectory comes close to the periodic orbit*, we can *substantially* increase the probability that a trajectory can be controlled. This can indeed be achieved since there exist stable and unstable directions at each point of the reference orbit on

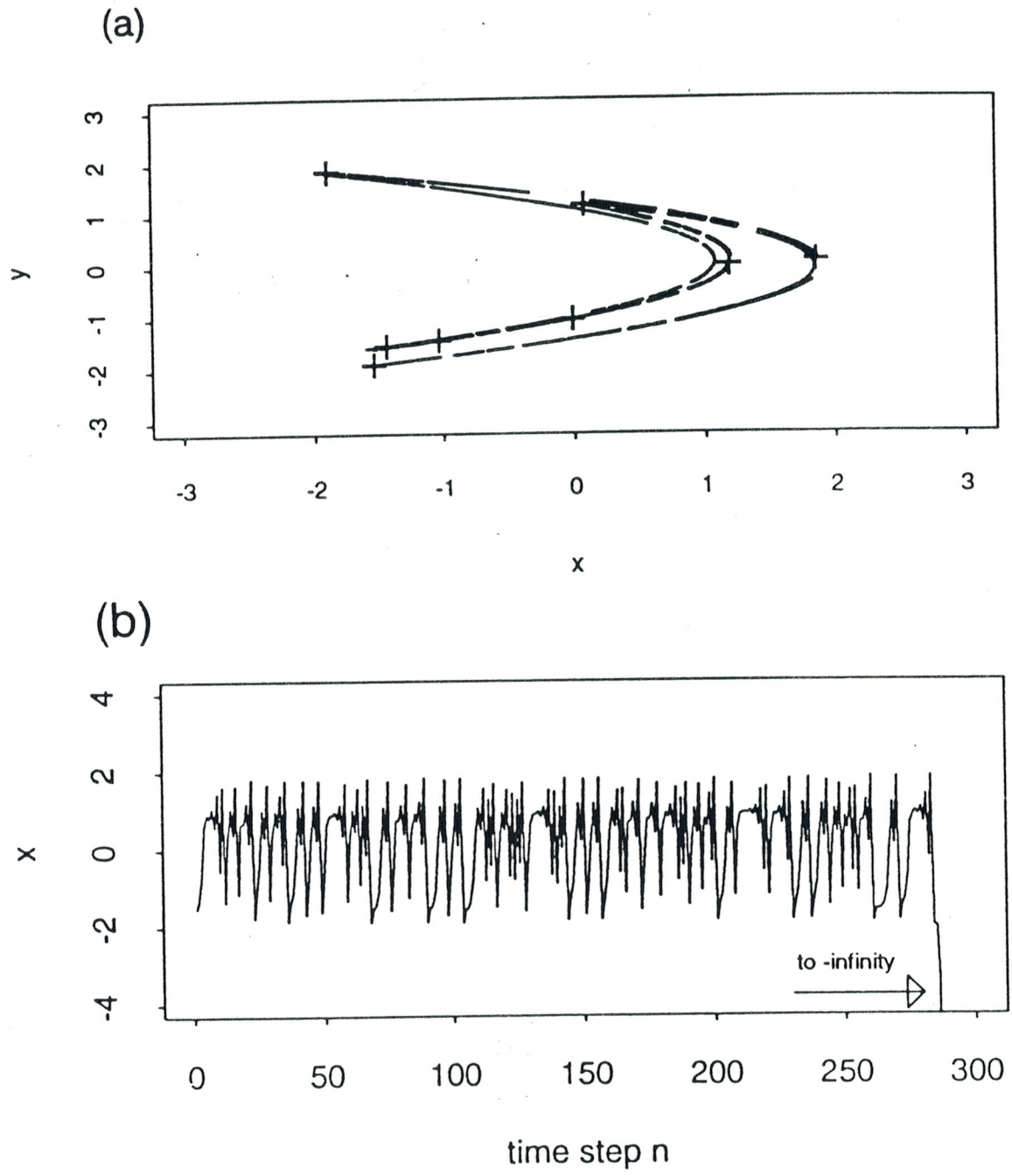

Figure 1: (a) A chaotic saddle for the Hénon map at $a = 1.5$. The crosses denote the locations of a period-8 orbit embedded in the chaotic saddle. (b) A time series of x_n for the case of transient chaos at $a = 1.5$. The trajectory starts from the initial condition: $(x_0, y_0) = (-1.711, 0.0)$. This trajectory wanders around the chaotic saddle of Fig. 1(b) for 288 iterates and escapes to an attractor at the negative infinity in subsequent iterates.

the chaotic saddle. Hence, in principle, controlling a trajectory near the reference orbit is equivalent to stabilizing a long unstable periodic orbit as in Ref. [1]. The longer the length of the reference orbit is, the larger the controllable probability for trajectories can be. We should mention that a method for stabilizing chaotic orbits on the attractor has been proposed and applied to the synchronization of two almost identical chaotic systems [9], and a method of creating desired chaotic orbits on a chaotic attractor has been implemented [10].

The organization of this paper is as follows. In Sec. 2 we review the method in Ref. [1] for controlling chaos by using the one-dimensional logistic map as an illustrative example. In Sec. 3, we discuss the method to stabilize a long chaotic reference orbit on the chaotic saddle for two-dimensional maps. In Sec. 4, we present numerical results for controlling transient chaos using the Hénon map and discuss our improvement in the probability that a randomly chosen initial condition can be controlled. In Sec. 5, we give concluding remarks.

2 Review of the Controlling Chaos Idea: A Simple Example

The basic idea in Ref. [1] for stabilizing unstable periodic orbits embedded in a chaotic attractor can be understood by considering a simple model system. We consider one of the best understood chaotic systems, the one-dimensional logistic map:

$$x_{n+1} = f(x_n, \lambda) = \lambda x_n(1 - x_n), \tag{1}$$

where x is restricted to the unit interval $[0, 1]$, and λ is a parameter. It is known that this map develops chaos via the period-doubling bifurcation route [11]. The period-doubling cascade accumulates at $\lambda = \lambda_\infty \approx 3.57$, after which chaos can arise.

