TRANSIENT CHAOS: A TYPE OF METASTABLE STATE

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Chaotic dynamics can often be observed for finite times only. This transient chaos represents a kind of nonequilibrium state different from the asymptotic steady state. The underlying nonattracting chaotic set in phase space is the fractal analog of a saddle point and is often called a chaotic saddle. Basic concepts and fundamental recent applications are reviewed. Besides discussing the analogy with nonequilibrium processes, we summarize how a chaotic saddle can be reconstructed in experiments. The phenomenon of chaotic transport in open hydrodynamical flows is discussed in some detail.

1. Introduction: What is Transient Chaos?

The appearance of chaos with finite lifetime is known as *transient chaos* (for a review see [1]), and provides an example of a kind of "nonequilibrium state" that cannot be understood as an asymptotic behavior. In such cases one observes a moving around of the system in an apparently chaotic manner and then, rather suddenly, a settling down to a steady state which is either a periodic or a chaotic motion (but of different type than the transients). Studying only the asymptotic behavior of such systems would mean loosing the interesting, chaotic part contained in the transients.

1.1. *Every-day-life examples*

1. *Chaos machines*. We all know the simple double-rotor-like battery-driven devices taylored for demonstrating permanent chaotic motion. In practice, however, nearly all of them settle down to a periodic motion after a chaotic swinging of no longer than one hour. This clearly shows the robustness of transient chaos that - as we shall see in the paper - is a more general phenomenon than stationary or permanent chaos.

2. *Shimmying wheels*. Trolleys (like the ones used in supermarkets or at airports) have wheels which can choose the direction of their rollings. Even at a constant speed of towing one can observe a periodic motion of the wheels' orientation ('shimmy'). At higher speeds this motion might become chaotic which is often of transient type: after a while the irregular oscillation of the wheel disappears and the motion becomes regular and stationary. A detailed mechanical model of this phenomenon has been investigated in the engineering literature [2]. Chaotic shimmying was

shown to be the consequence of the non-rigid suspension of the wheel's axis and the possibility of both rolling and slipping. The orientational angle of the wheel observed at a given position of its axis is governed by an approximate one-dimensional map whose form was determined numerically. In the case of transient chaos, there is an interval around an unstable fixed point of the map which is mapped partially outside of the interval. The motion in this interval is chaotic and might be very long, but outside of it the effect of a coexisting attracting point (the origin) becomes essential and oscillations gradually die out [2].

1.2. *Experiments*

The experimental investigation of transient chaos has received, in spite of its relevance, relatively little attention (for a few examples see [3] –[12]) and, with the exception of very recent efforts [11,12], has mainly concentrated on determining the most striking characteristics, the average chaotic lifetime only. Here we briefly discuss some of these experiments.

1. *Convection loop experiment* [4,9]. The apparatus consists of a loop-shaped reservoir filled with water that is heated from below along the lower semicircle and cooled from above. The system is the one-dimensional analog of the Rayleigh-Bénard convection problem. After convection sets in at sufficiently high heat flux, the velocity of the fluid along the loop changes its sign and modulus randomly. There is a broad range of system parameters where these chaotic oscillations do not last forever, but sooner or later (depending on the initial conditions) the system reaches a stage where oscillations cease to exist and a uniform rotation in one direction sets in.

2. *Chemical reactions preceeding thermal equilibrium*. It is well known that stirred chemical reactions in closed containers cannot be chaotic in a stationary manner since the system approaches thermal equilibrium corresponding to a fixed point attractor in the space of concentrations. It has been conjectured, however, that the *approach to equilibrium*, i.e., the kinetics before reaching the final attractor can be chaotic. This has to be a kind of transient chaos [13]. Recently, the conjecture has been verified with the example of the Belousov-Zhabotinsky reaction, and chaotic oscillations of the light transmission have been observed over several hours before a stage very close to thermal equilibrium was reached [10].

