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Renormalization Group Analysis of Relaxational Dynamics in Systems with Many-Component Order-Parameter II

Scaling Fields and Scaling Variables

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A general discussion of scaling fields and scaling variables in the dynamic renormalization group is given using path probability formalism. It is shown that scaling variables are the derivatives of the action with respect to scaling fields. The general ideas are illustrated on the multicomponent relaxational model in the large-n limit, where scaling fields and scaling variables are calculated explicitly and flow lines, crossover and universality are discussed. Critical points of higher order are also included in the investigation.

I. Introduction

Scaling fields [1, 2] and scaling variables [3] give the most concise formulation of the renormalization group transformation in describing static critical behaviour. Scaling fields are parameters with especially simple transformation rules and represent a special set of solutions of the non-linear renormalization group equations, scaling variables are random variables having well defined scaling dimensions. Complex phenomena, such as crossover [4] are most conveniently discussed in terms of scaling fields, while scaling variables help study certain properties of correlation functions [3]. The large-n system (n being the number of components of the order parameter field) [5] has served as a good example in understanding these general ideas in statics [3, 6-10].

Our purpose here is to investigate the general properties of scaling fields and variables in the framework of the dynamic renormalization group (DRG) and carry out explicit calculations in the non-trivial but exactly solvable model of the large-n system with purely relaxational dynamics. The DRG transformation in this model and related topics have been discussed in [11] and [12] ([12] will be referred hereafter as I).

The path probability formalism [13] with an additional response field [14-16] proved to be very convenient in constructing DRG (see also I). Using this technique a general discussion is given on dynamic scaling fields and variables. It is shown that the scaling variables can be derived by differentiating the action with respect to scaling fields.

Turning to the large-n limit we introduce an action, based on the results of I, which is more general than that of a simple Langevinian dynamics and provides a sufficiently wide parameter space for studying scaling fields and variables.

In general scaling fields, scaling variables and the corresponding exponents depend on which fixed point they are related to. For $2<d<4$ (d is the dimensionality of space) there exist two fixed points in the large-n system. Both the trivial and the nontrivial fixed point representations are worked out in this case. The scaling fields are constructed explicitly by means of appropriate generating functionals. The static scaling fields [9] are recovered as a subset of this manifold. The DRG transformation can be linearized around the fixed points, demonstrating the correctness of the general assumptions of RG procedures. The exponents of the dynamic scaling fields and the dimensions of the dynamic scaling variables depend on whether the order parameter is conserved or not. In order to illustrate

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some interesting features of the global solution in the large-n limit, DRG trajectories are determined and crossover phenomena are investigated in course of which it is shown that the attraction of the trivial fixed point is stronger when the order parameter is conserved. Furthermore, universality is also demonstrated.
The scaling variables are explicitly given in the large-n case. It is shown that the scaling variables are just the coefficients of the scaling fields in the generating functionals of scaling fields. Scaling products of scaling variables analogous to those obtained by Ma in statics [3] are also calculated.
Inherent in the model there are critical points of higher order, too, whose associated scaling fields and scaling variables are also deduced.
The outline of the paper is the following: Sect. II contains the general discussion of scaling fields and variables. In Sect. III the DRG transformation is derived for the large-n system. The scaling fields associated to both the trivial and the non-trivial fixed points are given in Sect. IV, where flow lines and crossover are also discussed. Section V is devoted to the determination of scaling variables in both representations.

II. Scaling Fields and Scaling Variables in the Dynamic Renormalization Group – A General Discussion

Let us consider a system the dynamics of which is described by the probability distribution

$$W=\exp \mathscr{A}\{\tilde{\phi}, \phi\} \tag{2.1}$$

where $\phi(x,t)=\{\phi_j(x,t)|j=1,2,\dots n\}$ denotes the n-component order parameter with momentum cut-off Λ, $\tilde{\phi}(x,t)$ is the corresponding response field and $\mathscr{A}\{\tilde{\phi},\phi\}$ represents the action functional of the process [14–16]. $\phi_j(x,t)$ and $\tilde{\phi}_j(x,t)$ or alternatively their Fourier components $\phi_{j,k,\omega}$ and $\tilde{\phi}_{j,k,\omega}$ (for the definition see I (2.12)) are the basic random variables. With the help of the probability distribution the average value of an arbitrary random variable B can be expressed as

$$\langle B\rangle_w=\int \delta\phi\,\delta\tilde{\phi}\, B \exp \mathscr{A} \Big/ \int \delta\phi\,\delta\tilde{\phi} \exp \mathscr{A} \tag{2.2}$$

where

$$\int \delta\phi\,\delta\tilde{\phi} \equiv \int \prod_{j,k<\Lambda,\omega} d\phi_{jk\omega}\, d\tilde{\phi}_{jk\omega}. \tag{2.3}$$

The probability distribution W is specified by a set of parameters μ (the parameters of the action). The dynamic renormalization group transformation R_b transforms μ to $\mu'=R_b\mu$. R_b is given by the relation

$$\begin{aligned} W' &=\exp \mathscr{A}\{\mu', \tilde{\phi}, \phi\} \\ &=\int \prod_{j,\Lambda/b<k<\Lambda,\omega} d\phi_{j,k,\omega}\, d\tilde{\phi}_{j,k,\omega} \\ &\cdot \exp \mathscr{A}\{\mu,\tilde{\phi},\phi\}\Big|_{\substack{\phi_{k,\omega}\to b^{y}\phi_{bk,b^z\omega} \\ \tilde{\phi}_{k,\omega}\to b^{\tilde{y}}\tilde{\phi}_{bk,b^z\omega}}}. \end{aligned} \tag{2.4}$$

The constants y, $\tilde{y}$ and z are determined by the requirement of the existence of a fixed point μ^* defined as

$$R_b\mu^*=\mu^*. \tag{2.5}$$

The values of y, $\tilde{y}$ and z may depend on which of the fixed points they belong to.
For the random variables $\phi_{k,\omega}$ and $\tilde{\phi}_{k,\omega}$ with $k<\Lambda/b$, W and W' are equivalent in the sense that

$$\begin{aligned} &\langle \phi_{k_1,\omega_1}\dots\phi_{k_m,\omega_m}\tilde{\phi}_{k_{m+1},\omega_{m+1}}\dots\tilde{\phi}_{k_l,\omega_l}\rangle_w \\ &=b^{my+(l-m)\tilde{y}}\langle \phi_{bk_1,b^z\omega_1}\dots\phi_{bk_m,b^z\omega_m}\tilde{\phi}_{bk_{m+1},b^z\omega_{m+1}}\dots \\ &\tilde{\phi}_{bk_l,b^z\omega_l}\rangle_{w'}. \end{aligned} \tag{2.6}$$