Consider the case $\lambda = 3.8$ shown in Fig. 2(a). The system is apparently chaotic for this value of λ and the chaotic attractor is contained in the interval $[0, 1]$. An important property of this chaotic attractor is that there exists *an infinite number of unstable periodic orbits embedded within it and they are dense in it.* For example, a fixed point $x^* \approx 0.7368$ and a period-2 orbit, $x(1) \approx 0.3737$, $x(2) \approx 0.8894$, where $x(1) = f[x(2)]$ and $x(2) = f[x(1)]$, are shown in Fig. 2(a).

Now suppose we want to avoid chaos at $\lambda = 3.8$. In particular, we want trajectories resulting from a randomly chosen initial condition x_0 to be as close as possible to the period-2 orbit shown in Fig. 2(a), assuming that this period-2 orbit gives the best system performance. Of course, we can choose the desired asymptotic state of the map to be any of the infinite number of unstable periodic orbits, if that periodic orbit gives the best system performance. To achieve this goal, we suppose that the parameter λ can be finely tuned in a very small range around the value $\lambda_0 = 3.8$, namely, we allow λ to vary in the range $[\lambda_0 - \delta, \lambda_0 + \delta]$, where $\delta << 1$. Due to the ergodicity of the chaotic attractor, the trajectory that begins from an arbitrary value of x_0 will fall, with probability one, into the neighborhood of the desired period-2 orbit at some later time. Because of the nature of chaos, the trajectory would diverge quickly from the period-2 orbit if we do not intervene. Our task is to program the parameter perturbations in such a way that the trajectory stays in the neighborhood of the period-2 orbit for as long as the control is present. The small parameter perturbations will be time-dependent in general.

The logistic map in the neighborhood of a periodic orbit can be approximated by a linear equation expanded around the periodic orbit. Let the target period-m orbit to be

controlled be $x(i)$, $i = 1, \ldots, m$, where $x(i+1) = f[x(i)]$ and $x(m+1) \equiv x(1)$. Assume that at time n, the trajectory falls into the neighborhood of the ith component of the period-m orbit. The linearized dynamics in the neighborhood of the $(i+1)$th component is then

$$\begin{aligned} x_{n+1} - x(i+1) &= \frac{\partial f(x,\lambda)}{\partial x}[x_n - x(i)] + \frac{\partial f(x,\lambda)}{\partial \lambda}(\Delta\lambda)_n \\ &= \lambda_0 [1 - 2x(i)][x_n - x(i)] + x(i)[1 - x(i)]\Delta\lambda_n. \end{aligned} \tag{2}$$

where the partial derivatives in (2) are evaluated at $x = x(i)$ and $\lambda = \lambda_0$. We require x_{n+1} to stay in the neighborhood of $x(i+1)$. Hence, we set $|x_{n+1} - x(i+1)| = 0$, which gives,

$$\Delta\lambda_n = \lambda_0 \frac{[2x(i) - 1][x_n - x(i)]}{x(i)[1 - x(i)]}. \tag{3}$$

Eq. (3) holds only when the trajectory x_n enters a small neighborhood of the period-m orbit, i.e., when $|x_n - x(i)| \to 0$. Hence, the required parameter perturbation $\Delta\lambda_n$ is small. Let the length of a small interval defining the neighborhood around each component of the period-m orbit be 2ϵ. In general, the required maximum parameter perturbation δ is proportional to ϵ. Since ϵ can be chosen to be arbitrarily small, δ also can be made arbitrarily small. However, as we will see later, the average transient time before a trajectory enters the neighborhood of the target periodic orbit depends on ϵ (or δ). A larger δ will mean a shorter average time for the trajectory to be controlled. Of course, if δ is too large, nonlinear effects become important and the linear control strategy (3) might not work. When the trajectory is outside the neighborhood of the target periodic orbit, we do not apply any parameter perturbation and the system evolves at its nominal parameter value λ_0. We usually set $\Delta\lambda_n = 0$ when $\Delta\lambda_n > \delta$. Note that the parameter perturbations $\Delta\lambda_n$ depend on x_n and are therefore time-dependent.

The above strategy for controlling the orbit is very flexible for stabilizing different periodic orbits at different times. Suppose we first stabilize a chaotic trajectory around the period-2 orbit shown in Fig. 2(a). Then we might wish to stabilize the fixed point in Fig. 2(a), assuming that the fixed point would correspond to a better system performance at a later time. To achieve this change of control, we simply turn off the parameter control with respect to the period-2 orbit. Without control, the trajectory will diverge from the period-2 orbit exponentially. We let the system evolve at the parameter value λ_0. Due to ergodicity, there comes a time when the chaotic trajectory enters a small neighborhood of the fixed point. At this time we turn on a new set of parameter perturbations calculated with respect to the fixed point. The trajectory can then be stabilized around the fixed point.

Fig. 2(b) shows a case where we first control the period-2 orbit and then the fixed point shown in Fig. 2(a). The initial condition is $x_0 = 0.28$. At time $n = 381$, the trajectory enters the neighborhood of the component $x(1)$ of the period-2 orbit. For subsequent iterations, parameter perturbations calculated from (3) are applied to stabilize the trajectory around the period-2 orbit. At time $n = 2200$, we choose to stabilize the trajectory around the fixed point, and hence we turn off the parameter perturbation at $n = 2200$. The trajectory quickly leaves the period-2 orbit and becomes chaotic. At time $n = 2757$, the trajectory falls into the neighborhood of the fixed point. Parameter perturbations calculated with respect to the fixed point are then turned on to stabilize the trajectory around the fixed point.

