

Renormalization Group Analysis of Relaxational Dynamics in Systems with Many-Component Order-Parameter I*

P. Szépfalusy^{1**} and T. Tél²

¹ Fachrichtung Theoretische Physik, Universität des Saarlandes, Saarbrücken, Federal Republic of Germany

² Institute for Theoretical Physics, Eötvös University, Budapest, Hungary

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General properties of the dynamic renormalization group transformation are studied by investigating the multicomponent relaxational model both with conserved and non-conserved order parameter in the large n limit. Exact expressions are given for the transformation of an infinite number of parameters. The strong dependence of all the dynamic quantities on whether the order parameter is conserved or not is illustrated. Critical points of higher order inherent in the model are also discussed. Explicit expressions for the action near the stable fixed points are derived. Different formulations of the dynamic renormalization group are compared and the conditions under which they are equivalent are found.

I. Introduction

The large- n limit of the Landau-Wilson model (n being the number of components of the order parameter) has proved to be an excellent tool for studying how the renormalization group transformation works in describing static critical behaviour [1–6]. Similarly its simplest time-dependent generalization, the large- n system with purely relaxational dynamics is expected to provide a convenient model for testing and expanding various general ideas related to the dynamic renormalisation group (DRG).

Although the dynamics of multicomponent systems has been discussed by several authors (see [7] and references therein) and the dynamic critical exponent has been determined up to order $1/n$ in the relaxational model [8, 9], an analysis using a Wilson type dynamic renormalisation group transformation in the large- n limit with successive elimination of short wavelength fluctuations has been carried out only recently [10, 11]. The parameter space on which the DRG transformation acts has played a central role in the procedure and turned out to be much

more complicated in the large- n limit than in the small ε ($\varepsilon \equiv 4 - d$) case.

The aim of the present paper is to generalize these latter investigations in different ways. First we include into the discussion the case of the conserved order parameter as well. In this way it is demonstrated that all the dynamic parameters depend on whether the order parameter is conserved or not. The model is rich enough to exhibit critical points of higher order the properties of which are also discussed.

An other aspect of this work is the comparison of different ways of the DRG calculations, namely the perturbative procedures carried out on the equation of motion [12–14] and alternatively in the path probability distribution expressed by the help of an action [15–20] and furthermore a saddle-point method in the latter formalism [10].

The couplings in the action which are local in space and time transform among themselves. The saddle-point method which has been used also in [10] enables us to follow the transformation of these couplings in a global way. On the other hand parameters related to the non-local couplings can be treated only by one of the perturbative methods.

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** On leave from the Central Research Institute for Physics, H-1525 Budapest. P.O.B. 49, Hungary

The main results are presented below along with the outline of the paper.

After a short discussion of the model and of the DRG procedure in the path probability formalism (Sect. II) we turn to the discussion of the non-perturbative treatment (saddle-point method) in Sect. III. The exact transformation is given for the local couplings. Even this restricted part of the parameter space contains an infinite number of static and dynamic parameters. Besides the ordinary critical point, the higher order critical points are also investigated. In each case the deviation of the action from its fixed point at T_c turns out to be a product of the non-linear scaling field with the largest exponent and of the corresponding eigenvector of the linearised transformation provided we are close to the fixed point. These scaling fields are purely static ones but the eigenvectors bear the marks of dynamics and depend on whether the order parameter is conserved or not.

By means of the perturbative method of Sect. IV the transformations of wavenumber- and frequency-dependent couplings are given. The transformed couplings are not all independent since fluctuation-dissipation theorems relate them. Two different representations of the parameters are presented; some of the details are relegated to Appendix A. It will be demonstrated that at higher order critical points, at T_c , the couplings originally local in space and time remain local after the transformation, too.

Comparing the DRG calculations carried out on the path probability and on the equation of motion it is found that they are equivalent only if higher order cumulants of the random coefficients generated in the equation of motion are also taken into consideration (Sect. V). This connection yields an interpretation of all the parameters in the action as different cumulants of the random vertices in the equation of motion. In the symmetry breaking phase the parameter space has to be further enlarged as it is demonstrated in Appendix B.

II. The Model and the DRG Transformation in the Path Probability Formalism

We consider a d -dimensional system of volume V the static behaviour of which is described by the Hamiltonian [2, 3]

$$\mathcal{H} = \int d^d x (a(\nabla\phi)^2 + U(\phi^2)), \quad (2.1)$$

where

$$(\nabla\phi)^2 = \frac{1}{2} \sum_{j=1}^n (\nabla\phi_j)^2, \quad \phi^2 = \frac{1}{2} \sum_{j=1}^n \phi_j^2. \quad (2.2)$$

$\phi = \{\phi_j | j=1, \dots, n\}$ denotes the n -component order parameter with wave number cut-off Λ . The function U is represented by a Taylor series as

$$U(\phi^2) = \sum_{m=1}^{\infty} \frac{u_{2m,2}}{2^m} (2\phi^2)^m, \quad (2.3)$$

where the coefficients $u_{2m,2}$ are of order n^{1-m} to ensure the existence of the limit $n \rightarrow \infty$.

The time variation of the order parameter is governed by the following Langevin type equations

$$\dot{\phi}_j = -L \frac{\delta \mathcal{H}}{\delta \phi_j} + \zeta_j, \quad j=1, 2, \dots, n. \quad (2.4)$$

ζ is assumed to be a random field with a Gaussian distribution and white spectrum. The bare coefficient L is constant for a non-conserved order parameter and is proportional to the Laplacian for a conserved one (models A and B in [7, 8]):

$$L = \Gamma(i\nabla)^c, \quad c=0 \text{ or } 2. \quad (2.5)$$

The stochastic process (2.4) can be described equivalently by the path probability functional $P\{\phi\}$ which reads

$$P\{\phi\} \propto \int \prod_{j=1}^n d\{\tilde{\phi}_j\} \exp \mathcal{A}\{\tilde{\phi}, \phi\},$$

where $\tilde{\phi}$ stands for the response field [21]. $\mathcal{A}\{\tilde{\phi}, \phi\}$ denotes the action [16–19]. In our case

$$\begin{aligned} \mathcal{A}\{\tilde{\phi}, \phi\} = & \int dt \int d^d x \sum_{j=1}^n (-\tilde{\phi}_j L \tilde{\phi}_j + i \tilde{\phi}_j (\dot{\phi}_j - a L \Delta \phi_j) \\ & + i \tilde{\phi}_j L \phi_j U^{(1)}(\phi^2) + 1/2 \Gamma K(c) U^{(1)}(\phi^2)) \end{aligned} \quad (2.6)$$

with

$$K(c) = K_d \int_0^{\Lambda} q^{c+d-1} dq, \quad (2.7)$$

$$K_d = 2^{1-d} \pi^{-d/2} / \Gamma(d/2). \quad (2.8)$$

Furthermore we introduced above the following notation

$$U^{(i)}(\phi^2) \equiv d^i U(\phi^2) / d(\phi^2)^i. \quad (2.9)$$

The last term of \mathcal{A} stems from the functional Jacobian [15, 18, 19] in the large- n limit.

The DRG transformation R_b is defined on the weight functional $\exp \mathcal{A}$ by integrating over shells in the space of wave numbers [17, 20]. The new action is determined by the equation

$$\begin{aligned} & \exp \mathcal{A}'\{\tilde{\phi}', \phi'\} \\ & = \int \prod_{j, \Lambda/b < k < \Lambda, \omega} d\tilde{\phi}_{j,k,\omega} d\phi_{j,k,\omega} \exp \mathcal{A}\{\tilde{\phi}, \phi\} \end{aligned} \quad (2.10)$$

with the rescaled quantities

$$\begin{aligned} x' &= b^{-1} x, & t' &= b^{-z} t, \\ \phi'(x', t') &= b^{-1+\eta/2+d/2} \phi(x, t), \\ \tilde{\phi}'(x', t') &= b^{-\bar{\gamma}} \tilde{\phi}(x, t). \end{aligned} \tag{2.11}$$

In (2.10) $\tilde{\phi}_{j,k,\omega}$ and $\phi_{j,k,\omega}$ denote the Fourier components of the response field and of the order parameter respectively:

$$\phi_j(x, t) = V^{-1/2} \sum_{k,\omega} \phi_{j,k,\omega} e^{i(kx-\omega t)}, \tag{2.12}$$

and similarly for $\tilde{\phi}_j(x, t)$. Here

$$\sum_{k,\omega} \equiv \sum_{k < A} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}. \tag{2.13}$$

This procedure shows a strong analogy to the static RG [22, 2].