The transformation of an arbitrary random variable B under R_b, $B\to B'$, is defined as follows:

$$\begin{aligned} &\langle B\,\phi_{k_1,\omega_1}\dots\tilde{\phi}_{k_l,\omega_l}\rangle_w \\ &-\langle B\rangle_w\langle\phi_{k_1,\omega_1}\dots\tilde{\phi}_{k_l,\omega_l}\rangle_w=b^{my+(l-m)\tilde{y}} \\ &\{\langle B'\,\phi_{bk_1,b^z\omega_1}\dots\tilde{\phi}_{bk_l,b^z\omega_l}\rangle_{w'} \\ &-\langle B'\rangle_{w'}\langle\phi_{bk_1,b^z\omega_1}\dots\tilde{\phi}_{bk_l,b^z\omega_l}\rangle_{w'}\} \end{aligned} \tag{2.7}$$

for any $k_1,\dots k_m,\dots k_l<\Lambda/b$, $\omega_1,\dots\omega_l$ and m, l.
Following from the semigroup property of the RG transformation one can construct a set of functions $g_i(\mu)$, similarly as in statics [1, 2], to each fixed point such that g_i transforms under R_b as

$$g_i'\equiv g_i(R_b\mu)=b^{y_i}\,g_i(\mu), \qquad i=1,2,\dots. \tag{2.8}$$

The quantities g_i are called scaling fields, the constant y_i is the exponent of g_i. The scaling fields form a more convenient set of parameters than μ but general procedures for the construction of the scaling fields are not known. We suppose here that the set $g\equiv(g_1,g_2,\dots)$ and the corresponding exponents y_i are given and use g instead of μ.
Generally the random variables are functions of ϕ and $\tilde{\phi}$ and also depend on the set of parameters g. In introducing the scaling variables as special random variables and discussing their properties we generalize the treatment followed by Ma in statics [3] to the situation in dynamics.
The random variable $\mathscr{D}$ which obeys the transformation rule

$$\mathscr{D}(g)\to(\mathscr{D}(g))'=b^{e}\,\mathscr{D}(g') \tag{2.9}$$

is called a scaling variable with scaling dimension $(-e)$. Let us consider

$$\mathscr{D}_i(g) = \partial \mathscr{A}(g)/\partial g_i \qquad i = 1, 2, \ldots. \tag{2.10}$$

Then

$$\mathscr{D}_i(g') = \partial \mathscr{A}(g')/\partial g'_i \tag{2.11}$$

where $\mathscr{A}(g') \equiv \mathscr{A}'$ represents the action functional after the transformation. Differentiating (2.6) with respect to g_i and using (2.2) and (2.7) the transformation rule

$$\mathscr{D}_i(g) \to (\mathscr{D}_i(g))' = b^{y_i} \mathscr{D}_i(g') \tag{2.12}$$

follows directly. It means that $\mathscr{D}_i(g)$ is a scaling variable with dimension $(-y_i)$.

It is worth mentioning that the quantity $\mathscr{D}_i \mathscr{D}_j$ is in general not a scaling variable. The following combination, however, which will be called the scaling product of $\mathscr{D}_i$ and $\mathscr{D}_j$

$$\{\mathscr{D}_i \mathscr{D}_j\} \equiv \mathscr{D}_i \mathscr{D}_j + \partial^2 \mathscr{A}/\partial g_i \partial g_j = \exp(-\mathscr{A}) \frac{\partial^2}{\partial g_i \partial g_j} \exp \mathscr{A} \tag{2.13}$$

gives a scaling variable. The corresponding dimension is $(-y_i - y_j)$. This can be shown by differentiating (2.6) twice with respect to g_i and g_j.

It is often convenient to use the so called local variables. $D_i(g; x, t)$ is a "local variable" related to $\mathscr{D}_i(g)$ if

$$\mathscr{D}_i(g) = \int d^d x \, dt \, D_i(g; x, t). \tag{2.14}$$

From relations (2.10) and (2.14) one expects that

$$D_i(g; x, t) = \partial A(g; \tilde{\phi}(x, t), \phi(x, t))/\partial g_i, \tag{2.15}$$

where $A(g; \tilde{\phi}, \phi)$ is defined by

$$\mathscr{A}\{g; \tilde{\phi}, \phi\} = \int d^d x \, dt \, A(g; \tilde{\phi}(x, t), \phi(x, t)), \tag{2.16}$$

is a local scaling variable with dimension $d + z - y_i$. Note, however, that the transformation rule

$$D_i(g; x, t) \to b^{-d-z+y_i} D_i(g'; x/b, t/b^z)$$

is correct only if D_i is a slowly varying function of x and t.

The local variable related to the scaling product (2.13) is

$$\{D_i(x_1, t_1) D_j(x_2, t_2)\} = D_i(x_1, t_1) D_j(x_2, t_2) + D_{i,j}(x_1, t_1) \, \delta(x_1 - x_2) \, \delta(t_1 - t_2) \tag{2.17}$$

where

$D_{i,j}(x, t) = \partial^2 A/\partial g_i \partial g_j$.

The associated scaling dimension is $2(d+z) - (y_i + y_j)$.

The scaling variables form a basis set, i.e., an arbitrary random variable can be expressed as a linear combination of the scaling variables. When the system is near its critical point only a few terms with the lowest dimensions play an important role in this series.

It is worth comparing the scaling fields and variables resulting from static and dynamic calculations, respectively. In the course of a dynamic calculation the set of scaling fields g can always be constructed in such a way that a subset of it, g_{st} depend only on the static parameters. The scaling fields of this subset correspond to those of a static calculation, up to a constant factor. On the other hand, there is no simple relationship between the scaling variables obtained in dynamics via differentiating the action with respect to the elements of the set g_{st} and the scaling variables deduced in statics.

III. The DRG in the Large-n Limit

In the first part of this work (I) we have started with an action functional corresponding to a simple Langevin type equation of motion. It has been shown there that DRG generates new couplings in the action. These are strongly related to the cumulants of the vertices which become random variables in the equation of motion under R_b (see Sect. V of I). We have determined the general form of the action arising after the DRG transformation in the large-n limit above T_c treating only couplings local in space and time which transform among themselves (see (3.1) of I).

In order to construct the complete set of scaling fields and scaling variables in this parameter space we start with the afore-mentioned action

$$\mathscr{A}\{\tilde{\phi}, \phi\} = \int d^d x \, dt \cdot \left[\sum_{j=1}^{n} \{-\tilde{\phi}_j L \tilde{\phi}_j + i \tilde{\phi}_j (\dot{\phi}_j - a L \Delta \phi_j)\} + Y(\phi^2, \varphi) \right] \tag{3.1}$$

where

$$\phi^2 = (1/2) \sum_{j=1}^{n} \phi_j^2 \tag{3.2}$$

$$\varphi = i \sum_{j=1}^{n} \tilde{\phi}_j L \phi_j + (n/2) \Gamma V^{-1} \sum_{k < \Lambda} k^c, \tag{3.3}$$

$L = \Gamma (i \nabla)^c$, $c = 0, 2$ in the case of a non-conserved and a conserved order parameter, respectively. V denotes the volume of the system.