In the presense of external noise, a controlled trajectory will occasionally be "kicked" out of the neighborhood of the periodic orbit. If this behavior occurs, we turn off the parameter perturbation and let the system evolve by itself. With probability one the

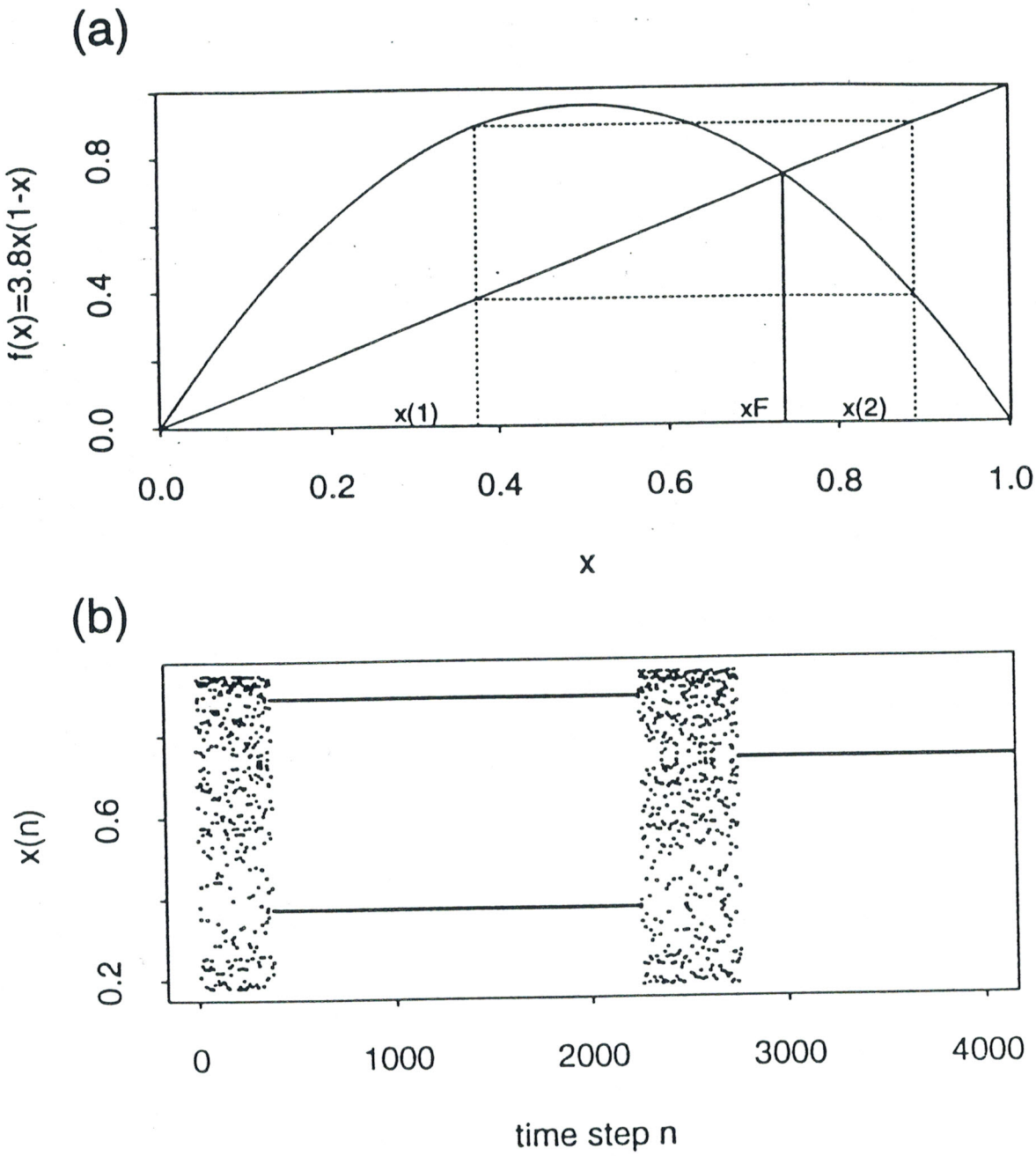

Figure 2: (a) The logistic map $x_{n+1} = f(x_n) = 3.8x_n(1 - x_n)$. An unstable fixed point and an unstable period-2 orbit are also shown. (b) Time series illustrating the control of the period-2 orbit and the fixed point in Fig. 2(a). The chaotic trajectory begins from $x_0 = 0.28$. At $n = 381$, the trajectory falls in an ϵ- neighborhood of the period-2 orbit, after which the parameter control is turned on to stabilize the trajectory around the period-2 orbit. At $n = 2200$, the control is turned off. At $n = 2757$, the chaotic trajectory comes close to the fixed point and is controlled in subsequent iterations. We choose $\epsilon = 10^{-3}$. The maximum allowed parameter perturbation is $\delta = 5 \times 10^{-3}$.

chaotic trajectory will enter the neighborhood of the target periodic orbit and be controlled again. This situation is illustrated in Fig. 3(a) where we control the period-2 orbit. The noise is modeled by an additive term in the logistic map of the form $\eta\sigma(n)$, where $\sigma(n)$ is a Gaussian distributed random variable with zero mean and unit standard deviation, and η is the noise amplitude. The effect of the noise is to turn a controlled periodic trajectory into an intermittent one in which chaotic phases (uncontrolled trajectories) are interspersed with laminar phases (controlled periodic trajectories). It is easy to verify that the averaged length of the laminar phase increases as the noise amplitude decreases, and the length tends to infinity as $\eta \to 0$.