III. Non-Perturbative Treatment (Saddle-Point Method)

The Parameter Space

Before carrying out the calculation we outline the structure of the parameter space. The form of the action we have started with, (2.6), turns out to be not sufficiently general i.e. after the RG transformation new terms arise. It can be proved that the parameters of the functional

$$\begin{aligned} \mathcal{A}\{\tilde{\phi}, \phi\} &= \int dt \int d^d x \left[\sum_{j=1}^n \{ -\tilde{\phi}_j L \tilde{\phi}_j \right. \\ &\left. + i \tilde{\phi}_j (\dot{\phi}_j - a L \Delta \phi_j) \} + Y(\phi^2, \varphi) \right], \end{aligned} \tag{3.1}$$

$$\varphi \equiv i \sum_{j=1}^n \tilde{\phi}_j L \phi_j + (n/2) \Gamma K(c) \tag{3.2}$$

span the whole parameter space involved (at least above T_c) provided we treat only local couplings in space and time*. The latter restriction is necessary in order that the saddle-point method be applicable. Y is represented by a double Taylor series

$$Y(\phi^2, \varphi) = \sum_{m=1}^{\infty} \sum_{1 \leq l \leq m} u_{2m, 2l} \varphi^l (2\phi^2)^{m-l} \tag{3.3}$$

with $u_{2m, 2l} = \mathcal{O}(n^{1-m})$ for all l 's. Initially

* These couplings transform among themselves in the large n limit. Some of the couplings treated here are related to frequency dependent ones as will be seen in the next Section

$$Y(\phi^2, \varphi) = \varphi U^{(1)}(\phi^2) \tag{3.4}$$

as it follows from (2.6). For $l=1$ the couplings in (3.3) agree with those in (2.3).

For the sake of convenience we introduce the notation for the derivatives of Y as follows

$$Y_{i,j}(\phi^2, \varphi) \equiv \partial^{i+j} Y(\phi^2, \varphi) / \partial(\phi^2)^i \partial\varphi^j. \tag{3.5}$$

$Y(\phi^2, 0)$ can be proved to be a constant after the transformation which will not be regarded as a parameter*. Thus $Y_{0,1}(\phi^2, \varphi)$ will be of importance since it specifies all the parameters besides a and Γ . Therefore we can specify the parameter space as

$$\begin{aligned} \mu &= (a, \Gamma, Y_{0,1}(\phi^2, \varphi)) \\ \text{or} \\ \mu &= (a, \Gamma, \{u_{2m, 2l} | m \geq l \geq 1\}). \end{aligned} \tag{3.6}$$

The static parameters given by (2.1) and (2.3)

$$\mu_{\text{st}} = (a, \{u_{2m, 2l} | m \geq 1\}) \tag{3.6a}$$

form of course an invariant subset in the parameter space. Finally we note the relationship

$$Y_{i,1}(\phi^2, 0) = U^{(i+1)}(\phi^2). \tag{3.7}$$

The Transformation and Fixed Points

The DRG transformation can be performed exactly in the large- n limit. The calculation goes on the same lines as in [10] where only the case of a non-conserved order parameter was treated, therefore we mention only the main steps of it. The integral in (2.10) over $\tilde{\phi}_{j,k,\omega}$ is straightforward since \mathcal{A} is quadratic in this variable. In carrying out the integral over $\phi_{k,\omega}$'s the saddle-point method turns out to be applicable.

For the first two parameters we obtain

$$a' = b^{-\eta} a, \quad \Gamma' = b^{-2+\eta-c+z} \Gamma \tag{3.8}$$

indicating that a finite fixed point can be achieved only if

$$\eta = 0, \quad z = 2 + c \tag{3.9}$$

which are well-known results of previous works [2, 8, 9].

The transformation of the function $Y_{0,1}(\phi^2, \varphi)$ couples to that of $Y_{1,0}(\phi^2, \varphi)$. For $d > 2$ it is found, using the definition (2.9), that

* This is why the sum over l in (3.3) starts with $l=1$

$$Y'_{0,1}(\phi^2, \varphi) = b^2 U^{(1)}(b^{2-d} Q(\phi^2, \varphi) + N_c), \quad (3.10)$$

$$Y'_{1,0}(\phi^2, \varphi) = b^{4-d} U^{(2)}(b^{2-d} Q(\phi^2, \varphi) + N_c) R(\phi^2, \varphi), \quad (3.11)$$

where ϕ^2 and φ denote the rescaled fields, furthermore

$$Q(\phi^2, \varphi) = \phi^2 - N_c + (n/2) \int_a^> (q^{c/2} S^{-1} - q^{-2}), \quad (3.12)$$

$$R(\phi^2, \varphi) = \varphi - (n/2) \int_a^> (q^{3c/2}(q^2 + Y'_{0,1}(\phi^2, \varphi)) S^{-1} - q^c) \quad (3.13)$$

with

$$S \equiv \{q^c(q^2 + Y'_{0,1}(\phi^2, \varphi))^2 - 2 Y'_{1,0}(\phi^2, \varphi)\}^{1/2}. \quad (3.14)$$

The constant N_c is given by

$$N_c = (n/2) K_d A^{d-2} / (d-2) \quad (3.15)$$

and

$$\int_a^> \equiv K_d \int_a^{Ab} dq q^{d-1}. \quad (3.16)$$

Since a and Γ do not transform they have been set equal to unity.

In the special case of $c=0$ Eq. (3.10)–(3.14) go over to the corresponding ones in [10]*.

An important feature of Eq. (3.10)–(3.15) is that at $\varphi=0$ they describe the transformation of the static parameters. In fact for $Y'_{0,1}(\phi^2, 0) \equiv t'(\phi^2)$ we recover the expression first obtained by Ma (see (4, 35) in [2]). This is a consequence of the fact that $Y'_{1,0}(\phi^2, 0) \equiv 0$ as it follows from (3.11).

The right hand sides of (3.10) and (3.11) can be expanded as

$$Y'_{0,1}(\phi^2, \varphi) = b^2 \sum_{j=0}^{\infty} (1/j!) U^{(j+1)}(N_c) \cdot b^{(2-d)j} [Q(\phi^2, \varphi)]^j, \quad (3.17)$$

$$Y'_{1,0}(\phi^2, \varphi) = b^{4-d} \sum_{j=0}^{\infty} (1/j!) U^{(j+2)}(N_c) \cdot b^{(2-d)j} [Q(\phi^2, \varphi)]^j R(\phi^2, \varphi). \quad (3.18)$$

The fixed point is generated by taking the limit $b \rightarrow \infty$. It can be seen from these equations that a necessary condition for the existence of a finite fixed point is

* Note that there are differences in the notations: $t(\phi^2)$, $v(\phi^2)$, $y(\phi^2, \varphi)$ and $X(\phi^2, \varphi)$ of [10] are here $U^{(1)}(\phi^2)$, $U^{(2)}(\phi^2)$, $Y'_{0,1}(\phi^2, \varphi)$ and $2 Y'_{1,0}(\phi^2, \varphi)$ respectively. Furthermore it can easily be seen that $b^{2-d} Q + N_c = b^{2-d} \phi^2 + \bar{\rho}$, where $\bar{\rho}$ is defined by (21) of [10] for $c=0$

$$U^{(1)}(N_c) = 0 \quad (3.19)$$

which specifies the critical surface in the parameter space. This is, of course, a well-known condition from statics [2].