3. *NMR laser and pendulum experiments*. These are the first experiments in which the chaotic set responsible for the transient behavior could be

reconstructed. This has been achieved by considering only those parts of several thousands transient signals that did *not* belong to the attractor. *NMR laser:* The low noise and drift free data of an NMR laser made this system appropriate for the experimental investigation of a variety of chaotic phenomena. The laser output is a time-dependent voltage, and the control parameter can be considered to be the modulation amplitude. By gluing several truncated signals (that contained points far away from the attractor only) together at a level corresponding to a background voltage very close to zero, a long artificial time series was created. It could then be investigated by standard methods of time series analysis worked out originally for permanent chaotic signals. This way not only the fractal chaotic set was reconstructed, but also its dynamical properties, like the average Lyapunov exponent or the correlation decay could be determined [11].

Pendulum: The parametrically driven damped pendulum is another example where high precision experiments can be carried out. The angle and the angular velocity of the pendulum were measured, and the damping constant played the role of the control parameter [12]. From several distinct truncated signals the chaotic set was reconstructed on a stroboscopic map. This experiment clearly demonstrates the fractality along the unstable foliation (cf. Section 2). The generalized dimensions of the chaotic set were also determined.

2. Basic Properties of Transient Chaos

2.1. *The chaotic set*

In transiently chaotic systems one finds a *nonattracting chaotic set* in the phase space. In physical cases characterized by *invertible* dynamics this set is a *chaotic saddle* [14,15,17,1]. Similarly to a chaotic attractor, a chaotic saddle also contains an *infinity* of unstable orbits, but is globally *not* attracting. In this paper we shall mainly consider systems with three-dimensional phase spaces, or equivalently, two-dimensional discrete-time dynamics. As an example, Fig. 1 shows the chaotic saddle of the Hénon map (at parameter values different from the usual ones $a = 1.4, b = 0.3$ belonging to the Hénon attractor).

This observation can be used to define transient chaos. We say that a dynamical process is chaotic in a transient sense if *there exists an underlying nonattracting chaotic set* in the phase space. As a consequence of the fractal structure and the infinity of unstable orbits, the chaotic saddle's *topological entropy has to be positive*, in contrast to the attractor's topological entropy.

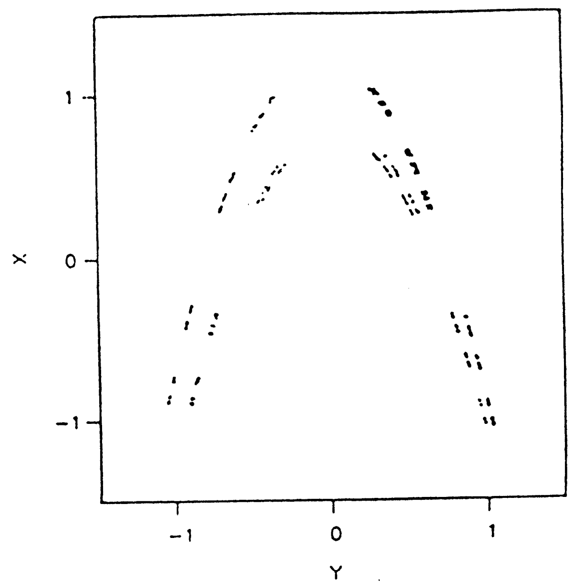

Fig. 1. The chaotic saddle of the Hénon map $x' = 1 - ax^2 + by, y' = x$ at parameters $a = 2, b = 0.3$ where no other invariant sets of the map exist. Trajectories escaping from this set go to minus infinity. Note the pronounced fractal structure resembling the direct product of two Cantor sets.