It is interesting to ask how many iterations is required for a chaotic trajectory to enter the neighborhood of the target periodic orbit and be controlled. Clearly, the smaller the value of ϵ (the size of the neighborhood), the longer it takes for the trajectory to be controlled. In general, the average transient time to achieve control $\tau(\epsilon)$ scales with ϵ as

$$\tau(\epsilon) \sim \epsilon^{-\gamma}, \tag{4}$$

where $\gamma > 0$ is a scaling exponent. For one-dimensional maps such as the logistic map, there usually exists a smooth probability distribution $\rho(x)$ for trajectory points on the attractor, which can roughly be defined as the frequency that a chaotic trajectory visits a small neighborhood of the point x on the attractor. In such a case, we have $\gamma = 1$, which can be seen as follows. The probability that a trajectory enters the neighborhood of one component (the ith component) of the periodic orbit is given by

$$P(\epsilon) = \int_{x(i)-\epsilon}^{x(i)+\epsilon} \rho[x(i)]\, dx \approx 2\epsilon\rho[x(i)]. \tag{5}$$

Therefore, $\tau(\epsilon) = 1/P(\epsilon) \sim \epsilon^{-1}$, and hence $\gamma = 1$. This behavior in illustrated in Fig. 3(b), where $\tau(\epsilon)$ is plotted on a logarithmic scale for the case of stabilizing the period-2 orbit shown in Fig. 2(a). Twenty values of ϵ were chosen in the range $[10^{-4}, 10^{-2}]$ on a logarithmic scale. For each ϵ, we randomly choose 2000 initial conditions (with uniform probability distribution) and calculate an average transient time. The slope of the straight line is approximately -1.02, indicating good agreement with the theoretical prediction that $\gamma = 1$. For higher dimensional chaotic systems, the exponent γ can be related to the eigenvalues of the target periodic orbit [1].

The idea of controlling chaos discussed in this section has attracted growing interest for controlling dynamical systems and has been extended to higher dimensional dynamical systems [12], Hamiltonian systems [13], the control of transient chaos [8] and chaotic scattering [14], and the synchronization of chaotic systems [9]. It also has been successfully implemented in various physical experiments (see ref. [13] for a partial list of controlling chaos experiments). In the following two sections we will extend these ideas to controlling transient chaos.

3 Method of Stabilizing a Chaotic Reference Orbit

We consider a transient chaotic system that can be described by two-dimensional maps on the Poincaré surface of section,

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, p), \tag{6}$$

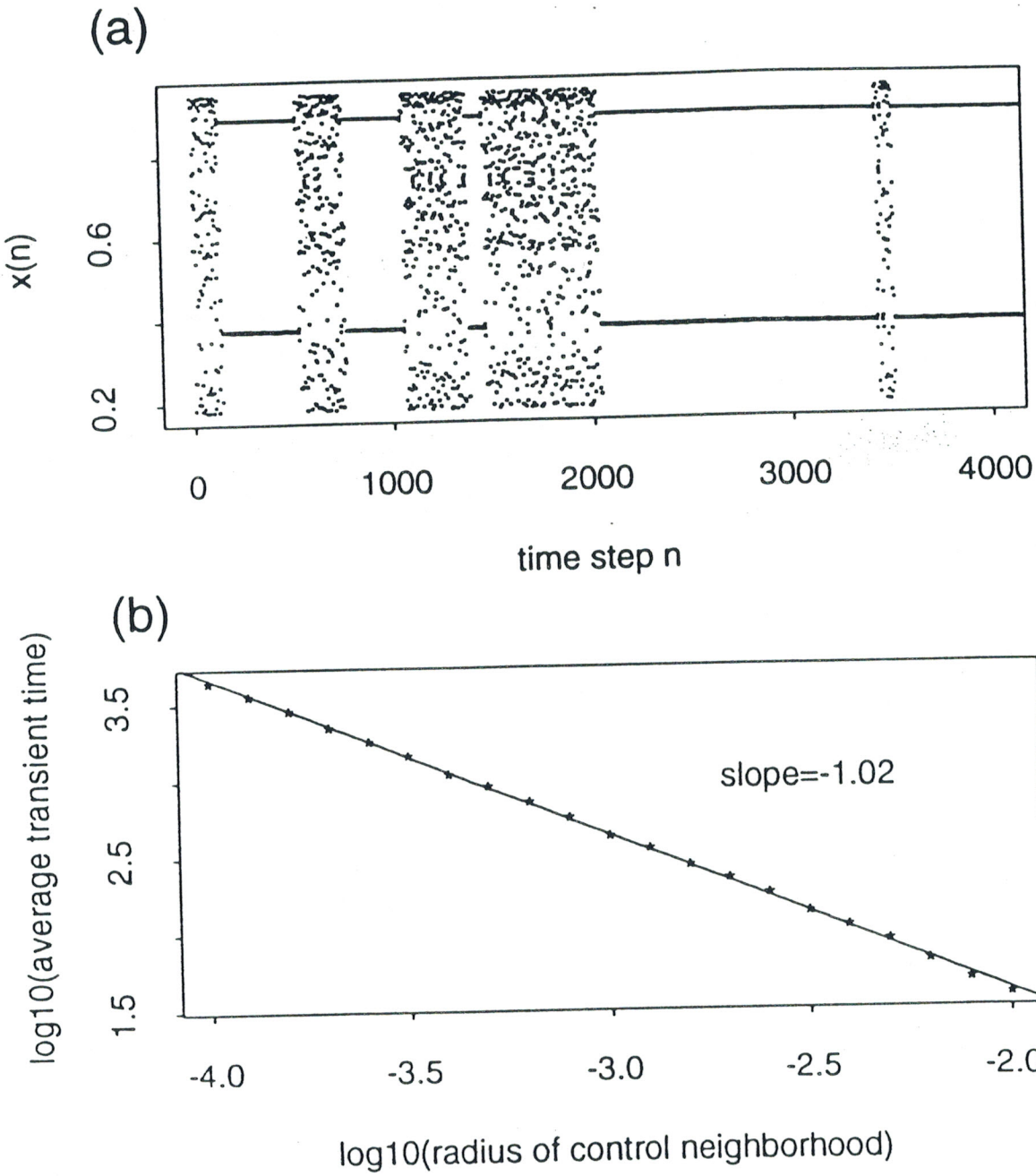

Figure 3: (a) The effect of additive noise modeled by $2.6 \times 10^{-4}\sigma_n$, where σ_n is a Gaussian random variable with zero mean and unit standard deviation. The noise can occasionally kick the controlled trajectory out of the neighborhood of the periodic orbit. (b) Log-log plot of the average time to achieve control $\tau(\epsilon)$ versus ϵ, the size of the controlling neighborhood. Twenty values of ϵ are chosen on a logarithmic scale. For each ϵ, 2000 random initial conditions uniformly distributed in $[0, 1]$ are chosen to compute $\tau(\epsilon)$. The scaling relation between $\tau(\epsilon)$ and ϵ is well fitted by $\tau(\epsilon) \sim \epsilon^{-1}$.

where $\mathbf{x}_n \in \mathbf{R}^2$, p is an externally controllable parameter. For p values considered in this paper, we assume that Eq. (1) possesses only non-attracting chaotic saddles. We require the parameter perturbations to be small, *i.e.*,

$$|\Delta p| \equiv |p - p_0| < \delta, \tag{7}$$

where p_0 is some nominal parameter value, δ is a small number defining the range of parameter perturbations.