For $d > 4$ the Gaussian fixed point is reached

$$Y_G^*(\phi^2, \varphi) \equiv 0.$$

Here and the following the fixed point values of the quantities are marked by an asterisk.

Below four dimensions let us discuss first the case when $U^{(2)}(N_c) > 0$. Then it follows from (3.17) and (3.18) that $Y'_{0,1}$ and $Y'_{1,0}$ can approach finite fixed point expressions only if Q and R tend to zero in the limit $b \rightarrow \infty$. Using (3.12) and (3.13) the fixed functions $Y_{0,1}^*$ and $Y_{1,0}^*$ are given by the coupled integral equations:

$$\phi^2 = N_c - (n/2) K_d \int_A^{\infty} (q^{c/2} (S^*)^{-1} - q^{-2}) q^{d-1} dq, \quad (3.20)$$

$$\varphi = (n/2) K_d \int_A^{\infty} (q^{3c/2} (q^2 + Y_{0,1}^*(\phi^2, \varphi)) \cdot (S^*)^{-1} - q^c) q^{d-1} dq \quad (3.21)$$

with

$$S^* \equiv \{q^c(q^2 + Y_{0,1}^*(\phi^2, \varphi))^2 - 2 Y_{1,0}^*(\phi^2, \varphi)\}^{1/2}. \quad (3.22)$$

The fixed point expression of the action is given by the general relationship

$$Y(\phi^2, \varphi) = \int_0^{\varphi} Y_{0,1}(\phi^2, \varphi) d\varphi. \quad (3.23)$$

The transformation of the function $Y_{0,1}(\phi^2, \varphi)$ specifies, at least in principle, the transformations and fixed point values of all the parameters $u_{2m, 2l}$. It is more convenient however to consider the different derivatives of Y taken at $\phi^2 = N_c$ and $\varphi = 0$ as an other set of parameters. The fixed point values of some of these quantities are listed below:

$$(1/2) Y_{1,1}^*(N_c, 0) = A^{4-d} (4-d) (n K_d)^{-1},$$

$$(1/2) Y_{0,2}^*(N_c, 0) = A^{2-c-d} (4-d)^2 (6+c-d)^{-1} (n K_d)^{-1},$$

$$(1/8) Y_{2,1}^*(N_c, 0) = A^{6-2d} (4-d)^3 (6-d)^{-1} (n K_d)^{-2},$$

$$(1/4) Y_{1,2}^*(N_c, 0) = A^{4-c-2d} \frac{(4-d)^3}{(n K_d)^2} \left\{ \frac{4(4-d)}{(6+c-d)(6-d)} - \frac{3}{8+c-d} \right\},$$

$$(1/6) Y_{0,3}^*(N_c, 0)$$

$$= A^{2-2c-2d} \frac{2(4-d)^3}{(nK_d)^2} \left\{ \frac{2(4-d)^2}{(6+c-d)^2(6-d)} - \frac{3(4-d)}{(8+c-d)(6+c-d)} + \frac{1}{10+2c-d} \right\}, \quad (3.24)$$

$$Y_{j,0}^*(N_c, 0) = 0, \quad j = 1, 2, \dots$$

These formulae illustrate the strong dependence of the dynamic quantities on whether the order parameter is conserved or not.

Higher Order Critical Points

Under special conditions we can arrive at fixed points describing higher order critical points (see for a discussion of higher order critical points [23]). Namely for

$$U^{(j)}(N_c) = 0, \quad 1 \leq j < \sigma$$

and

$$U^{(\sigma)}(N_c) > 0, \quad \sigma = 3, 4, \dots \quad (3.25)$$

and for $d > 2\sigma/(\sigma - 1)$ the recursion relations (3.17), (3.18) lead to the Gaussian fixed point, which in this case specifies a critical point of order σ . (The ordinary critical point for $d > 4$ would correspond to $\sigma = 2$). Note that $Q(N_c, 0) = R(N_c, 0) = 0$, which ensures that for $1 \leq j < \sigma$ $U^{(j)}(N_c) \equiv Y'_{j-1,1}(N_c, 0)$ are also zero if (3.25) is fulfilled.

Let us discuss first the tricritical point, many of the static properties of which have been studied by several authors for large n [4, 5, 24–29]. The recursion relations (3.17) and (3.18) do not lead to a finite fixed point below three dimensions corresponding to a tricritical point*. It is in accordance with the results of Emery [25] who has found that for $d < 3$ the critical line terminates at a critical end point and a first order phase transition takes place when decreasing $U^{(2)}(N_c)$ already for small positive values of it. There is no finite fixed point of the recursions (3.17) and (3.18) at $d = 3$ either.* This feature can be contrasted with the behaviour of the recursions at $d = 4$ where the Gaussian fixed point is reached in the limit $b \rightarrow \infty$. The reason behind this is that Q and R have contributions proportional to $\ln b$ at $d = 4$, while they develop no logarithmic singularities at $d = 3$ even when the condition of tricriticality ($U^{(2)}(N_c) = 0$) is fulfilled. It is expected, however, that such logarithms show up when calculating to order $1/n$. Then a fixed point describing tricritical behaviour at $d = 3$ may be

* These statements hold also, if we consider the recursion relations at $\phi = 0$, which corresponds to statics

obtained by taking first the limit $b \rightarrow \infty$. and subsequently the limit $n \rightarrow \infty$. Such a procedure could account for the features of the exact solution of the model for equilibrium properties [25] according to which there exists a tricritical point in three dimensions without, however, logarithmic corrections to the mean field behaviour. This possibility is also in accordance with the results by Stephen et al. [30], who pointed out that the critical region (in which the logarithmic corrections arise) shrinks to zero when $n \rightarrow \infty$ in three dimensions at the tricritical point. It is expected that similar situation occurs in case of the critical point of σ^{th} order at $d_\sigma = 2\sigma/(\sigma - 1)$.

The Large- b Behaviour of the Transformation

The properties of the transformation near the fixed points can be deduced from (3.10), (3.11). It is found that for $2 < d < 4$ at T_c in leading order in b

$$Y'_{0,1}(\phi^2, \phi) - Y_{0,1}^*(\phi^2, \phi) = b^{d-4} g \partial(Y_{0,1}^* Y_{1,0}^*) / \partial \phi, \quad (3.26)$$

$$Y'_{1,0}(\phi^2, \phi) - Y_{1,0}^*(\phi^2, \phi) = b^{d-4} g \partial(Y_{0,1}^* Y_{1,0}^*) / \partial(\phi^2), \quad (3.27)$$

where

$$g = [Y_{1,1}^*(N_c, 0)]^{-1} - [Y_{1,1}(N_c, 0)]^{-1} \quad (3.28)$$

and $Y_{1,1}^*(N_c, 0)$ is given by (3.24). Then it follows from (3.23) and (3.26) that for large b

$$Y'(\phi^2, \phi) - Y^*(\phi^2, \phi) = b^{d-4} g Y_{0,1}^*(\phi^2, \phi) Y_{1,0}^*(\phi^2, \phi). \quad (3.29)$$

It should be emphasized that the expressions (3.26), (3.27) and (3.29) are not restricted to such starting points in the parameter space which lie near the fixed point. As a matter of fact due to the special form of the initial action (2.6) the starting point could not generally be chosen in the vicinity of the fixed point.

It can be shown that g is a non-linear scaling field*, which transforms at T_c as $g' = g b^{d-4}$. It is interesting to point out that $Y_{0,1}^*(\phi^2, \phi) Y_{1,0}^*(\phi^2, \phi)$ is an eigen-operator of the linearised transformation around the non-trivial fixed point.