2.2. *Invariant manifolds*

A chaotic saddle is the chaotic analog of a saddle point that repells trajectories from its neighborhood, but has nevertheless an invariant subspace of volume zero along which it is attractive. This subspace is called the *stable manifold* of the chaotic saddle. The stable manifold provides a fractal foliation of the phase space (see Fig. 2). Similarly, there is an invariant subspace along which the chaotic set repells out those exceptional trajectories that have hit it infinitely long time ago, the set's *unstable* manifold. More precisely, this can also be defined as the stable manifold of the time reversed dynamics. Sometimes it is worth considering the chaotic saddle as the intersection of its invariant manifolds.

2.3. *Fractal properties*

A chaotic saddle has *more pronounced fractal properties* than a chaotic attractor. Due to the escape, the chaotic saddle has holes on all scales *along its unstable manifold*, too. The saddle of the Hénon map appears to be the direct

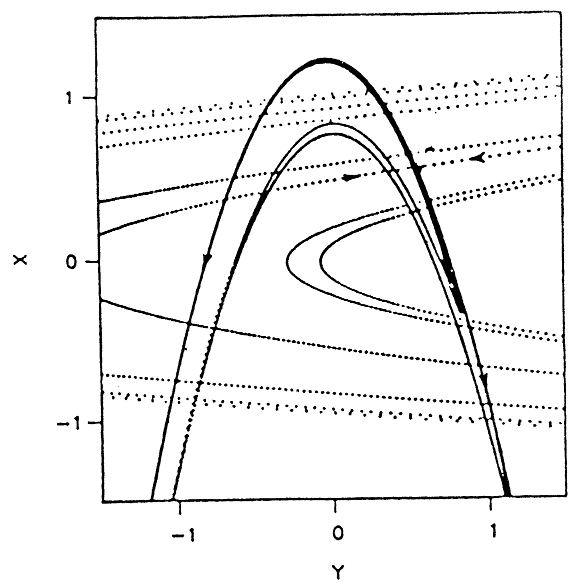

Fig. 2. The stable (dotted lines) and unstable (full lines) manifolds of the Hénon map chaotic saddle at $a = 2, b = 0.3$.

product of two Cantor sets. The full fractal dimension D_0 of the saddle can thus be written as the sum of the Cantor sets' dimensions: $D_0 = D_0^{(s)} + D_0^{(u)}$, where $D_0^{(s)}$ ($D_0^{(u)}$) is called the saddle's *partial* fractal dimension along the stable (unstable) direction [15]. Because the invariant manifolds provide a foliation of the phase space with smooth curves, their fractal dimension is obtained as the corresponding partial dimension plus unity. Thus, for the fractal dimension of the stable (unstable) manifold we find [17] $d^{(s)} = 1 + D_0^{(u)}$ ($d^{(u)} = 1 + D_0^{(s)}$). For a chaotic attractor the partial dimension along the unstable manifold is always unity (no escape), thus its stable manifold is space filling in certain regions at least. This is why chaotic attractors have finite basins of attractions. A chaotic saddle's stable manifold, and consequently its basin of attraction, is, however, a fractal of *zero volume*. Chaotic saddles typically coexist in dissipative cases with an attractor, and in Hamiltonian systems with an exit channel leading to an asymptotic motion. Trajectories starting from randomly chosen initial points then approach the attractor or the asymptotic state with probability one. Because of the saddle's stable foliation, however, they might come close to the saddle and stay in its vicinity for a longer time. What is observable numerically or experimentally is not the chaotic

saddle but rather a *small neighborhood* of it. This results in the appearance of a chaotic motion for finite times. The actual time spent around the saddle depends sensitively on initial conditions but the average transient lifetime is well defined.