Let $\{\mathbf{y}_n\}$ $(n = 0, 1, 2, \ldots, N)$ denote a long reference orbit on the chaotic saddle obtained by the PIM-triple method [6]. Now generate the orbit $\{\mathbf{x}_n\}$ to be stabilized around the reference orbit. Randomly pick an initial condition $\mathbf{x}_0$, assume that the orbit point $\mathbf{x}_n$ $(n \geq 0)$ falls in a small neighborhood of the point $\mathbf{y}_k$ of the reference orbit on the chaotic saddle at time step n. Without loss of generality, we set $k = n$ on the reference orbit. In this small neighborhood, the linearization of Eq. (1) is applicable. We have, thus,

$$\mathbf{x}_{n+1}(p_n) - \mathbf{y}_{n+1}(p_0) = \mathbf{J} \cdot [\mathbf{x}_n(p_0) - \mathbf{y}_n(p_0)] + \mathbf{K}\Delta p_n, \tag{8}$$

where $\Delta p_n = p_n - p_0$, $\Delta p_n \leq \delta$, $\mathbf{J}$ is the 2×2 Jacobian matrix and $\mathbf{K}$ is a two-dimensional column vector,

$$\mathbf{J} = \mathbf{D}_{\mathbf{x}}\mathbf{F}(\mathbf{x}, p)|_{\mathbf{x}=\mathbf{y}_n, p=p_0}, \quad \mathbf{K} = \mathbf{D}_p\mathbf{F}(\mathbf{x}, p)|_{\mathbf{x}=\mathbf{y}_n, p=p_0}. \tag{9}$$

Without control, *i.e.*, $\Delta p_n = 0$, the orbit $\mathbf{x}_i$ $(i = n+1, \ldots)$ diverges from the reference orbit $\mathbf{y}_i$ $(i = n+1, \ldots)$ exponentially. Our task is to program the parameter perturbations Δp_n in such a way that the trajectory $\mathbf{x}$ stays near the reference orbit on the chaotic saddle (or equivalently, $|\mathbf{x}_i - \mathbf{y}_i| \to 0|$) for subsequent iterates $i \geq n+1$.

For each reference orbit point on the chaotic saddle, there exist both a stable and an unstable direction [15]. (For higher dimensional maps, there may be several stable and unstable directions. The algorithm to control chaos in such cases is more complicated [12] and will not be discussed here.) The existence of the stable and unstable directions at each reference orbit point can be seen as follows. Let us choose a small circle of radius ε around some orbit point $\mathbf{y}_n$ and map this circle to $\mathbf{y}_{n-1}$ by the inverse map. In a Cartesian coordinate system with the origin at $\mathbf{y}_{n-1}$, the deformed circle can be expressed as $A(dx')^2 + B(dx')(dy') + C(dy')^2 = 1$ which is typically an ellipse. Here A, B and C are functions of the entries of the inverse Jacobian matrix at $\mathbf{y}_n$. This deformation from a circle to an ellipse means that distance along the major axis of the ellipse at $\mathbf{y}_{n-1}$ contracts as a result of the map. Similarly, the image of a circle at $\mathbf{y}_{n-1}$ under $\mathbf{F}$ is typically an ellipse at $\mathbf{y}_n$, which means that distance along the inverse image of the major axis of the ellipse at $\mathbf{y}_n$ expands under $\mathbf{F}$. Thus the major axis of the ellipse at $\mathbf{y}_{n-1}$ and the inverse image of the major axis of the ellipse at $\mathbf{y}_n$ approximate the stable and unstable directions at $\mathbf{y}_{n-1}$. It should be noted that, typically, the stable and unstable directions are not orthogonal to each other. In nonhyperbolic chaotic systems they can coincide [15].

To achieve control, it is neccessary to calculate the stable and unstable directions along the reference orbit. We will use an algorithm developed in Ref. [15]. This numerical method, however, requires that the Jacobian matrix of the map be explicitly known. The stable and unstable directions are then stored together with the reference orbit, and they will be used to compute the parameter perturbations applied at each time step.

To find the stable direction at a point $\mathbf{y}$, we first iterate this point forward N times under the map $\mathbf{F}$ and get a trajectory $\mathbf{F}^1(\mathbf{y})$, $\mathbf{F}^2(\mathbf{y})$, ..., $\mathbf{F}^N(\mathbf{y})$. Now imagine we put a circle of arbitrarily small radius ε at the point $\mathbf{F}^N(\mathbf{y})$. If we iterate this circle backward once, the circle becomes an ellipse at the point $\mathbf{F}^{N-1}(\mathbf{y})$ with the major axis along the stable direction of the point $\mathbf{F}^{N-1}(\mathbf{y})$. We continue iterating this ellipse backwards, while