The action given by (3.1) corresponds to a general equation of motion the vertices of which are deltacorrelated random variables with non-Gaussian distributions. The parameters $u_{2m,2l}$ defined as the Taylor coefficients of the function $Y(\phi^2, \varphi)$ (for the definition see I (3.3)) form one possible representation of the parameter space μ. The quantities $u_{2m,2}$ denote the average values of random coefficients of terms like ϕ^{2m-1} in the equation of motion, while the parameters $u_{2m,2l}$, $l>1$ are related to the higher order cumulants of the random coefficients (for more details see Sect. V of I).

$Y(\phi^2, \varphi)$ can be any function with the restriction

$$Y(\phi^2, 0) = \text{constant} \equiv Y(N_c, 0). \tag{3.4}$$

This is required by causality which can be proved by similar arguments as in Appendix C of [16]. Equation (3.4) insures that the response function loop cancels the contribution of the functional Jacobian. Later the constant $Y(N_c, 0)$ in (3.4) will be chosen to be zero.

To perform the DRG transformation (2.4) we decompose the fields into two parts

$$\phi_j \to \phi_j + \hat{\phi}_j, \qquad j=1,2,\dots n, \tag{3.5}$$

where ϕ_j on the right hand side involves only wave numbers smaller than Λ/b, while $\hat{\phi}_j$ contains the large wave number components. A similar separation is valid also for $\tilde{\phi}_j$. Since n is large and both $\hat{\phi}^2$ and $\hat{\varphi}$ are sums of n terms, the relative fluctuations of these quantities are small, that is

$$\hat{\varphi} - \langle\hat{\varphi}\rangle_b \ll \mathcal{O}(n), \qquad \hat{\phi}^2 - \langle\hat{\phi}^2\rangle_b \ll \mathcal{O}(n), \tag{3.6}$$

where $\langle\dots\rangle_b$ denotes the average over field variables with wave numbers between Λ/b and Λ.

It follows from (3.6) that the action functional can be expanded in an appropriate way in powers of $\hat{\varphi} - \langle\hat{\varphi}\rangle_b$ and $\hat{\phi}^2 - \langle\hat{\phi}^2\rangle_b$ making possible to perform the DRG transformation by simple Gaussian integrations. The details of the calculations are relegated to the Appendix. Before discussing the result we recall two notations already used in I:

$$Y_{i,j}(\phi^2, \varphi) \equiv \partial^{i+j} Y(\phi^2, \varphi)/\partial(\phi^2)^i \, \partial\varphi^j \tag{3.7}$$

$$\int_q^{>} \equiv K_d \int_{\Lambda}^{\Lambda b} dq\, q^{d-1}, \tag{3.8}$$

where $K_d(2\pi)^d$ is the area of the d-dimensional unit sphere. The DRG transformation can more conveniently be given for the two first partial derivatives of Y instead of Y. We obtain for them

$$Y'_{1,0}(\phi^2, \varphi) = b^{4+c}\, Y_{1,0}(b^{2-d} Q(\phi^2, \varphi) + N_c\, b^{-d-c} R(\phi^2, \varphi)), \tag{3.9a}$$

$$Y'_{0,1}(\phi^2, \varphi) = b^2\, Y_{0,1}(b^{2-d} Q(\phi^2, \varphi) + N_c\, b^{-d-c} R(\phi^2, \varphi)), \tag{3.9b}$$

where

$$Q(\phi^2, \varphi) = \phi^2 - N_c + (n/2) \int_q^{>} (q^{c/2} S^{-1} - q^{-2}), \tag{3.10a}$$

$$R(\phi^2, \varphi) = \varphi - (n/2) \int_q^{>} (q^{3c/2}(q^2 + Y'_{0,1}(\phi^2, \varphi))\, S^{-1} - q^c), \tag{3.10b}$$

with

$$S \equiv \{q^c (q^2 + Y'_{0,1}(\phi^2, \varphi))^2 - 2\, Y'_{1,0}(\phi^2, \varphi)\}^{1/2}, \tag{3.11}$$

$$N_c = (n/2)\, K_d\, \Lambda^{d-2}/(d-2). \tag{3.12}$$

Since the parameters a and Γ do not transform (a similar situation has been found in I, see (3.8), (3.9) of I) they have been set equal to unity in equations (3.9)–(3.11).

An important feature of the recursion relations (3.9) is that at $\varphi = 0$ they describe the transformation of the static parameters as it can also be seen by comparing (3.9) with the results of I. This feature can be traced back to the consequence of (3.4)

$$Y_{1,0}(\phi^2, 0) \equiv 0. \tag{3.13a}$$

In the special case $Y(\phi^2, \varphi) = \varphi\, U^{(1)}(\phi^2)$, which corresponds to the initial action used in I, the recursion relations (3.9) reduce to equations (3.10)–(3.11) of I.

The fixed point is generated by taking the limit $b \to \infty$. It can easily be seen that the critical surface is specified by

$$Y_{0,1}(N_c, 0) = 0. \tag{3.13b}$$

For $d>4$ always the Gaussian fixed point is reached:

$$Y_G^*(\phi^2, \varphi) \equiv 0. \tag{3.14}$$

For $2<d<4$ the Gaussian fixed point exists but it can be stable only if $Y_{0,2}(N_c, 0) = 0$. If, however, $Y_{0,2}(N_c, 0) > 0$ a new non-trivial fixed point arises which is stable. The fixed functions $Y^*_{0,1}$ and $Y^*_{1,0}$ associated with this fixed point are determined by the same equations as in I ((3.20) and (3.21) of I) where we started from a different initial action. This is a manifestation of universality. The connection between Y^* and its derivatives is given as follows:

$$Y^*(\phi^2, \varphi) = (\phi^2 - N_c)\, Y^*_{1,0} + \varphi\, Y^*_{0,1} - (n/2) \int_{\Lambda}^{\infty} K_d\, q^{d-1+c} \{[(q^2 + Y^*_{0,1})^2 - 2\, Y^*_{1,0}\, q^{-c}]^{\frac{1}{2}} - (q^2 + Y^*_{0,1}) + Y^*_{1,0}\, q^{-2-c}\}\, dq. \tag{3.15}$$

IV. Scaling Fields in the Large-*n* Limit

In order to construct the set of scaling fields one should find a function of the random variables which follows a simple transformation under R_b.
For this purpose let us consider the Legendre transformation of $Y(\phi^2, \varphi)$:

$$\tilde{Z}(z_1, z_2) = Y - (\phi^2 - N_c)\, z_1 - \varphi z_2, \tag{4.1}$$

which is regarded as a function of the variables

$$z_1 \equiv Y_{1,0} = \partial Y/\partial \phi^2, \qquad z_2 \equiv Y_{0,1} = \partial Y/\partial \varphi. \tag{4.2}$$

Note that for convenience the notation z_1 and z_2 will be used for $Y_{1,0}$ and $Y_{0,1}$, respectively, in this Section. It follows from (4.1) that

$$\phi^2 - N_c = -\partial \tilde{Z}/\partial z_1, \qquad \varphi = -\partial \tilde{Z}/\partial z_2. \tag{4.3}$$