2.4. *The escape rate*

When distributing a large number N_0 of initial points in a region overlapping with the chaotic saddle (but not overlapping with the attractor(s)), the trajectories emanating from these points will leave the region with certainty. Those with initial points close enough to a branch of the stable manifold have a long time before escaping. Thus the number of points $N(t)$ staying still in the preselected neighborhood after a time t (t can also denote discrete time) is a function monotonously descreasing to zero. How rapidly it decreases is an important characteristics of the saddle. Often one finds that the decay is exponential for long times [15], i.e.,

$$N(t) \sim N_0 e^{-\kappa t}. \tag{1}$$

for $t \to \infty$. In what follows we shall deal with such hyperbolic cases. The positive number κ is called the *escape rate* of the saddle and turns out to be independent of the shape of the region containing the initial points. The escape rate measures the saddle's strength of repulsion: a large escape rate marks a rather unstable saddle. Conversely, $\tau = 1/\kappa$ is considered to be the average lifetime of transient chaos.

2.5. *The natural distribution and Lyapunov exponents*

Just like on chaotic attractors, there exist invariant distributions on chaotic saddles, too. The so-called *natural distribution* [15,1] on a saddle is obtained by distributing an *ensemble* of points around it and following those with long lifetimes. The frequency of visiting different regions of the saddle by these trajectories defines the natural probability measure. One can then speak about characteristic numbers taken with respect to this measure. The largest average Lyapunov exponent λ on the saddle is positive, and is of basic importance since it characterizes the typical divergence of nearby trajectories staying for long times around this set. The Lyapunov exponent describes the local instability of the saddle, while the escape rate is a global measure of instability. It is natural to believe that these quantities are intimately related.

2.6. *Fundamental relations*

A basic relation [15] connecting the escape rate with metric properties says that

$$\kappa = (1 - D_1^{(u)})\lambda = \lambda - K_1 \tag{2}$$

where $D_1^{(u)}$ is the *information* dimension (which is typically very close to the fractal dimension) along the unstable direction, and K_1 represents the metric or Kolmogorov-Sinai entropy of the saddle. (For a chaotic attractor $\kappa = 0$ and we recover the well-known relations $D_1^{(u)} = 1$ and $K_1 = \lambda$.) As a consequence, for a saddle

$$D_1^{(u)} = 1 - \frac{\kappa}{\lambda}, \tag{3}$$

that is, the deviation of the partial information dimension along the unstable manifold from unity is given by the ratio of the global and local instabilities' measures. This deviation is the stronger the larger is the escape rate measured on the Lyapunov time scale. The partial information dimension $D_1^{(s)}$ along the stable direction turns out to be λ/λ' times $D_1^{(u)}$, where λ' is the modulus of the negative average Lyapunov exponent [15]. This leads to a generalization of the Kaplan-Yorke formula valid for transient chaos. The saddle's full information dimension can be written as the sum of the partial dimensions [16,15]

$$D_1 = (1 - \frac{\kappa}{\lambda})(1 + \frac{\lambda}{\lambda'}) = K_1(\frac{1}{\lambda} + \frac{1}{\lambda'}). \tag{4}$$

These relations connect fractal and dynamical properties in a unique and elegant fashion.

3. Types of Transient Chaos

3.1. *Type I — in coexistence with simple attractors*

The cleanest form of transient chaos occurs when the system has a periodic attractor coexisting with a chaotic saddle [14,15]. Then the asymptotic behaviour is simple, the maximal Lyapunov exponent on the attractor is negative. The transients, however, exhibit chaotic properties, and depending on the parameter can be rather persistent. Cases when the attractor is at infinity (see e.g., the Hénon example) or open Hamiltonian systems when the role of the attractor(s) is taken over by different exit possibilities are also of this type. In such cases the dynamical complexity (and the positive Lyapunov exponent and metric entropy around the saddle) would be completely missed when studying the asymptotic properties only.

3.2. *Type II — in coexistence with chaotic attractors*

The situation when a chaotic attractor coexists with a chaotic saddle is a more complicated and more interesting case. Then two different kinds of chaos are present. The motion around the saddle is, however, typically *more* chaotic than on the attractor (both the Lyapunov exponent and the metric entropy are larger), as, e.g. in the NMR laser experiment [11]. Thus, studying asymptotic properties only would lead to neglecting the dominant chaotic part of the full complex dynamics.