at the same time normalizing the ellipse's major axis to be of the order ε. When we iterate the ellipse all the way back to the point $\mathbf{y}$, the ellipse becomes very thin with its major axis along the stable direction at point $\mathbf{y}$ if N is large enough. It should be mentioned that this same method can also be used to compute the stable and unstable directions along unstable periodic orbits. For an unstable period-k orbit, we choose $N = mk$ so that N is large, where m is an integer. In practice, instead of using a small circle, we take a unit vector at the point $\mathbf{F}^N(\mathbf{y})$ since the Jacobian matrix of the *inverse map* $\mathbf{F}^{-1}$ rotates a vector in the tangent space of $\mathbf{F}$ towards the stable direction. Thus, we iterate a unit vector backward to the point $\mathbf{y}$ by multiplying by the Jacobian matrix of the inverse map at each point on the *already existing* orbit. We normalize the vector after each multiplication to the unit length. For sufficiently large N, the unit vector so obtained at $\mathbf{y}$ is a good approximation of the stable direction at $\mathbf{y}$. A key point in the calculation is that we do *not* actually calculate the inverse Jacobian matrix along the trajectory *by* iterating the point $\mathbf{F}^N(\mathbf{y})$ backwards using the inverse map $\mathbf{F}^{-1}$. The reason is that if we do so, the trajectory will usually diverge from the original trajectory $\mathbf{F}^N(\mathbf{y})$, $\mathbf{F}^{N-1}(\mathbf{y})$, ..., $\mathbf{F}^1(\mathbf{y})$ after only a few backward interations. What we do is to store the inverse Jacobian matrix at every point of the orbit $\mathbf{F}^i(\mathbf{y})$ $(i = 1, \ldots, N)$ when we iterate forward the point $\mathbf{y}$ beforehand.

Similarly, to find the unstable direction at point $\mathbf{y}$, we first iterate $\mathbf{y}$ backward under the inverse map N times to get a backward orbit $\mathbf{F}^{-j}(\mathbf{y})$ with $j = N, \ldots, 1$. We then choose a unit vector at point $\mathbf{F}^{-N}(\mathbf{y})$ and iterate this unit vector forward by multiplying by the Jacobian matrices. The final vector at point $\mathbf{y}$ is a good approximation of the unstable direction at that point if N is large enough. Again, to avoid divergence from the original trajectory, we do not actually iterate the inverse map. What we do in this case is to choose $\mathbf{y}$ to be the end point of a forward orbit, all the points before $\mathbf{y}$ are the inverse images of $\mathbf{y}$ and we store the Jacobian matrix of forward map at those points.

The method so described is efficient. For instance, the error between the calculated and real stable or unstable directions is on the order of 10^{-10} for chaotic saddles in the Hénon map if $N = 20$ [15].

Let $\mathbf{e}_{s(n)}$ and $\mathbf{e}_{u(n)}$ be the stable and unstable unit vectors at $\mathbf{y}_n$ and, $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ be the corresponding unit contravariant vectors that satisfy $\mathbf{f}_{u(n)} \cdot \mathbf{e}_{u(n)} = \mathbf{f}_{s(n)} \cdot \mathbf{e}_{s(n)} = 1$ and $\mathbf{f}_{u(n)} \cdot \mathbf{e}_{s(n)} = \mathbf{f}_{s(n)} \cdot \mathbf{e}_{u(n)} = 0$. To stabilize $\{\mathbf{x}_n\}$ around $\{\mathbf{y}_n\}$, we require the next iteration of $\mathbf{x}_n$, after falling into a small neighborhood around $\mathbf{y}_n$, to lie on the stable direction at $\mathbf{y}_{(n+1)}(p_0)$, *i.e.*,

$$[\mathbf{x}_{n+1} - \mathbf{y}_{(n+1)}(p_0)] \cdot \mathbf{f}_{u(n+1)} = 0. \tag{10}$$

Substituting Eq. (8) into Eq. (10), we obtain the following expression for the parameter perturbation,

$$\Delta p_n = \frac{\{\mathbf{J} \cdot [\mathbf{x}_n - \mathbf{y}_n(p_0)]\} \cdot \mathbf{f}_{u(n+1)}}{-\mathbf{K} \cdot \mathbf{f}_{u(n+1)}}. \tag{11}$$

It is understood in Eq. (11) that if $\Delta p_n > \delta$, we set $\Delta p_n = 0$.

After a trajectory is stabilized around the reference orbit, we monitor the trajectory to see if it gets close to the target periodic orbit. To guarantee that the trajectory will always approach the target periodic orbit at later times, a possible strategy is to let the end point of the long reference orbit be in the neighborhood of the target periodic orbit. As soon as the controlled chaotic trajectory is in the vicinity of the target periodic orbit, a new set of parameter perturbations computed with respect to the periodic orbit is turned on to stabilize the trajectory around it. The new parameter perturbations can be computed similarly [Eq. (11)], except that the stable, unstable and their corresponding

contravariant vectors are now associated with the target periodic orbit. These directions can be calculated using the same method discussed above [13].

4 Numerical Results for the Hénon Map

Figures 4(a-b) show an example of applying our algorithm to the chaotic saddle of the Hénon map shown in Fig. 1(a). We use a reference orbit on the chaotic saddle of length $N = 10017$. The maximally allowed parameter perturbation is $\delta = 0.01$ and the size of the small neighborhood around each point on the reference orbit is chosen to be $\epsilon = 0.005$. We can choose both δ and ϵ arbitrarily, as long as they are small. We start the trajectory to be stabilized with initial condition: $(x_0, y_0) = (0.5, -0.1)$. After 4 initial iterates, the trajectory falls into the neighborhood of a point of the reference orbit $[(x, y) \approx (-1.8393, 1.8387)]$. When this occurs, parameter control based on Eq. (6) is turned on to stabilize the trajectory around the reference orbit. At the time step $n = 846$, the controlled chaotic trajectory comes into the vicinity of the period-8 orbit, at which time we immediately turn on a new set of parameter perturbations calculated with respect to the period-8 orbit. The trajectory stays in the neighborhood of the period-8 orbit in subsequent iterations as long as the parameter perturbation is present. Figure 4(b) shows values of the parameter perturbations applied. Numerically, the controlled trajectory rapidly converges to the reference orbit both after $n = 4$ (stabilized around the reference orbit) and after $n = 846$ (stabilized around the period-8 orbit). After a few iterates, the parameter perturbations required become extremely small (around 10^{-10}).