The transformed quantity $\tilde{Z}'$ is defined similarly

$$\tilde{Z}'(z_1', z_2') = Y' - (\phi^2 - N_c)\, z_1' - \varphi z_2', \tag{4.4}$$

$$\phi^2 - N_c = -\partial \tilde{Z}'/\partial z_1', \qquad \varphi = -\partial \tilde{Z}'/\partial z_2'. \tag{4.5}$$

The connection between $\tilde{Z}'$ and $\tilde{Z}$ is given as

$$\tilde{Z}'(z_1', z_2') = b^{d+2+c}\, \tilde{Z}(b^{-4-c} z_1', b^{-2} z_2') + F(z_1', z_2'), \tag{4.6}$$

where

$$F(z_1, z_2) = -(n/2) \int_q^{>} f(q; z_1, z_2) \tag{4.7a}$$

with

$$f(q; z_1, z_2) = q^c \{[(q^2 + z_2)^2 - 2 z_1 q^{-c}]^{\frac{1}{2}} - (q^2 + z_2) + z_1 q^{-2-c}\}. \tag{4.7b}$$

The transformation rule (4.6) can easily be verified using (3.9), (4.3), (4.5) and the fact, that

$$\partial f/\partial z_1' = Q - (\phi^2 - N_c), \quad \partial f/\partial z_2' = R - \varphi,$$

where Q and R are defined by (3.10). The recursion relation (4.6), (4.7) with z_1 and z_2 as independent variables is equivalent to the transformation (3.9)-(3.12) where ϕ^2 and φ are regarded as independent variables.
In order to rewrite our results in a more convenient form, let us introduce the function

$$Z(z_1, z_2) = \tilde{Z}(z_1, z_2) - \tilde{Z}^*(z_1, z_2), \tag{4.8a}$$

where

$$\tilde{Z}^*(z_1, z_2) = -(n/2) \int_{\Lambda}^{\infty} K_d q^{d-1} f(q; z_1, z_2)\, dq \tag{4.8b}$$

is the expression of $\tilde{Z}$ at the non-trivial fixed point. This can be directly seen by substituting $\tilde{Z}^*$ into (4.6). Furthermore it can be shown using (A.11) and (3.9) that $\tilde{Z}'$ goes over to $\tilde{Z}^*$ for large b if one starts from the region of attraction of the non-trivial fixed point. It is worth mentioning that $Z^*(z_1^*, z_2^*)$ is just the Legendre transformation of $Y^*(\phi^2, \varphi)$. The function Z defined by (4.8) transforms in a simple way under R_b:

$$Z'(z_1', z_2') = b^{d+2+c}\, Z(b^{-4-c} z_1', b^{-2} z_2') \tag{4.9}$$

as a consequence of (4.6), (4.7). Equation (4.9) will play an important role in deriving the scaling fields and scaling variables.

Scaling Fields Associated with the Non-Trivial Fixed Point

Let us consider the quantities $g_{\alpha\beta}$ defined by the Taylor expansion

$$Z(z_1, z_2) = \sum_{\alpha \geq 0,\, \beta \geq 0} g_{\alpha,\beta}\, z_1^\alpha z_2^\beta. \tag{4.10}$$

It follows from the transformation rule (4.9) and from the definition (2.8) that the parameters $g_{\alpha\beta}$ are scaling fields: $g'_{\alpha,\beta} = b^{y_{\alpha,\beta}} g_{\alpha,\beta}$ with

$$y_{\alpha,\beta} = d + 2 + c - (4 + c)\,\alpha - 2\beta. \tag{4.11}$$

It is clear from (4.8a) that the scaling fields $g_{\alpha\beta}$ are associated with the non-trivial fixed point, where all $g_{\alpha,\beta} = 0$.
As a consequence of (3.4) and (4.2) $z_1 \equiv 0$ if $\varphi = 0$. Thus from (4.1) we obtain, that

$$Z(0, z_2) = \text{constant} \equiv Y(N_c, 0). \tag{4.12}$$

Comparing (4.12) with (4.10) one concludes that the scaling fields $g_{0,\beta}$ with $\beta > 0$ can never be present. It is worth discussing here shortly the role of the quantity $g_{0,0} = Y(N_c, 0)$. This scaling field increases with b as b^{d+z}. It is related to the constant term of the action (to the normalization factor of the probability distribution W). The role of $g_{0,0}$ is analogous to that of the static scaling field g_0 with exponent d (which represents the non-singular part of the free energy [2]). Similarly to g_0 neither $g_{0,0}$ influences the critical behaviour of the system, therefore from now on, we shall set $Y(N_c, 0)$ equal to zero. Thus in what follows the sum over α in (4.10) will start with $\alpha = 1$.
It will be more convenient to use the derivatives of Z:

$$\tau_i(z_1, z_2) \equiv -\partial Z/\partial z_i, \qquad i = 1, 2. \tag{4.13}$$

According to (4.8a) we can write

$$\tau_i(z_1, z_2) = \tilde{\tau}_i(z_1, z_2) - \tilde{\tau}_i^*(z_1, z_2) \tag{4.14a}$$

where

$$\tilde{\tau}_i = \partial \tilde{Z}/\partial z_i, \quad \tilde{\tau}_i^* = \partial \tilde{Z}^*/\partial z_i, \quad i=1,2. \tag{4.14b}$$

From (4.10)

$$\tau_1(z_1, z_2) = -\sum_{\alpha>0,\,\beta\geq 0} g_{\alpha,\beta}\,\beta z_1^\alpha z_2^{\beta-1}, \tag{4.15}$$

$$\tau_2(z_1, z_2) = -\sum_{\alpha>0,\,\beta\geq 0} g_{\alpha,\beta}\,\alpha z_1^{\alpha-1} z_2^\beta \tag{4.16}$$

At the non-trivial fixed point $\tau_1^*(z_1, z_2) = \tau_2^*(z_1, z_2) \equiv 0$. Using the explicit expressions (4.3) and (4.7b), (4.8b) one easily sees that these conditions and (3.20)–(3.22) of I are equivalent (the difference lies only in the choice of independent variables).

As it has been mentioned, the transformation of the static parameters is recovered from the recursion relations at $\varphi=0$. Since for $\varphi=0$ we have $z_1=0$ it follows from (4.16) that the expansion

$$\tau_2(0, z_2) = -\sum_{\beta\geq 0} g_{1,\beta} z_2^\beta \tag{4.17}$$

generates the static scaling fields. Comparing this with equation (3.8b) of [9] one finds, that $g_{1,\beta} = (\beta+1)\, g_{(\beta+1)C}$, where $g_{\beta C}$ denotes the static scaling fields associated with the non-trivial fixed point as determined by Zannetti and Di Castro.