4. Occurrence of Transient Chaos

Here we give a list of phenomena from the class of deterministic dynamical systems that are inevitably related to transient chaos.

4.1. *Periodic windows*

Periodic windows are ubiquitous in the chaotic regime of dynamical systems [18]. In such windows chaos is present in the sense that there exists an infinity of periodic orbits but their union is not neccessarily attractive. Transient chaos of Type I (II) occurs inside (outside) the period doubling regime of these windows where the attractor is a cycle of length 2^n with an integer n. Since the total measure of windows is finite in the parameter space, just like that of strictly chaotic cases, the probability to find transient chaos is on the *same* order as that of permanent chaos.

4.2. *Crises*

Transient chaos can also be a sign of permanent chaos to be born, hence the name "preturbulence" or "metastable chaos" [19]. More generally, all types of crisis configurations: attractor destructions, explosions or mergings [14] are accompanied with long lived transient chaos. Large attractors born at crises incorporate into themselves the chaotic saddles existing before. Consequently, the dynamical properties of the saddle are partially inherited by the large attractor. This is the origin for the scaling behaviour found in such systems [20]. The average time τ_1 trajectories of the large attractor spend in the region where the saddle existed (and continuos to exist) is practically the same as the average lifetime of transient chaos in the pre-crisis regime (cf. NMR laser experiment [11]). The average time τ_2 of visiting the pre-crisis attractor, however, changes as a negative power of the control parameter difference $a - a_c$. This is nothing but the transient chaotic lifetime on a newborn chaotic saddle into which the pre-crisis chaotic attractor turned at crisis. Transient chaos can thus provide a backbone of the motion on composed attractors [21,22]. More generally, one can select any finite subregion of the phase space and determine the set of orbits never escaping this region. If the subregion overlaps with an attractor (or the chaotic sea in Hamiltonian cases), these orbits form a fractal chaotic saddle responsible for the transient chaotic dynamics exhibited by trajectories not bounded to this subregion forever [23].

4.3. *Fractal boundaries*

Fractal basin boundaries [24] are another common properties of dynamical systems. If two or more simple or chaotic attractors coexist, trajectories may hesitate for a long time to which of the attractors they will asymptote to. On fractal basin boundaries such trajectories exhibit transient chaos. In fact, fractal basin boundaries turned out to be *stable manifolds of chaotic saddles* [25].

Uncertainties in initial conditions on fractal boundaries lead to enhanced uncertainties in the final state. This can be characterized by an uncertainty exponent α [24]: the fraction $f(\epsilon)$ of uncertain initial conditions typically decreases with the box size ϵ according to a power law: $f(\epsilon) \sim \epsilon^{\alpha}$. The fractal dimension of the intersection of the saddle's stable manifold with a one-dimensional line is given by $d = 1 - \alpha$. In two-dimensional maps $\alpha = 1 - D_0^{(u)}$. It has recently been shown that in higher dimensional systems the boundaries can be widely interwoven [26]. Such *riddled basins* are characterized by near-zero uncertainty exponents. This means that an increase in the resolution hardly improves the prediction of the system's final outcome. Simultaneously, the fractal dimension of the set of intersecting points of the stable manifold with a one-dimensional line is a rather dense fractal with dimension close to unity.

4.4. *Controlling chaos*

The OGY method of controlling permanent chaos [27] is based on the requirement that control sets in if the trajectory visits a preselected target region. The set of points never reaching this target region form a fractal subset whose escape rate determines the time needed to achieve control. Thus, the OGY method converts the motion on a chaotic attractor into a kind of transient chaos before control sets in. The OGY method has been extended to controlling transient chaos, too, both in dissipative and in Hamiltonian cases [28,29].

5. Applications of Transient Chaos

The concept of transient chaos has proved to be rather useful outside the realm of dynamical systems, as well. Below we give a few examples belonging to entirely different fields of physics. We can just mention here that interesting applications can be found in other disciplines, too, among which epidemiology is a notable example [30].