* The scaling field g is of purely static character (i.e. it is a combination of static parameters only, as can be seen from (3.28) and (3.7), and agrees with what has been determined in the RG analysis of the statics of the model [4, 5]. The complete hierarchy of the scaling fields including the dynamic ones will be given in a subsequent paper [31]

Similarly we can investigate the transformation in the neighbourhood of the trivial fixed point in the case of higher order critical points. We obtain for a σ^{th} order critical point for large b

$$Y'(\phi^2, \varphi) = b^{2+(2-d)(\sigma-1)}(1/(\sigma-1)!) \cdot U^{(\sigma)}(N_c)(\phi^2 - N_c)^{\sigma-1} \varphi \quad (3.30)$$

provided d is larger than d_σ .

It turns out that the non-linear scaling field with the largest exponent in this case is proportional to $U^{(\sigma)}(N_c)$ at T_c (and the exponent is $2+(2-d)(\sigma-1)$). Moreover $(\phi^2 - N_c)^{\sigma-1} \varphi$ is just the corresponding eigenvector of the linearised transformation.

The fact that the deviation of the action from its fixed point expression for large b at T_c is the product of the non-linear scaling field with the largest exponent and the corresponding eigenvector of the linearised transformation is expected to be valid for all cases when there is only one non-linear scaling field at T_c which belongs to the largest exponent.

IV. Perturbative Method in the Path Probability Formalism

The multiple integral (2.10) can be evaluated perturbationally as well [17, 20]. In order to do this we decompose the action into a harmonic part and the interaction

$$\mathcal{A}_I\{\tilde{\phi}, \phi\} = \int dt \int d^d x \sum_{j=1}^n \{u_{4,2} i \tilde{\phi}_j L \phi_j 2 \phi^2 + u_{4,2} \Gamma K(c) \phi^2 + \dots\}. \quad (4.1)$$

The only non-vanishing averages over the harmonic part are given in terms of Fourier components (2.12) as

$$-i \Gamma k^c \langle \phi_{j,k,\omega} \tilde{\phi}_{j,-k,-\omega} \rangle_0 \sim G^{(0)}(k, \omega) = \Gamma k^c (-i\omega + \Gamma k^c (a k^2 + u_{2,2}))^{-1}, \quad (4.2)$$

$$\langle \phi_{j,k,\omega} \phi_{j,-k,-\omega} \rangle_0 \sim C^{(0)}(k, \omega) = (2/\omega) \text{Im} G^{(0)}(k, \omega), \quad (4.3)$$

where $G^{(0)}$ denotes the free response function and $C^{(0)}$ is the corresponding correlation function.

To represent the contributions to the integral

$$\int_{j, A/b < k < A, \omega} \prod d\tilde{\phi}_{j,k,\omega} d\phi_{j,k,\omega} \exp \mathcal{A}_0 \sum_{l=0}^{\infty} \mathcal{A}_I^l / l! \quad (4.4)$$

it is convenient to introduce diagrams. In the large- n limit only graphs with a maximal number of closed loops survive because every loop involves a factor n . The topological structure of these diagrams correspond to that of the static ones [3]. Figure 1 shows some of the diagrams coming from the $l=3$ term of (4.4).

Looking at the graphs one concludes that only those vertices give contributions in which both legs belong-

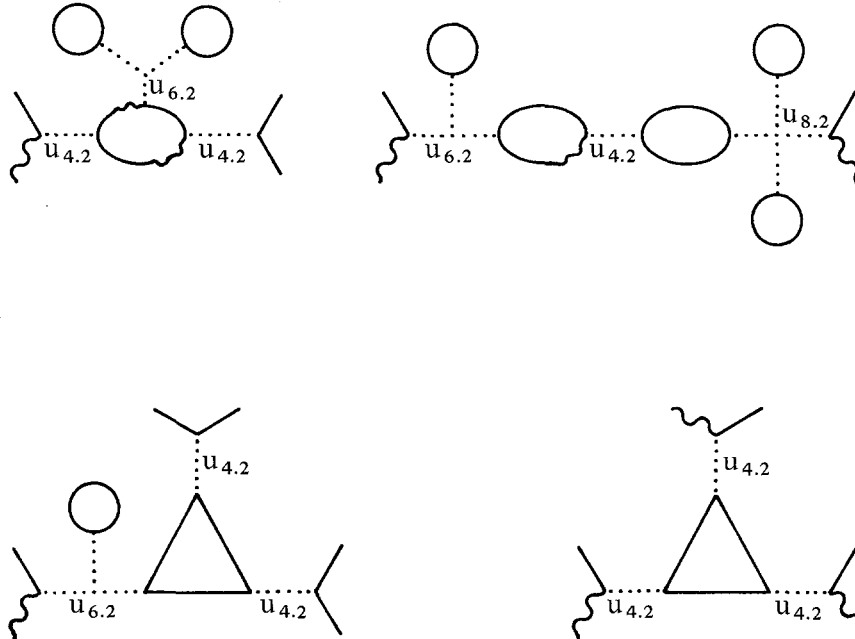


Fig. 1. Some diagrams in the DRG calculation of the large- n limit. Notations: — $C^{(0)}(k, \omega)$, — $G^{(0)}(k, \omega)$. External legs: $\sim i \Gamma k^c \tilde{\phi}_{j,k,\omega}$, — $\phi_{j,k,\omega}$ with $k < A/b$. Internal lines carry momenta between A/b and A

ing to one endpoint carry either large or small momenta. If in addition the k - and ω -dependent parts of the diagrams are not considered the net wavenumber and frequency transfers of the potential lines are zero. As a matter of fact, in the language of diagrams this argument justifies the approximations done when performing the DRG transformation by the non-perturbative method (see Eqs. (11) and (13) in [10]). The steps of the calculation can be summarised in the following. Infinite subsets of graphs are summed up by using self consistently dressed correlation and response functions and an effective interaction, containing the series of bubble graphs*. After introducing new scales corresponding to (2.11), choosing \tilde{y} as $\eta/2 - 1 - d/2$ in order to ensure a finite coefficient of the term $i\tilde{\phi}'\phi'$ and making use of the linked cluster theorem we arrive at a new action containing couplings non-local in space and time. Let

$$u'_{2m, 2l}(k_1, \dots, k_{m-1}; \omega_1, \dots, \omega_{m-1}) \quad (4.5)$$

denote the coupling described by a graph with l external wavy lines and $2m-l$ external straight lines. We can write

$$\mu = (a, \Gamma, \{u_{2m, 2l}(k_1, \dots, k_{m-1}; \omega_1, \dots, \omega_{m-1}) | m \geq l \geq 1\}) \quad (4.6)$$

indicating that all parameters of the functions $u_{2m, 2l}$ (for instance the Taylor coefficients) are to be considered as elements of the parameter space.

It is easy to see that graphs containing $K(c)$ (the contribution of the Jacobian) do not determine new independent parameters**. In the previous treatment this has been reflected in that, that always the combination φ , (3.2), has been involved.

The lack of terms like $\tilde{\phi}'^2$, $\tilde{\phi}'^3$ etc. in \mathcal{A}' has the important consequence that the noise in the equation of motion remains Gaussian (see for the form of the action in case of non-Gaussian noise [19]). This is one of the simplifying features of the dynamics in the large- n limit.