The next step is to find the relation between the scaling fields and the parameters in the action. Here it is convenient to use the representation of the parameter space as follows.*

$$\mu = \{U_{2m,\,2l} \mid m \geq l \geq 1\}, \tag{4.18}$$

where $U_{2m,\,2l}$ is defined by

$$Y(\phi^2, \varphi) = \sum_{m=1}^{\infty} \sum_{1\leq l\leq m} U_{2m,\,2l}\, \varphi^l [2(\phi^2 - N_c)]^{m-l} \tag{4.19}$$

(see also Sect. IV of I). In order to obtain the scaling fields $g_{\alpha,\beta}$ we return in (4.16) to the variables ϕ^2 and φ, calculate the derivatives of the equation and evaluate the expressions at $\phi^2 = N_c$, $\varphi = 0$.

Although above T_c the expressions obtained are very complicated they become tractable at T_c, where both z_1 and z_2 vanish at $\phi^2 = N_c$, $\varphi = 0$ due to (3.13). By this reason in the following only the scaling fields on the critical surface ($T = T_c$) will be discussed. This means that the relevant scaling field $g_{1,0}$ with exponent $y_{1,0} = d-2$ is chosen to be zero. (It can be seen from (4.11) that there is no other relevant scaling field for $2<d<4$).

* Since a and Γ do not transform they are not involved

The first few irrelevant scaling fields are obtained as

$$g_{1,1} = (1/2)(U_{4,2}^{*-1} - U_{4,2}^{-1}), \tag{4.20}$$

$$g_{2,0} = (1/4)(U_{4,4}\, U_{4,2}^{-2} - U_{4,4}^*\, U_{4,2}^{*-2}), \tag{4.21}$$

$$\begin{aligned} g_{1,2} &= (1/2)(U_{6,2}\, U_{4,2}^{-3} - U_{6,2}^*\, U_{4,2}^{*-3}),\\ g_{2,1} &= (1/4)(U_{6,4}\, U_{4,2}^{-3} - U_{6,4}^*\, U_{4,2}^{*-3})\\ &\quad -(U_{6,2}\, U_{4,4}\, U_{4,2}^{-4} - U_{6,2}^*\, U_{4,4}^*\, U_{4,2}^{*-4}). \end{aligned} \tag{4.22}$$

The fixed point values $U_{2m,\,2l}^*$ in (4.20)–(4.22) can be read off from (3.24) of I using

$$U_{2m,\,2l} = \frac{1}{m!\, 2^{m-l}} \binom{m}{l}\, Y_{m-l,\,l}(N_c, 0). \tag{4.23}$$

In the large-n limit one can explicitly see, that in the immediate neighbourhood of the fixed point the DRG transformation can be linearized which is in accord with the general assumptions of RG procedures. In the linear approximation the scaling fields $g_{\alpha,\beta}$ go over to the linear scaling fields $\mu_{\alpha,\beta}$:

$$\begin{aligned} \mu_{1,1} &= (1/2)\, \delta U_{4,2}\, U_{4,2}^{*-2}\\ \mu_{2,0} &= (1/4)\, \delta U_{4,4}\, U_{4,2}^{*-2} - \mu_{1,1}\, U_{4,4}^*\, U_{4,2}^{*-1}\\ \mu_{1,2} &= (1/2)\, \delta U_{6,2}\, U_{4,2}^{*-3} - 3\mu_{1,1}\, U_{6,2}^*\, U_{4,2}^{*-2}\\ \mu_{2,1} &= (1/4)\, \delta U_{6,4}\, U_{4,2}^{*-3} - 2\mu_{1,2}\, U_{4,4}^*\, U_{4,2}^{*-1}\\ &\quad -4\mu_{2,0}\, U_{6,2}^*\, U_{4,2}^{*-2}\\ &\quad -\mu_{1,1}(3U_{6,4}^*\, U_{4,2}^* + U_{6,2}^*\, U_{4,4}^*)\, U_{4,2}^{*-3}, \end{aligned} \tag{4.24}$$

where $\delta U_{2m,\,2l} \equiv U_{2m,\,2l} - U_{2m,\,2l}^*$.

It is of particular interest that near the fixed point an explicit expression can be given for Y in terms of the linear scaling fields. Namely, here $g_{\alpha,\beta}$ in (4.10) can be replaced by the small quantity $\mu_{\alpha,\beta}$ and thus in a linearized calculation one can put z_1^* and z_2^* for z_1 and z_2, respectively, on the right hand side of (4.10). (Here we have returned to the variables ϕ^2 and φ). Furthermore expanding $\tilde{Z}^*$ around z_1^*, z_2^* we find using (3.15) that in leading order the left hand side of (4.10) is equal to $Y - Y^*$ and consequently

$$Y(\phi^2, \varphi) - Y^*(\phi^2, \varphi) = \sum_{\alpha>0,\,\beta\geq 0} \mu_{\alpha,\beta}\, z_1^{*\alpha} z_2^{*\beta}. \tag{4.25}$$

This expression suggests that the quantities $z_1^{*\alpha} z_2^{*\beta} \equiv Y_{1,0}^{*\alpha}\, Y_{0,1}^{*\beta}$ are eigenoperators of the linearized DRG transformation. We shall return to this question in Sect. V.

Equation (4.10) makes possible to obtain the recursion relations for large b independently how far away from the non-trivial fixed point on the critical surface we started. For this purpose let us consider (4.10) after the transformation where $\tilde{Z}'$ is given by (4.4). For large b it is sufficient to keep only the term with the largest exponent on the right hand side and

z_1' and z_2' can be replaced by their fixed point expressions. On the left hand side since $z_1'-z_1^*$ and $z_2'-z_2^*$ are small quantities $\tilde{Z}^*$ can again be expanded around z_1^* and z_2^* and for $2<d<4$ we obtain at T_c in leading order that

$$Y'(\phi^2,\varphi)-Y^*(\phi^2,\varphi)=g_{1,1}' z_1^* z_2^*$$

in accordance with (3.29) of I got by a direct calculation. The scaling field g of I corresponds to $g_{1,1}$ in the present notation.

Scaling Fields Associated with the Trivial Fixed Point at Ordinary and Higher Order Critical Points

In order to determine the scaling fields associated with the trivial Gaussian fixed point we consider the Legendre transformation of Z with τ_1 and τ_2 defined by (4.13), (4.14) as independent variables:

$$\Sigma(\tau_1,\tau_2)=Z+\tau_1 z_1+\tau_2 z_2. \tag{4.26}$$

As a consequence of the transformation of Z, (4.9), the following recursion relation is valid for Σ:

$$\Sigma'(\tau_1',\tau_2')=b^{d+2+c}\Sigma(b^{-d-c}\tau_1', b^{2-d}\tau_2'). \tag{4.27}$$

Owing to this equation the quantities $g_{\alpha,\beta}^{(G)}$ defined by the expansion

$$\Sigma(\tau_1,\tau_2)=\sum_{\alpha\geqq 0,\beta\geqq 0} g_{\alpha,\beta}^{(G)}\tau_1^\alpha\tau_2^\beta \tag{4.28}$$

are scaling fields with exponents

$$y_{\alpha,\beta}^{(G)}=d+2+c-(d+c)\alpha-(d-2)\beta. \tag{4.29}$$

The scaling fields $g_{\alpha,\beta}^{(G)}$ are associated with the trivial fixed point where of course $z_{1G}^*(\tau_1,\tau_2)=z_{2G}^*(\tau_1,\tau_2)\equiv 0$.