The probability that a randomly chosen initial condition can be controlled, $P(N, \epsilon)$, depends both on the length of the reference orbit N and the size ϵ of the small region around each reference point. Figure 5(a) shows the $P(N, \epsilon)$ versus N curve, where $\epsilon = 0.005$. This curve is calculated by varying N systematically and randomly choosing 10^4 initial conditions with uniform probability distribution in the square region of Fig. 1(a) for each fixed N value. The probability is given by the ratio between the number of initial conditions that approach the reference orbit before escaping to infinity and the total number of initial conditions chosen (10^4). For small N values, say $N < 800$, $P(N, \epsilon)$ increases approximately linearly. The reason is that the probability that a trajectory enters the neighborhood of the chaotic saddle is approximately proportional to the total area of the small circles surrounding all the reference orbit points. This area is approximately $\pi\epsilon^2 N$ when overlaps between neighboring circles are small. As N increases further, the overlaps between neighboring circles become significant, thereby causing $P(N, \epsilon)$ to saturate. In fact, when $N > 1000$, $P(N, \epsilon)$ increases very slowly. For $N = 1000$, $P(N, \epsilon) = 0.546$. For $N > 10000$, we have $P(N, \epsilon) > 0.66$. We expect the optimum length of the reference orbit to be $N \sim 1/\epsilon$, the point when overlap starts to become significant. If a trajectory is directly stabilized around the period-8 orbit without being stabilized around the reference orbit, the probability that an initial condition can be controlled is *only 0.04* as shown by the lower straight line in Fig. 5(a). Thus, *by using a reference orbit of length about* 1000, *a factor of more than 10 improvement in this probability can be achieved.* The relation between $P(N, \epsilon)$ and ϵ for fixed $N = 8000$ is shown by the upper curve in Fig. 5(b). As a contrast, the lower curve in Fig. 5(b) shows the same probability when no reference orbit is used to stabilize the period-8 orbit.

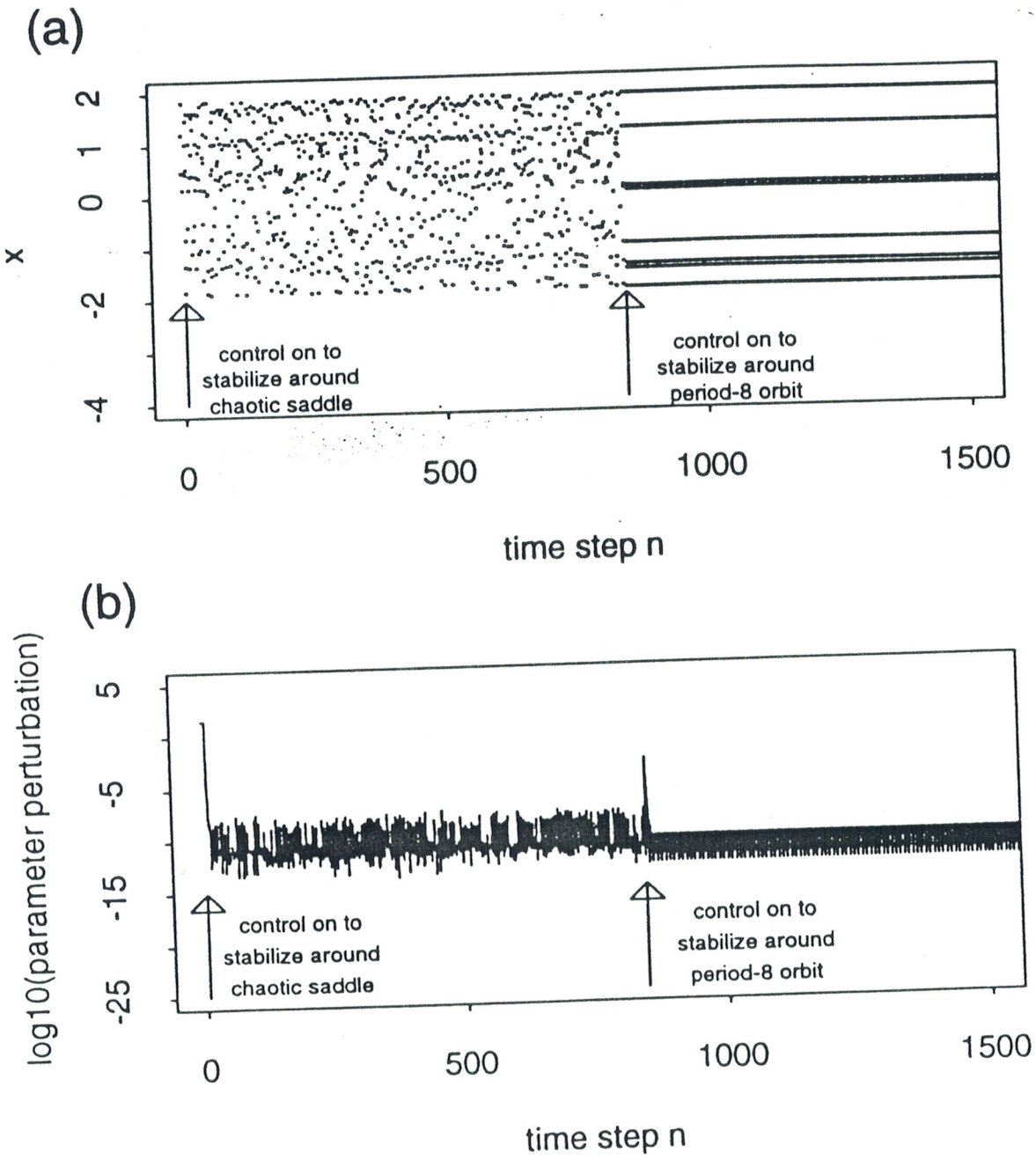

Figure 4: (a) An example of stabilizing the period-8 orbit shown in Fig. 1(a). A trajectory starts with the initial condition $(x_0, y_0) = (0.5, -0.1)$. At time step $n = 4$, it falls in a neighborhood of one point on the reference orbit. Parameter control is turned on to stabilize the trajectory around the reference orbit for $n \geq 4$. At $n = 846$, the controlled chaotic trajectory gets close to the period-8 orbit. A new set of parameter perturbations calculated with respect to the period-8 orbit is turned on to stabilize the trajectory. For $n \geq 846$, the controlled motion is period-8. (b) The time-dependent parameter perturbations applied at each time step.