The results for the first few couplings $u_{2m, 2l}$ are given in Appendix A. Here, instead, we introduce an other representation of the parameters in order to show the connection between the results of the present and previous Sections more directly. Let us define:

$$U_{2m, 2l} = u_{2m, 2l} + \sum_{j=1}^{\infty} u_{2(m+j), 2l} \binom{j+m-l}{j} (2N_c)^j. \quad (4.7)$$

* The bubble graphs here consist of one response and one correlation function

** This feature should be compared with the role of the Jacobian in other types of perturbation expansion [18, 19]

It is worth while mentioning that the linear combination (4.7) has a well-defined meaning in the language of the usual perturbation expansion*. $U_{2m, 2l}$ is a sum of vertices of the same type (i.e. with $2m$ external lines, among them l response field lines) where the j^{th} term of the series comes from $u_{2(m+j), 2l}$ but j pair of the order parameter lines are closed to form fully dressed Hartree loops of the correlation function (whose contribution gives the average value of ϕ^2) evaluated at T_c . The correlation function in this formalism is given by (4.2), (4.3) but with $U_{2, 2}$ instead of $u_{2, 2}$. Since $U_{2, 2} = U^{(1)}(N_c)$ it is zero at T_c (see (3.19)), and the contribution of a Hartree loop turns out to be simply the constant N_c , (3.15), which has frequently occurred in our calculation previously. The self-loop of the response function, on the other hand, cancels the contribution of the Jacobian as required by causality [18, 19]. In other words this means that the average of φ vanishes and this is why a summation over the second indices of u is not involved in (4.7). Figure 2 shows as an example how $U_{4, 2}$ can be represented by diagrams. New vertices similar in spirit to those defined above have been used in calculating static and dynamic critical indices at critical points of higher order just below the critical dimensionality d_σ (see [32, 33]).

When performing the DRG transformation we start with the new parameters $U_{2m, 2l}$ and find that the corresponding transformed quantities become k - and ω -dependent, $U'_{2m, 2l}(k_1, \dots, k_{m-1}; \omega_1, \dots, \omega_{m-1})$, where k_1, \dots and ω_1, \dots denote the rescaled wavenumbers and frequencies, respectively.

For $U'_{4, 2}(k, \omega)$ one obtains

$$U'_{4, 2}(k, \omega) = \frac{b^{4-d}(1/2) U^{(2)}(I_b^{(1)})}{1 + b^{4-d}(n/2) U^{(2)}(I_b^{(1)}) I_b^{(2)}(k, \omega)}, \quad (4.8)$$

where $U^{(2)}$ is defined by (2.9) and $I_b^{(1)} = b^{2-d} Q(N_c, 0) + N_c$ with Q as given by (3.12) and

$$\begin{aligned} I_b^{(2)}(k, \omega) &= \int_{A < q < Ab, A < |k-q| < Ab} (q^2 + U'_{2, 2})^{-1} ((k-q)^2 + U'_{2, 2})^{-1} \\ &\cdot \frac{q^c (q^2 + U'_{2, 2}) + (k-q)^c ((k-q)^2 + U'_{2, 2})}{-i\omega + q^c (q^2 + U'_{2, 2}) + (k-q)^c ((k-q)^2 + U'_{2, 2})} \\ &\cdot \frac{d^d q}{(2\pi)^d}. \end{aligned} \quad (4.9)$$

Here a and Γ have been set equal to unity and $U'_{2, 2}$ can easily be deduced from (3.10) as

** As contrasted to the RG calculation, here the momenta of the internal lines are not restricted to a shell but run from zero to the cut-off

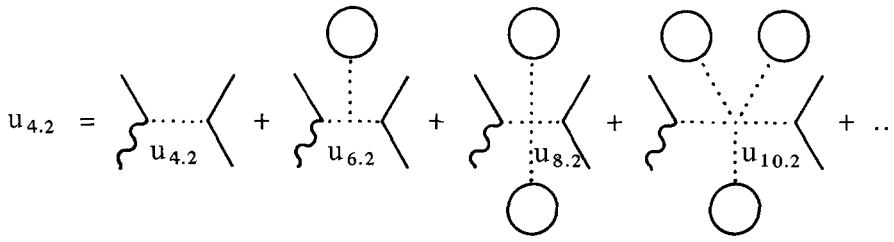


Fig. 2. Graphic representation of the parameter $U_{4,2}$. The loops denote the average value of ϕ^2 at T_c i.e. N_c .

$$U'_{2,2} = b^2 U^{(1)}(I_b^{(1)}). \tag{4.10}$$

The meaning of the quantities $I_b^{(1)}$ and $I_b^{(2)}$ can be given as follows. $I_b^{(1)} - b^{2-d} N_c$ and $b^{4-d} I_b^{(2)}$ represent the contributions of the Hartree loop and the bubble graph respectively, in which the integration over wave-numbers runs in the interval $(\Lambda/b, \Lambda)$. At $T_c U'_{2,2} = 0$ and $I_b^{(1)} = N_c$.

It is of interest that $U'_{4,4}$ can be expressed by the formula

$$U'_{4,4}(k, \omega) = (-2/\omega) \text{Im } U'_{4,2}(k, \omega) \tag{4.11}$$

representing a fluctuation-dissipation theorem. The transformation of $U_{6,2}(k_1, k_2; \omega_1, \omega_2)$ is given in Appendix A.

Finally we mention that the general relationship

$$U_{2m,2l}(0, 0, \dots; 0, 0, \dots) = \frac{1}{m! 2^{m-l}} \binom{m}{l} Y_{m-l,l}(N_c, 0)$$

holds (both for the starting and for the transformed quantities), and establishes a connection between the parameters of the present and previous Sections. This set of parameters is very convenient in describing the linear and non-linear scaling fields [31]. An example is given by (3.28).

Provided the initial parameters lie on the critical surface ($U^{(1)}(N_c) = 0$) and $U^{(2)}(N_c)$ is positive the fixed function $U_{4,2}^*(k, \omega) = [n I_\infty^{(2)*}(k, \omega)]^{-1}$ is obtained, where of course, $I_\infty^{(2)*}$ should be taken at the fixed point value of $U_{2,2}$ which is zero. On the other hand for $U^{(2)}(N_c) = 0$ (tricritical point) and for $d > 4$ the trivial fixed function $U_{4,2}^* \equiv 0$ is obtained. The fixed point expression of $U_{4,4}^*(k, \omega)$ follows from (4.11).

For the deviation from the non-trivial fixed point we find for large b and T_c

$$U'_{4,2}(k, \omega) - U_{4,2}^*(k, \omega) = g b^{d-4} 2 [U_{4,2}^*(k, \omega)]^2 \tag{4.12}$$

indicating that all parameters of $U'_{4,2}$ have the same large- b behaviour when approaching the fixed point, and each one is proportional to the scaling field g given by (3.28). For every $\Delta U_{2m,2l}(k_1, \dots; \omega_1, \dots)$ similar relation is valid as (4.12).

An other question of particular interest is how different initial actions modify the transformation. This can be studied by adding various perturbative terms to (4.1). For example starting with an additional interaction $u_{4,4} \phi^2$ we find instead of (4.11)

$$U'_{4,4}(k, \omega) = (-2/\omega) \text{Im } U'_{4,2}(k, \omega) + b^{d-6-c} 4 u_{4,4} |U'_{4,2}(k, \omega)|^2 / [U^{(2)}(I_b^{(1)})]^2, \tag{4.13}$$

which for large b yields the same transformation as if we had started with zero value of $u_{4,4}$. This feature turns out to be characteristic for all other $U'_{2m,2l}(k_1, \dots; \omega_1, \dots)$ ($m, l > 2$) too which can be traced back to the fact that there is only one irrelevant scaling field which belongs to the largest exponent, that is to $d-4$. It is worth mentioning that this is the reason of obtaining in the previous Section the same large- b behaviour of the transformation as if we had started with a general $Y(\phi^2, \varphi)$.

Let us shortly discuss the structure of the parameter space, and compare the methods of Sects. III and IV. Quantities like the expansion coefficients of the k - and ω -dependence of a coupling of given order and the set $\{U_{2m,2l}(0, 0, \dots, 0, 0, \dots)\}$ define two subspaces in the parameter space. The non-perturbative treatment enables us to give the transformation of the parameters in the latter subspace by studying their generating function, $Y_{0,1}$. The perturbative calculation on the other hand yields the transformation of the other subspace in a compact way. Some of the parameters in the two subspaces are related by fluctuation dissipation theorems.