Following from the requirement of causality, (3.13), $\tau_1=0$ if $\varphi=0$, $(z_1=0)$ thus $\Sigma(0,\tau_2)=Y(N_c,0)$. Comparing this with (4.28) one concludes that $g_{0,\beta}^{(G)}=0$ for all positive β-s. Furthermore $g_{0,0}^{(G)}=g_{0,0}=Y(N_c,0)$ which vanishes due to our previous choice of $Y(N_c,0)$. So the sum over α in (4.28) will start with $\alpha=1$.

The subset of the static scaling fields is related to the value $\alpha=1$ also in this case

$$g_{1,\beta}^{(G)}=(\beta+1)\,g_{(\beta+1)t},$$

where $g_{\beta t}$ denotes the static scaling fields associated with the trivial fixed point determined in [9].

When deducing the relation between the scaling fields $g_{\alpha,\beta}^{(G)}$ and the elements of the parameter space (4.18) we return to the variables ϕ^2 and φ in Σ. The quantitites $g_{\alpha,\beta}^{(G)}$ are obtained by differentiating (4.28) and evaluating the expressions at $\phi^2=N_c$ and $\varphi=0$. Here again only the scaling fields on the critical surface will be discussed i.e., the relevant scaling field $g_{1,0}^{(G)}$ with exponent $y_{1,0}^{(G)}=2$ will be chosen to be zero.

The expressions for the scaling fields $g_{1,1}^{(G)}$, $g_{2,0}^{(G)}$ and $g_{1,2}^{(G)}$ can be summarized in the formula

$$g_{\alpha,\beta}^{(G)}=(-1)^\alpha g_{\alpha,\beta}\, g_{1,1}^{-(\alpha+\beta)} \tag{4.30}$$

$g_{2,1}^{(G)}$, however, can not be expressed in terms of the scaling fields associated with the non-trivial fixed point in a simple way since

$$\begin{aligned} g_{2,1}^{(G)}&=2(U_{6,4}U_{4,2}^{*3}-U_{6,4}^*U_{4,2}^3)(U_{4,2}^*-U_{4,2})^{-3}\\ &\quad -8(U_{6,2}U_{4,2}^{*2}-U_{6,2}^*U_{4,2}^2)\\ &\quad \cdot(U_{4,4}U_{4,2}^*-U_{4,4}^*U_{4,2})(U_{4,2}^*-U_{4,2})^{-4} \end{aligned} \tag{4.31}$$

where $U_{2m,2l}^*$ denotes the fixed point values at the non-trivial fixed point. Note that the scaling field $g_{1,1}^{(G)}$ is a relevant one for $d<4$ since $y_{1,1}^{(G)}=4-d$. This reflects the instability of the trivial fixed point: for $g_{1,1}^{(G)}\neq 0$ the non-trivial fixed point is reached as b goes to infinity, while $g_{1,1}'^{(G)}$ diverges (crossover phenomena will be discussed in the next subsection in more details). When $g_{1,1}^{(G)}$ is chosen to be zero, which according to (4.20) and (4.30) is possible only if $U_{4,2}=0$, the trivial fixed point becomes stable for $d>3$. This situation corresponds to a tricritical point. For $d<3$ $g_{1,2}^{(G)}$ is also a relevant scaling field with $y_{1,2}^{(G)}=6-2d$. If, however, also $g_{1,2}^{(G)}$ is chosen to be zero $(U_{4,2}=U_{6,2}=0)$ the trivial fixed point becomes stable already for $d>8/3$. This specifies a critical point of fourth order.

In general at a critical point of order σ $U_{2m,2}=0$ for $m<\sigma$ and $U_{2\sigma,2}>0$ are required. In this case $g_{1,m}^{(G)}=0$ for $m<\sigma-1$ and it is easy to deduce the most relevant scaling field $g_{1,1}^{(\sigma)}$ related to this critical point

$$g_{1,1}^{(\sigma)}=2^{(\sigma-1)}U_{2\sigma,2}, \quad \text{with } y_{1,1}^{(\sigma)}=2-(d-2)(\sigma-1).$$

It indicates that the trivial fixed point is stable for $d>d_\sigma=2\sigma/(\sigma-1)$. Further scaling fields $g_{\alpha,\beta}^{(\sigma)}$ can be obtained from relations like (4.30), (4.31) and from the condition $U_{2m,2}=0$ for $m<\sigma$.

The linear scaling fields $\mu_{\alpha,\beta}^{(G)}$ of the trivial fixed point are given as the linearized expressions of the scaling fields $g_{\alpha,\beta}^{(G)}$. Sufficiently close to the trivial fixed point $g_{\alpha,\beta}^{(G)}$ in (4.28) can be replaced by the small quantity $\mu_{\alpha,\beta}^{(G)}$. As a consequence of (4.1), (4.8) and (4.26) in leading order in z_1 and z_2 $\Sigma=Y$ and thus near the trivial fixed point

$$Y(\phi^2,\varphi)=\sum_{\alpha>0,\beta\geqq 0}\mu_{\alpha,\beta}^{(G)}\varphi^\alpha(\phi^2-N_c)^\beta. \tag{4.32}$$

It has been used that τ_1 and τ_2 in this approximation go over to φ and $\phi^2 - N_c$, respectively, as it follows from (4.13), (4.14). Comparing (4.19) and the generating function of the linear scaling fields (4.32) one obtains the relation

$$\mu^{(G)}_{\alpha,\beta} = 2^\beta\, U_{2(\alpha+\beta),\,2\alpha}. \tag{4.33}$$

It follows from (4.32) that $\varphi^\alpha(\phi^2 - N_c)^\beta$ is an eigenoperator of the linearized DRG transformation near the trivial fixed point.

Finally it is of interest that equation (4.28) applied after the transformation makes possible to obtain the recursion relations of Y for large b, provided the trivial fixed point is stable. At a critical point of order σ we find in leading order in b at T_c that for $d > d_\sigma$

$$Y'(\phi^2, \varphi) = g'^{(G)}_{1,\sigma-1}\, \varphi(\phi^2 - N_c)^{\sigma-1} \tag{4.34}$$

in accordance with (3.30) of *I*.

Finally we want to stress that the trivial fixed point representation and the non-trivial fixed point representation of the scaling fields are equivalent for $2 < d < 4$ as far as the ordinary critical point is concerned and one can use that set of scaling fields which appears to be more convenient. A non-Gaussian fixed point does not exist above four dimensions in case of the ordinary critical point and in any dimensions in case of higher order critical points and consequently only the scaling fields associated with the trivial fixed point are meaningful in these situations.

Crossover and Trajectories

The phenomenon of crossover is described in the renormalization group picture as an effect of fixed points of competing stability. In order to analyse crossover one needs to know the complete set of scaling fields. As for the statics of the model in the large-n limit the crossover has already been discussed [9].