5.1. *Chaotic scattering*

For scattering processes in open Hamiltonian systems the only way chaos can appear is in the form of transients, because of the incoming and outgoing motion's asymptotic freedom. Trajectories are then trapped in a given region of the configuration space for a while. Because of the Hamiltonian character of the dynamics the chaotic saddle has a special symmetry that implies the coincidence of the stable and unstable manifolds' dimensions. In systems with two degrees of freedom or for two dimensional area preserving maps

$$D_0^{(u)} = D_0^{(s)} \qquad D_1 = 2(1 - \frac{\kappa}{\lambda}). \tag{5}$$

A detailed characterization of the trapping process is based on the so-called time delay function. It tells us how the time $\tau(b)$ spent around the chaotic saddle depends on the impact parameters b of initial conditions taken along a line. A unique sign of chaotic scattering is the rather *irregular* appearance of the time delay function. The function $\tau(b)$ is singular in points that lie on branches of the saddle's stable manifold. Consequently, the time delay function has singularities on a fractal whose dimension is the partial fractal dimension of the saddle. The phenomenon of chaotic scattering [31] has attracted great interest recently: the field of its application ranges from chemical reactions, via acustics and elasticity to celestial mechanics.

5.2. *Chaotic mixing in open hydrodynamical flows*

The passive advection of tracer particles (e.g., small dye droplets) in open hydrodynamical flows with uniform inflow and outflow velocities is one of the most appealing applications of chaotic scattering [32] - [36]. The reason is that in two-dimensional incompressible fluids the tracer dynamics turns out to be Hamiltonian with the plane of the flow as its phase space. Thus, the fractal structures appearing in the phase space become now observable *by naked eye*. This phenomenon is thus a good candidate for being the first experimental investigation of chaotic scattering. The chaotic saddle consists now of an infinity of unstable tracer orbits, restricted to a finite domain of the flow. Tracers injected into the flow before this domain are either transported away quickly or, if being sufficiently close to the stable manifold, stay a long time around the saddle. These particles finally escape very close to the unstable manifold. Thus the *unstable manifold of the saddle becomes a direct physical observable*: the object traced out by particles with long life times. Its fractal dimension is $d^{(u)} = 1 + D_0^{(s)} = 1 + D_0^{(u)}$ and is definitely less than two. Transient dynamics is thus accompanied with pronounced and persistent fractality. The latter is reflected by the singularity distribution of the particles' time delay function, too. These general features do not depend on whether the fluid is viscid or not. The key ingredients are the time dependence and incompressibility of the flow.

As an example we consider a model of the so-called 'leapfrogging' motion of smoke rings. If the rings have the same sense of rotation and move along the same axis, the rear vortex ring attempts to pass through the front one. The leading ring then widens due to the mutual interaction and travels more slowly. Simultaneously, the other ring shrinks, travels faster and penetrates the first one. This process is then repeated continuously, with some period. In [35] we examined the two-dimensional analog of this process: the advection in the field of two coaxial pairs of ideal point vortices of the same strength, which also exhibit a strictly periodic motion of some period T. It is worth then investigating the particle dynamics on a stroboscopic map taken at integer multiples of T. Figure 3 shows the chaotic saddle of the map, as well as its unstable manifold, in a frame comoving with the vortex system. The manifold has been obtained by injecting a droplet of tracers in front of the system and plotting its shape after a long time. It qualitatively agrees with a photograph of smoke rings taken in a laboratory experiment [37].