Finally a remark is in order on the bare coupling coefficient determined by the so-called matching condition [34]. In the language of the renormalisation group a general formulation can be given for the matching condition. It corresponds to a special choice of the initial parameters which set the scaling field g (i.e. the irrelevant scaling field with the largest exponent) equal to zero. This procedure kills off the transients in the parameter space which would transform as $b^{-\varepsilon}$ ($\varepsilon \equiv 4-d$) for large b , that is the transients which become slow near four dimensions. The matching condition is very simple in terms of the new

parameter $U_{4,2}$ since the requirement $g=0$ can be fulfilled only if

$$U_{4,2} = U_{4,2}^*(0, 0)$$

as can be seen from (3.28). In the special case when we start with the usual ϕ^4 model e.g. with

$$U(\phi^2) = u_{2,2} \phi^2 + u_{4,2} (\phi^2)^2 \tag{4.14}$$

we have $u_{4,2} = U_{4,2}$, and thus the matching condition requires a special bare coupling constant $u_{4,2c}$ as

$$u_{4,2c} = U_{4,2}^*(0, 0).$$

It is worth while comparing $u_{4,2c}$ with the fixed point expression of $u_{4,2}(0, 0)$ (see (A.2)) for small ε . We obtain that they agree up to order ε^2 in accordance with the results of [35], but they differ already in a term of order ε^3 .

Closing this Section let us turn to a short discussion of the case of higher order critical points. The corresponding recursions can be read off from the general ones (for example (4.8), (A.5)). At a σ^{th} order critical point (see (3.25)) we find at T_c that

$$U'_{2m,2l}(k_1, \dots; \omega_1, \dots) \equiv 0$$

for $m < \sigma, l \leq m$,

$$U'_{2\sigma,2l}(k_1, \dots; \omega_1, \dots) = b^{2+(2-d)(\sigma-1)} (2^{1-\sigma}/(\sigma-1)!) U^{(\sigma)}(N_c) \delta_{l,1} \tag{4.15}$$

valid for arbitrary b . Furthermore for large b

$$U'_{2m,2}(k_1, \dots; \omega_1, \dots) = b^{2+(2-d)(m-1)} (2^{1-m}/(m-1)!) U^{(m)}(N_c), \quad m > \sigma. \tag{4.16}$$

It is apparent that k - and ω -dependences do not arise in the couplings at higher order critical points at T_c which is due to the fact that the quantity $U'_{4,2}(k, \omega)$ vanishes at T_c . A consequence of (4.15) is that at a critical point of order σ the scaling field with the largest exponent, $2+(2-d)(\sigma-1)$, is just $U^{(\sigma)}(N_c)$ as already mentioned at the end of the last Section. Of course for $d > d_\sigma$ the Gaussian fixed point is obtained.

V. DRG Transformation Carried out on the Equation of Motion

If the DRG transformation is defined on the equation of motion it involves the following steps [12-14]. a) Eliminate the variables $\phi_{j,k,\omega}$ with $\Lambda/b < k < \Lambda$ by solving their equations of motion in terms of the remaining ones and the random forces and substituting the solutions in the remaining equations. b) Re-

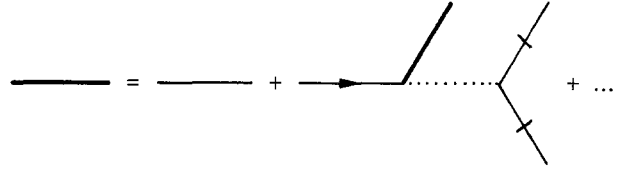


Fig. 3. Graphic representation of the equation of motion after the first iteration in step a. Notations: — $\phi_{j,k,\omega}$, — $\phi_{j,k,\omega}^{(0)}$ $= G^{(0)} \zeta_{j,k,\omega} / \Gamma k^c$, — $G^{(0)}(k, \omega)$, \dots $(-u_{4,2})$. Lines with a slash carry wave numbers larger than Λ/b

place the variables $\phi_{j,k,\omega}$, $k < \Lambda/b$ by new ones corresponding to (2.11).

Through this procedure we arrive at new equations of motion which are more complicated than the starting ones, they contain higher order couplings and random vertices. The parameter space should contain not only the mean values of these vertices but all higher order cumulants among them [36, 14].

Our model serves as a good example to illustrate this general concept. Let us start with the ϕ^4 model, (4.14). Step a) can be performed by an iterative solution of Eq. (2.4). The result of the first iteration is represented in Fig. 3 indicating that the coefficient of $\phi_{j,k,\omega}$ on the r.h.s. has become a random variable the full specification of which requires the determination of all its cumulants [14]. As a matter of fact none of these cumulants can be neglected in the large- n limit. An other important difference between our case and a calculation up to order ε is that we must not stop after the first iteration as the successive steps generate cumulants of the same order of magnitude as the first one. Fortunately the leading contributions in n can be summed up yielding closed expressions for the cumulants. Figure 4 shows some examples in a graphic representation: $\hat{u}_{2m,2l}$ denotes the contribution of diagrams with l external response function lines and $2m-l$ order parameter lines.

Comparing the graphs of the perturbation expansion for the action and those just discussed, one immediately recognizes that their contributions are essentially identical. If weighting factors are also taken into consideration the following simple relation is found

$$\hat{u}_{2m,2l}(k_1, \dots, k_{m-1}; \omega_1, \dots, \omega_{m-1}) = l u_{2m,2l}(k_1, \dots, k_{m-1}; \omega_1, \dots, \omega_{m-1}). \tag{5.1}$$

This relationship indicates that the two definitions of DRG are equivalent only if the higher cumulants of the random vertices in the equation of motion are taken into account as elements of the parameter space. Moreover it provides a physical interpretation of the new couplings generated by the DRG in the action.