Let us examine the b dependence of the simplest dynamic parameter $U_{4,4}$ at T_c. (As a consequence of (4.33) $U_{4,4}$ is at the same time proportional to the linear scaling field $\mu^{(G)}_{2,0}$.) It follows from (4.20), (4.21) and (4.30) that

$$\begin{aligned} U_{4,4}/U^*_{4,4} &= \mu^{(G)}_{2,0}/\mu^{*(G)}_{2,0} \\ &= [g^{(G)}_{2,0}/\mu^{*(G)}_{2,0} + (g^{(G)}_{1,1}/\mu^{*(G)}_{1,1})^2](1 + g^{(G)}_{1,1}/\mu^{*(G)}_{1,1})^{-2} \end{aligned} \tag{4.35}$$

where $\mu^{*(G)}_{\alpha,\beta}$ denotes the value of the linear scaling field associated with the trivial fixed point taken at the non-trivial fixed point. Equation (4.35) indicates that for $d > 4$ or in the case of $g^{(G)}_{1,1} = 0$ for $2 < d < 4$

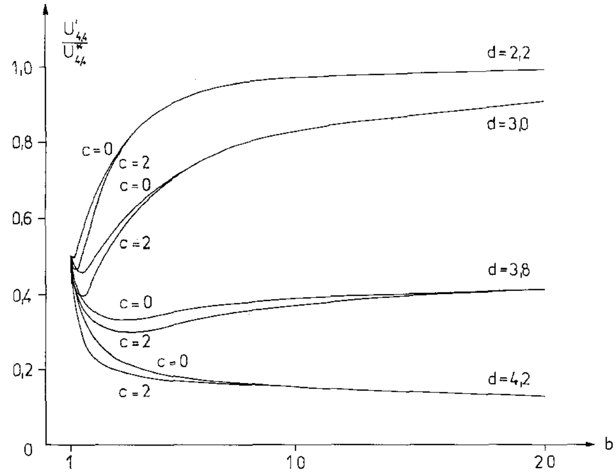

Fig. 1. U'_{44} as a function of b with $g^{(G)}_{2,0} = \mu^{*(G)}_{2,0}$ and $g^{(G)}_{1,1} = \mu^{*(G)}_{1,1}$ as initial values at $b = 1$

(tricritical point) $U'_{4,4}$ tends to zero as b goes to infinity. However, if we start from a point of the parameter space which corresponds to small, finite scaling fields in the trivial fixed point representation, (4.35) describes a crossover between the Van Hove behaviour and the true critical behaviour. In this case for b values near $b = 1$ $U'_{4,4} \sim g'^{(G)}_{2,0} = b^{2-d-c} g'^{(G)}_{2,0}$ i.e., $U'_{4,4}$ decreases first since the attraction of the trivial fixed point is decisive but with increasing b values the relevant scaling field $g'^{(G)}_{1,1}$ starts to dominate and finally $U'_{4,4} \to U^*_{4,4}$, thus the non-trivial fixed point is reached for $b \to \infty$. In order to characterize the approach to this fixed point it is more appropriate to use $\mu^{(G)}_{2,0}$.

Figure 1 illustrates the b dependence of $U'_{4,4}$ at T_c for a given initial value in different dimensions both for conserved ($c = 2$) and for non-conserved order parameter: At increasing dimensionalities (for $d < 4$) the approach of the non-trivial fixed point is continually slowing down and the tendency ceases at $d = 4$. In the case of a conserved order parameter the initial decrease, i.e., the attraction of the trivial fixed point is always stronger then for a non-conserved order parameter.

The knowledge of the scaling fields makes also possible to determine the flow lines of the DRG trajectories in the parameter space. Here we shall discuss the subspace spanned by the simplest static parameter ($U_{4,2}$) and the simplest dynamic one ($U_{4,4}$) at T_c. Due to (4.30)

$$\begin{aligned} U_{4,2}/U^*_{4,2} &= \mu^{(G)}_{1,1}/\mu^{*(G)}_{1,1} \\ &= (g^{(G)}_{1,1}/\mu^{*(G)}_{1,1})(1 + g^{(G)}_{1,1}/\mu^{*(G)}_{1,1})^{-1}. \end{aligned} \tag{4.36}$$

From (4.35) and (4.36) the desired relation between $U'_{4,2}$ and $U'_{4,4}$ can easily be deduced.

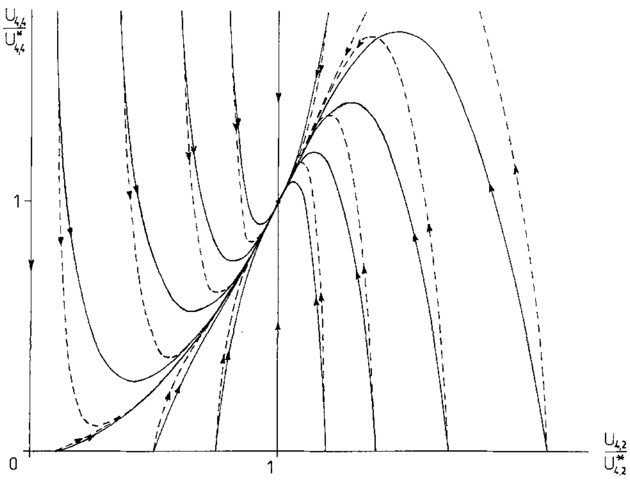

Fig. 2. Flow diagram for $d=3$ at $T=T_c$. The solid and dashed lines correspond to the case of the non-conserved and conserved order parameter, respectively

Figure 2 shows the DRG trajectories in the subspace $U_{4,2}$, $U_{4,4}$ in three dimensions for conserved and non-conserved order parameter, respectively. The flow lines approach the parabola $U_{4,4}/U^*_{4,4}=(U_{4,2}/U^*_{4,2})^2$ along which they tend to the point (1, 1) corresponding to the non-trivial fixed point. The origin represents, of course, the trivial fixed point. In the case of a conserved order parameter the flow lines are steeper before reaching the parabola. Similar situations are found also in other dimensions for $2<d<4$; with increasing dimensions the flow lines are steeper and steeper before reaching the parabola.

Finally it is to be noted that the special initial action corresponding to a usual Langevin type equation of motion (treated in *I*) is represented by the straight line $U_{4,4}=0$ on Fig. 2. It can be seen that the behaviour in the immediate vicinity of the non-trivial fixed point (the large b behaviour) is independent from the initial conditions. This is again a manifestation of universality.

V. Scaling Variables in the Large-n Limit

From the general relations derived in Sect. II and from the scaling fields deduced in the previous section it is straightforward to determine the scaling variables in the large-n limit.