5.3. *Random systems*

Renormalization group transformations or, more generally, length scaling or decimation procedures when applied to models of disordered systems often lead to unbounded iterations. A classical example is provided by the ***phonon spectrum of fractals***. Imagine a two-dimensional Sierpinski gasket lattice of point masses linked by identical strings and allowed to move perpendicularly to the plane of the lattice. A decimation procedure applied to frequency ω of the collective motion yields [38] a new frequency $\omega' = \omega(5 - \omega)$. This is the logistic map at a parameter where it generates transient chaos. A quantity of central interest is the spectrum of eigenfrequencies. The invariant set of the frequency iteration is obviously part of the spectrum. Thus, each point of the

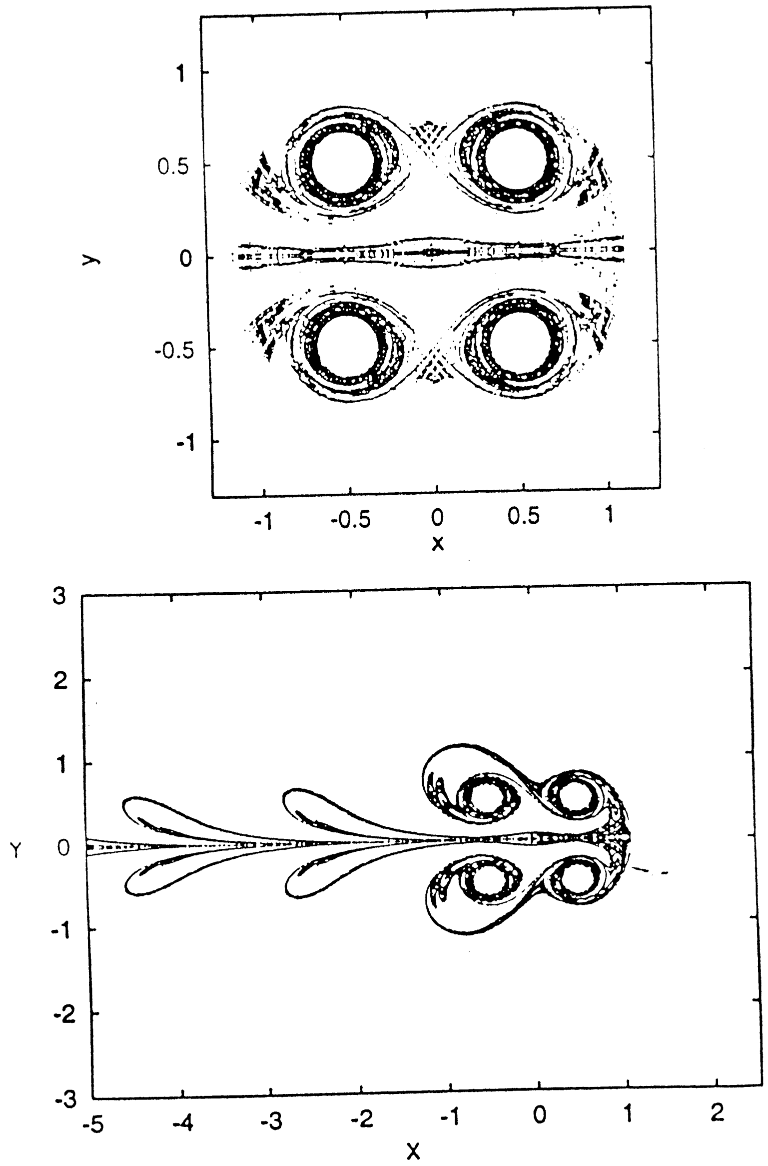

Fig. 3. Top: chaotic saddle around four leapfrogging ideal vortices as appearing in the plane of the flow. Bottom: the unstable manifold of the same saddle. The snapshots are taken in both cases when the two upper vortices fall on the same horizontal line. The white regions around the vortex centers are never reachable by tracers coming form outside. They can be considered as the vortex cores from the point of view of the tracer dynamics.

chaotic set represents an element of the vibration spectrum which exhibits, therefore, a *Cantor set structure*. More recent examples are provided by the theory of random walks in random environments [39], and the learning process of neural networks with random input data [40].