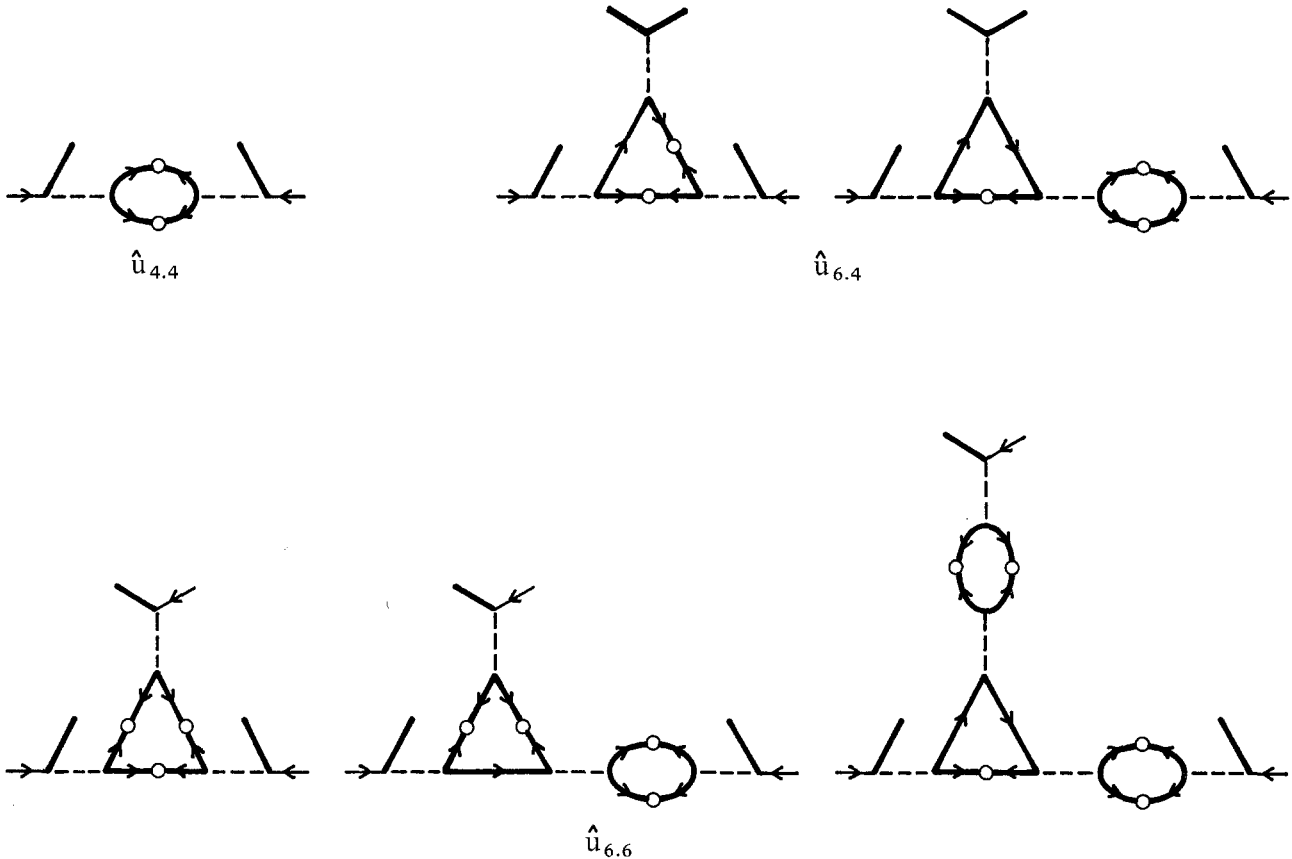


Fig. 4. Diagrams of a few second and third cumulants of the equation of motion after the DRG procedure. The thick lines \rightarrow and $\rightarrow\leftarrow$ denote dressed propagators with arc of the form of (4.2) and (4.3) but with $u_{2,2}$ replaced by $u'_{2,2}/b^2$ and with $k > \Lambda/b$. The broken line represents the effective interaction (A.2)

Thus our investigation shows explicitly how the simple Langevinian form of the equation of motion is lost after applying the DRG. New vertices appear which are random variables. The distribution of these stochastic quantities is non-Gaussian since the higher order cumulants do not vanish. They are not even delta correlated, exhibit especially ω dependences indicating the role of memory effects. On the other hand the character of the original random noise does not change in the large- n limit. In the present description it follows from the fact that diagrams containing only ingoing external lines do not contribute in leading order.

Appendix B shows how the structure of the equation of motion is modified when the DRG is applied in the presence of symmetry breaking. The main difference lies in the more complicated k - and ω -dependences of the couplings and in the appearance of new types of cumulants. As straightforward consequences of these facts a and Γ do not remain constant in the equation for the longitudinal component of the order parameter (B.2) and the Langevin noise loses its white spectrum (B.3).

VI. Discussion

We have studied here the working of the DRG in the large- n limit for dimensions $2 < d < 4$. The discussion has included critical points of arbitrary order.

One of the main general conclusions of our investigation is that the parameter space should be extended to include a manifold of couplings with two indices:

$$\{U_{2m, 2l} | m=1, 2, \dots; l \leq m\}.$$

It is obvious that this property is not restricted to large n . As a matter of fact at arbitrary n even further parameters may arise, for instance the coefficients of terms like $\tilde{\phi}^4$. So in general the DRG does not preserve the Langevinian form and Markoffian character of the original equation of motion, but the noise and the vertices become coloured stochastic quantities with non-Gaussian distributions.

The physical interpretation of the parameters in the action i.e. the correspondence between them and the parameters of the equation of motion (e.g. (5.1)) is

valid for arbitrary n indicating that there can be a connection between stochastic differential equations and path probabilities even in more general cases than that of the Langevin equation.

Concerning the properties of the parameters it is worth mentioning that as can be easily seen, the parameters of the action, $U_{2\sigma, 2\tau}$ or $u_{2\sigma, 2\tau}$ become dimensionless at dimensionality $d_{\sigma, \tau} = [2\sigma - (2+c)(\tau - 1)]/(\sigma - 1)$ where again $c=0, 2$ for the case of non-conserved and conserved order parameters, respectively. The following properties can immediately be concluded: $d_{\sigma, 1} = d_\sigma$ and correspondingly gives the marginal dimension of the critical point of order σ ($d_\sigma > 2$); $d_{\sigma, 2} = 2$ for arbitrary σ provided the order parameter is not conserved; in all other cases $d_{\sigma, \tau} < 2$ for $\tau \geq 2$; for a fixed τ in the limit $\sigma \rightarrow \infty$ $d_{\sigma, \tau}$ converges to 2. These properties indicate the special role of dimension 2.

Our investigations in the large- n model have been restricted to $d > 2$. In this regime none of the dynamic parameters become marginal.

$$J_b^{(3)}(k_1, k_2; \omega_1, \omega_2) = \int q^c (k_1 - q)^c (k_1 - k_2 - q)^c [B_1 B_2 B_3]^{-1} \cdot \frac{-\omega_1 B_1 B_2 - \omega_2 B_2 B_3 - (\omega_1 - \omega_2) B_1 B_3 - i(B_1 + B_2)(B_2 + B_3)(B_1 + B_3)}{(\omega_1 + i(B_1 + B_2))(\omega_2 + i(B_2 + B_3))(\omega_1 - \omega_2 + i(B_1 + B_3))} \frac{d^d q}{(2\pi)^d} \quad (A.6)$$

Further characteristics of the dynamic renormalization group transformation in the large- n limit, particularly the determination of the linear and non-linear scaling fields and scaling variables will be discussed in a subsequent paper.

Helpful discussions with Prof. G. Meissner and Dr. N. Menyhárd are gratefully acknowledged. One of us (P.Sz.) would like to express his thanks to Prof. G. Meissner also for his hospitality at the University of Saarland.

Appendix A

Recursions of $u_{2, 2}$, $u_{4, 2}$, $u_{6, 2}$ and $U_{6, 2}$

We write down here the transformation of the first terms of the set $\{u'_{2m, 2l}(k_1, \dots; \omega_1, \dots)\}$ obtained by the perturbative method starting with the local interaction given by (2.6). Thus

$$u'_{2, 2} = b^2 U^{(1)}(J_b^{(1)}), \quad (A.1)$$

where $J_b^{(1)} = b^{2-d} Q(0, 0) + N_c$ and Q has been defined in (3.12). Furthermore

$$u'_{4, 2}(k, \omega) = \frac{b^{4-d}(1/2) U^{(2)}(J_b^{(1)})}{1 + b^{4-d}(n/2) U^{(2)}(J_b^{(1)}) J_b^{(2)}(k, \omega)} \quad (A.2)$$

where $J_b^{(2)}(k, \omega)$ can be expressed by the same integral as $I_b^{(2)}(k, \omega)$ (see (4.9)) but $U'_{2, 2}$ is replaced by $u'_{2, 2}$. $u'_{4, 4}(k, \omega)$ is determined by the fluctuation-dissipation theorem*

$$u'_{4, 4}(k, \omega) = (-2/\omega) \text{Im } u'_{4, 2}(k, \omega). \quad (A.3)$$

If we represent $u'_{4, 2}(k, \omega)$ by the series

$$u'_{2, 2}(k, \omega) = \sum_{\alpha, \beta} c'_{\alpha, \beta} (k^2)^\alpha (i\omega)^\beta \quad (A.4)$$

then (A.3) requires

$$c'_{0, 1} = (-1/2) u'_{4, 4}(0, 0)$$

For $u'_{6, 2}(k_1, k_2; \omega_1, \omega_2)$ we find

$$u'_{6, 2}(k_1, k_2; \omega_1, \omega_2) = u'_{4, 2}(k_1, \omega_1) u'_{4, 2}(k_2, \omega_2) \cdot u'_{4, 2}(k_1 - k_2; \omega_1 - \omega_2) \{n J_b^{(3)}(k_1, k_2; \omega_1, \omega_2) + b^{d-6} U^{(3)}(J_b^{(1)}) [U^{(2)}(J_b^{(1)})]^{-3}\} \quad (A.5)$$

with

where the momenta q , $|k_1 - q|$, $|k_1 - k_2 - q|$ are restricted to the interval $(\Lambda/b, \Lambda)$ and

$$B_1 \equiv q^c (q^2 + u'_{2, 2}),$$

$$B_2 \equiv (k_1 - q)^2 ((k_1 - q)^2 + u'_{2, 2}),$$

$$B_3 \equiv (k_1 - k_2 - q)^2 ((k_1 - k_2 - q)^2 + u'_{2, 2}). \quad (A.7)$$

The expression of $U_{6, 2}(k_1, k_2; \omega_1, \omega_2)$ can be obtained from (A.5)-(A.7) if we write U instead of u and I_b instead of J_b in these formulae.