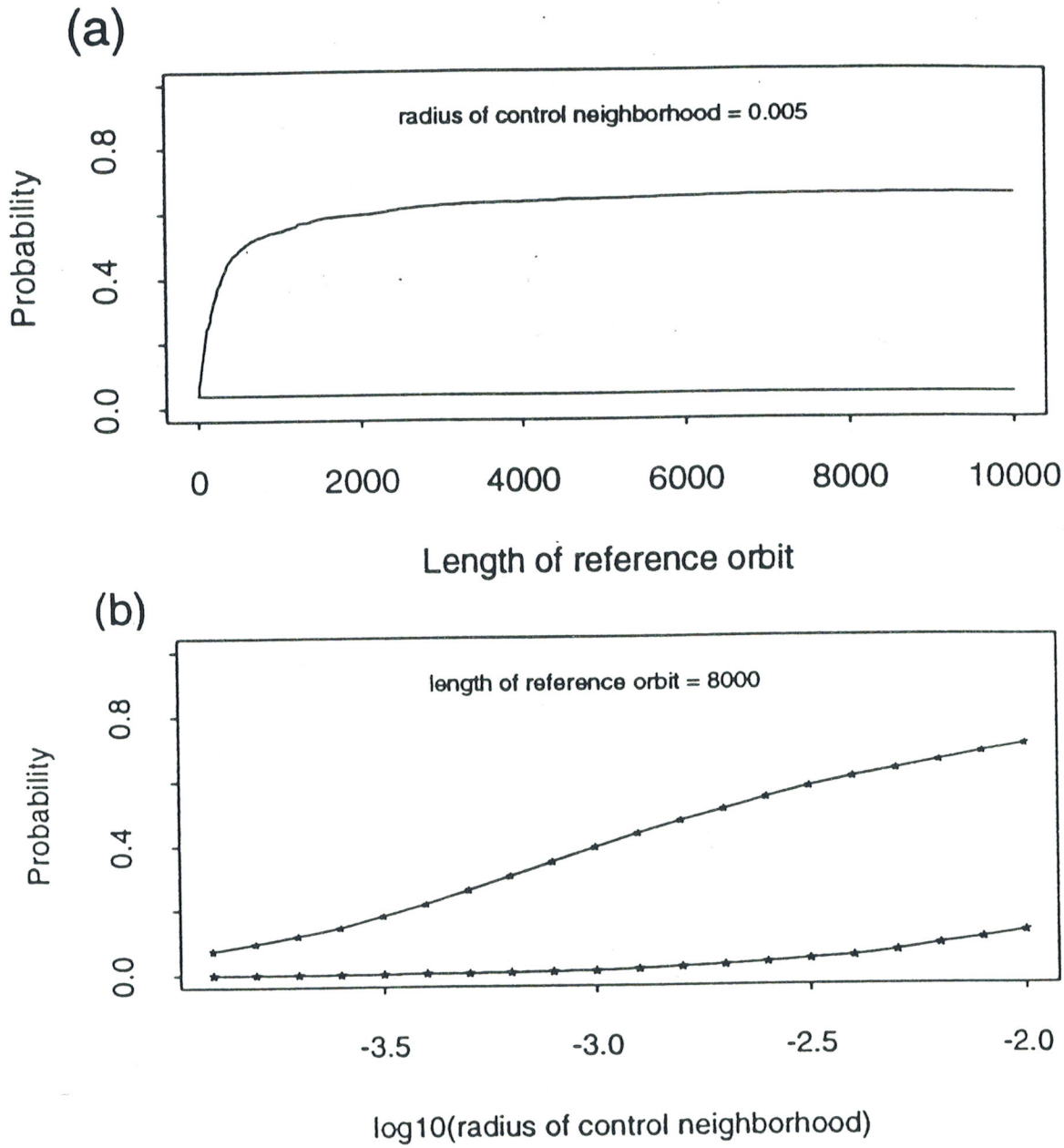

Figure 5: (a) For fixed $\epsilon = 0.005$ (the radius that defines the controlling neighborhood), the probability $P(N, \epsilon)$ that a randomly chosen initial condition can be controlled versus N, the length of the reference orbit (the upper curve). This probability increases initially with N and saturates for large N. The asymptotic value of $P(N, \epsilon)$ is approximately 0.66. The lower straight line represents the probability that the trajectory is directly stabilized around the period-8 orbit. The value of this probability is only 0.04. Therefore, by using a reference orbit of length about 1000, a factor of more than 10 improvement in this probability can be achieved. (b) For fixed $N = 8000$, $P(N, \epsilon)$ versus ϵ curve (upper curve). The lower curve is the same probability versus ϵ when the trajectory is directly stabilized around the period-8 orbit.

5 Conclusions

In this work we have devised a scheme to stabilize unstable periodic orbits embedded in chaotic saddles. By using a chaotic reference orbit on the chaotic saddle, the probability that a trajectory can eventually be stabilized around the target periodic orbit is significantly enhanced as compared with the case where no reference orbit is used. The novel feature is that we explicitly use the geometrical stucture, *i.e.*, stable and unstable directions, along a long reference orbit on the chaotic saddle to achieve the control.

The method we presented can also be applied to convert transient chaos into sustained chaos [16]. By constructing an arbitrarily long reference orbit on the chaotic saddle, we can make other trajectories stay in the neighborhood of this reference orbit for as long as we wish by applying small parameter control. In this sense, non-attracting trajectories in the neighborhood of the chaotic saddle are transformed into stable chaotic trajectories.

Our method may have applications in engineering problems such as the "voltage-collapse" that occurs in electrical power systems [17]. For a particular type of voltage collapse, the phenomenon is that the power supply system suddenly breaks down after exhibiting complicated dynamical bahavior resembling that of transient chaos [Fig. 1(b)]. Theoretical models for this type of voltage collapse suggest that transient chaos may be the culprit [17]. Therefore, the conversion of a transient chaotic trajectory into a sustained chaotic or periodic trajectory would prevent the voltage collapse.

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