5.4. *Noise induced chaos*

In systems subjected to external random forces the form of the attractor observed might depend on the noise intensity. The phenomenon when a system with simple periodic attractors turns to be chaotic at sufficiently strong noise is called noise induced chaos [41]. It has recently be interpreted as a consequence of transient chaos [42]. In such systems there is always a chaotic saddle coexisting with the simple attractors. At increasing noise intensity the saddle becomes embedded, together with the simple attractors, into a noisy chaotic attractor. Another interesting effect of weak noise is that it decreases the escape rate from a saddle, i.e., it stabilizes transient chaos [43].

5.5. *Transport phenomena*

Diffusion and other transport phenomenona can be interpreted as consequences of chaotic scattering and transient chaos [44]. This new deterministic way of describing transport phenomena in a single particle picture is based on the idea of considering an open Hamiltonian system that is of finite but large extent L along a given direction. The phase space is low-dimensional but of large linear size. One can calculate the escape rate from the chaotic saddle and show that it can be connected with transport coefficients [44,45,46]. This escape rate formalism has previously been applied to cases corresponding to a symmetric random walk only. When the system is also subjected to a bias causing a nonzero average velocity, it has recently been shown [47] that the escape rate can be written as

$$\kappa = \frac{1}{2}\frac{v^2}{D} + \frac{1}{2}\frac{D\pi^2}{L^2}. \qquad (6)$$

Here v and D are the (constant) drift and diffusion coefficients of the system on a macroscopic level that also appear in the Fokker-Planck equation governing the macro dynamics. This formula says that escape is responsible for two different ways of transport: drift and diffusion, and that the corresponding coefficients can be extracted via finite size scaling. It is worth mentioning that other characteristics of the chaotic scattering process cannot be expressed solely by means of macro parameters. It is the escape rate alone that has a well defined thermodynamical limit.

5.6. *Supertransients*

Transient chaos also occurs in spatiotemporal dynamical systems having high-dimensional phase spaces. These transients differ from their low-dimensional counterparts in that the average lifetime can be extremely long before settling down into a final attractor which is usually nonchaotic. Often the lifetime grows exponentially with the system size L, i.e,

$$\kappa \sim \exp\left(-aL^{\sigma}\right) \qquad (7)$$

where a and σ are positive constant, the latter on the order of unity. The orgin of such *supertransients* [48,49] is not yet fully understood. Their presence implies that the observation of the systems's actual attractor is practically impossible. It has recently been shown [50] that supertransients are accompanied with very dense chaotic saddles: their stable manifolds have fractal dimensions arbitrary close to the phase space dimension. In some other cases the escape rate decreases slower; as a negative power of the system size [48,51].

6. Conceptual Novelties

- Transient chaos plays a similar role in the realm of chaotic processes as an unstable equilibrium point in mechanical systems. Starting an ensemble of trajectories around such a hyperbolic fixed point, particles escape from any neighborhood of it with a rate governed by the positive Lyapunov exponent. A chaotic saddle is a fractal set of such hyperbolic points. It is, however, globally *less* repelling than the component periodic orbits because the escape rate is less than the average Lyapunov exponent by $D_1^{(u)}\lambda$. The fractal structure thus stabilizes the saddle, and a dense set implies rather small escape rates.
- Transient chaos is an example for a phenomenon where long time single particle and short time ensemble averages are different. An ensemble of trajectories staying around the chaotic saddle yields averages characterizing the saddle, different from the long time asymptotics.
- Transient chaos might also shed new light on previous observations. For instance, the famous "period-3 implies chaos" statement [52] means in modern terminology: period-3 implies transient chaos.

In conclusion, we believe that transient chaos is an essential part of nonlinear dynamics accessible in experiments, too, and is needed to the correct interpretation of many interesting phenomena. It is especially well suited to characterize nonequilibrium processes preceding the approach to steady states.

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