For the large- b behaviour at T_c around the non-trivial fixed point ($2 < d < 4$) we find

$$\Delta u'_{2, 2} = g b^{d-4} u_{2, 2}^* u_{4, 2}^*(0, 0),$$

$$\Delta u'_{4, 2}(k, \omega) = g b^{d-4} \{2 [u_{4, 2}^*(k, \omega)]^2 + 4 u_{2, 2}^* u_{6, 2}^*(k, 0; \omega, 0)\},$$

$$\Delta U_{6, 2}(k_1, k_2; \omega_1, \omega_2) = g b^{d-4} \{2 U_{6, 2}^*(k_1, k_2; \omega_1, \omega_2) \cdot \{U_{4, 2}^*(k_1, \omega_1) + U_{4, 2}^*(k_2, \omega_2) + U_{4, 2}^*(k_1 - k_2, \omega_1 - \omega_2)\}\}. \quad (A.8)$$

The fixed point expressions occurring in (A.8) can be deduced from (A.1), (A.2) and (A.7).

* Similar fluctuation dissipation theorems have been discussed by Bausch and Halperin [37]

Appendix B

DRG in the Presence of Symmetry Breaking in Case of a Non-Conserved Order Parameter

In the presence of a homogeneous time-independent external field H coupled to the first component of the order parameter the average of ϕ_1 becomes non-zero, $\langle \phi_1(x, t) \rangle = M(H)$. According to the equation of state [3] $M = \mathcal{O}(n^{1/2})$. In order to study quantities of the same order of magnitude in the equation of motion it is convenient to separate the contribution of M by substituting

$$\phi_{1,k,\omega} - M(H) V^{1/2} \delta_{k,0} 2\pi \delta(\omega) \rightarrow \phi_{1,k,\omega},$$

which yields different equations of motion for the longitudinal component ($j=1$) and for the transverse ones ($j \geq 2$). In both cases vertices with an even power of ϕ are also involved.

After applying the DRG a number of new dynamic parameters arise, not necessarily independent ones. We recall here only a few of them for the case when we start from the ϕ^4 model (4.14). The transverse parameters $u'_{2,2T}$, a'_T , Γ'_T are given by similar transformations as (A.1) with $U^{(1)}(x) = u_{2,2T} + u_{4,2} 2x$ and (3.8). Initially $u_{2,2T} = u_{2,2} + u_{4,2} M^2$. If a_T and Γ_T are chosen to be unity $u'_{4,2T}(k, \omega)$ is determined by (A.2) but $u_{2,2}$ is to be replaced by $u'_{2,2T}$. An interesting novel feature of the transformation is the k - and ω -dependence of the averaged longitudinal two-leg vertex (see Fig. 5a)

$$u'_{2,2L}(k, \omega) = u'_{2,2T} + 2u'_{4,2T}(k, \omega) M'^2 \quad (\text{B.1})$$

where

$$M' = b^{d/2-1} M.$$

The latter relation indicates that $M=0$ is required on the critical surface. As a consequence of (B.1) neither a'_L nor Γ'_L remains unchanged by the DRG transformation

$$a'_L = a + \left. \frac{\partial u'_{2,2L}(k, 0)}{\partial k^2} \right|_{k=0},$$

$$\frac{1}{\Gamma'_L} = \frac{1}{\Gamma} + \left. \frac{\partial u'_{2,2L}(0, \omega)}{\partial(-i\omega)} \right|_{\omega=0}. \quad (\text{B.2})$$

Both a'_L and $\Gamma'_L{}^{-1}$ diverge when $b \rightarrow \infty$ for a non-zero M .

Investigating the second cummulants an other new feature is found. Namely the diagram shown on Fig. 5b indicates that the longitudinal noise loses its white character since its auto-correlation function

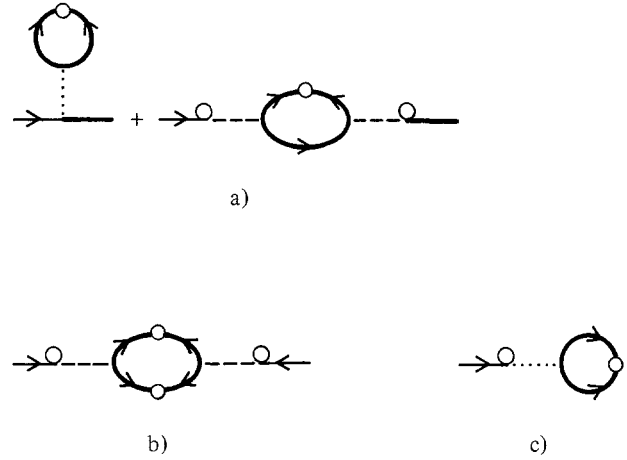


Fig. 5a-c. Some typical diagrams in the ordered phase. The circle at a vertex denotes M . The thick lines \longrightarrow and \longleftarrow represent the transverse response and correlation functions given by (4.2) and (4.3), respectively, but with $u'_{2,2T}/b^2$ instead of $u_{2,2}$ and with $k > A/b$. The broken and dotted lines stand for the series of transverse bubble graphs and for $u_{4,2}$, respectively

becomes non-local in time. The contribution of this graph is determined by the fluctuation-dissipation theorem

$$\hat{u}_{2,4L}(k, \omega) = (-2/\omega) \text{Im} u'_{2,2L}(k, \omega) \quad (\text{B.3})$$

implying as a special case the relation

$$\hat{u}'_{2,4L}(0, 0) = 2(1/\Gamma'_L - 1/\Gamma), \quad (\text{B.4})$$

which is expected also on physical grounds.

Finally it should be noted how the derivation of the equation of state is included in this calculation. Let us perform the limit $b \rightarrow \infty$ in the equation of motion but without introducing new scales. This procedure corresponds to solving the problem for fields with $k=0$, $\omega=0$. After averaging the longitudinal equation we obtain

$$M \left(u_{2,2T} + u_{4,2} n \int_0^A \frac{K_d q^{d-1} dq}{q^2 + \lim_{b \rightarrow \infty} (u'_{2,2T}/b^2)} \right) = H, \quad (\text{B.5})$$

where the integral arises from the diagram of Fig. 5c. Comparing this relation to the expression of $u'_{2,2T}$ we find

$$\lim_{b \rightarrow \infty} (u'_{2,2T}/b^2) = H/M,$$

which implies that (B.5) is equivalent to the equation of state [3]. This procedure, however, can not be considered as a step of the DRG transformation rather it is a complementary calculation.

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P. Szépfalusy
 Fachrichtung Theoretische Physik 11.1
 Universität des Saarlandes
 Im Stadtwald
 D-6600 Saarbrücken
 Federal Republic of Germany

T. Tél
 Institute for Theoretical Physics
 Eötvös University, P.O.B. 327
 Budapest
 Hungary