Scaling Variables Associated with the Non-Trivial Fixed Point

It follows from (2.15), (2.16) and (3.1) that the local scaling variables $D_{\alpha,\beta}$ are obtained as

$$D_{\alpha,\beta}(x,t)=\partial Y(g;\tilde{\phi}(x,t),\phi(x,t))/\partial g_{\alpha,\beta}, \tag{5.1}$$

where $g_{\alpha,\beta}$ is defined by (4.10). In order to find an explicit g-dependence it is convenient to express Y from (4.1) and (4.8). Due to (4.13) and (4.14) the partial derivative of Y can be given as

$$\partial Y/\partial g_{\alpha\beta}=\partial Z/\partial g_{\alpha,\beta},$$

which means according to (4.10) that

$$D_{\alpha,\beta}(x,t)=[Y_{1,0}(\phi^2(x,t),\varphi(x,t))]^{\alpha}\cdot[Y_{0,1}(\phi^2(x,t),\varphi(x,t))]^{\beta} \tag{5.2}$$

with dimension

$$(4+c)\alpha+2\beta,\qquad \alpha=1,2,\ldots\ \beta=0,1,\ldots.$$

Also the scaling product of these variables can easily be worked out. From (2.20)

$$\begin{aligned}D_{\alpha',\beta';\alpha,\beta}&=\frac{\partial^2 Y}{\partial g_{\alpha',\beta'}\partial g_{\alpha,\beta}}=\frac{\partial}{\partial g_{\alpha',\beta'}}(Y^{\alpha}_{1,0}Y^{\beta}_{0,1})\\&=\alpha Y^{\alpha-1}_{1,0}Y^{\beta}_{0,1}\,\partial D_{\alpha',\beta'}/\partial\phi^2\\&\quad+\beta Y^{\alpha}_{1,0}Y^{\beta-1}_{0,1}\,\partial D_{\alpha',\beta'}/\partial\varphi;\end{aligned} \tag{5.3}$$

which gives by (2.19)

$$\begin{aligned}\{D_{\alpha',\beta';\alpha,\beta}\}&=\{D_{\alpha,\beta;\alpha',\beta'}\}\\&=D_{\alpha,\beta}(x_1,t_1)[1+\delta(x_1-x_2)\,\delta(t_1-t_2)\\&\quad\cdot(\alpha Y^{-1}_{1,0}\partial/\partial\phi^2+\beta Y^{-1}_{0,1}\partial/\partial\varphi)]D_{\alpha',\beta'}(x_2,t_2).\end{aligned} \tag{5.4}$$

The corresponding dimension is

$$(4+c)(\alpha+\alpha')+2(\beta+\beta').$$

Finally we note that near the non-trivial fixed point the random variables

$$Y^{*\alpha}_{1,0}Y^{*\beta}_{0,1}\qquad \alpha=1,2,\ldots\ \beta=0,1,\ldots$$

i.e., the eigenvalues of the linearized DRG transformation are the local scaling variables as it follows also from (4.25).

Scaling Variables Associated with the Trivial Fixed Point

In this case the local scaling variables are given by

$$D^{(G)}_{\alpha,\beta}(x,t)=\partial Y(g^{(G)};\tilde{\phi}(x,t),\phi(x,t))/\partial g^{(G)}_{\alpha,\beta} \tag{5.5}$$

where $g^{(G)}_{\alpha,\beta}$ is defined by (4.28). Now it is convenient to express Y from (4.26). Since $\partial\Sigma/\partial\tau_1=Y_{0,1}$ and $\partial\Sigma/\partial\tau_2=Y_{1,0}$ we obtain

$$\partial Y/\partial g^{(G)}_{\alpha,\beta}=\partial\Sigma/\partial g^{(G)}_{\alpha,\beta}$$

which according to (4.28) gives

$$D^{(G)}_{\alpha,\beta}(x,t) = [\tau_1(\phi^2(x,t),\varphi(x,t))]^\alpha[\tau_2(\phi^2(x,t),\varphi(x,t)]^\beta \tag{5.6}$$

with dimension

$$(d+c)\alpha+(d-2)\beta, \quad \alpha=1,2,\dots\ \beta=0,1,\dots.$$

The scaling product can be derived similarly as in the previous case.
Close to the trivial fixed point the random variables

$$\varphi^\alpha(\phi^2-N_c)^\beta, \quad \alpha=1,2,\dots\ \beta=0,1,\dots$$

are the local scaling variables as it follows also from (4.32).
The set of scaling variables obtained is complete in the sense that any local random variable which is given by a power series in ϕ^2 and φ can be expressed as a linear combination of the scaling variables $D_{\alpha,\beta}$ or $D^{(G)}_{\alpha,\beta}$.

Table 1.

Exponent	Scaling field	
	Non-conserved o.p. ($c=0$)	Conserved o.p. ($c=2$)
$d-2$	g_{10}	g_{10}
$d-4$	g_{11}	g_{11}
$d-6$	g_{12} g_{20}	g_{12}
$d-8$	g_{13} g_{21}	g_{13} g_{20}
$d-10$	g_{14} g_{22} g_{30}	g_{14} g_{21}
$d-12$	g_{15} g_{23} g_{31}	g_{15} g_{22}
$d-14$	g_{16} g_{24} g_{32} g_{40}	g_{16} g_{23} g_{30}

Table 2.

Exponent ($d=3$)	Scaling field	
	Non-conserved o.p. ($c=0$)	Conserved o.p. ($c=2$)
2	$g^{(G)}_{10}$	$g^{(G)}_{10}$
1	$g^{(G)}_{11}$	$g^{(G)}_{11}$
0	$g^{(G)}_{12}$	$g^{(G)}_{12}$
-1	$g^{(G)}_{13}$ $g^{(G)}_{20}$	$g^{(G)}_{13}$
-2	$g^{(G)}_{14}$ $g^{(G)}_{21}$	$g^{(G)}_{14}$
-3	$g^{(G)}_{15}$ $g^{(G)}_{22}$	$g^{(G)}_{15}$ $g^{(G)}_{20}$
-4	$g^{(G)}_{16}$ $g^{(G)}_{23}$ $g^{(G)}_{30}$	$g^{(G)}_{16}$ $g^{(G)}_{21}$

VI. Summary and Discussion

We have determined here the complete set of scaling fields and scaling variables in a non-trivial dynamic model. Both the trivial and the non-trivial fixed point representations have been worked out.
The difference whether the order parameter is conserved or not is clearly reflected in the scaling fields. From Table 1 which gives the hierarchy of scaling fields associated with the non-trivial fixed point, the following properties can immediately be seen: for a conserved order parameter the exponents of the dynamic scaling fields are smaller (in absolute values larger) than the corresponding ones in the other case. There are scaling fields sharing the same dimensionality. The degree of this degeneracy is smaller when the order parameter is conserved. As for the scaling fields associated with the trivial fixed point there is only one scaling field belonging to one exponent in general. If however, the dimensionality of space is a rational number degeneracy may occur. In $d=3$ we obtain the hierarchy indicated on Table 2. From (4.29) one sees that in the case of a conserved order parameter is the attraction of the trivial fixed point stronger, provided we start from its vicinity. This fact may play an important role in crossover phenomena also in systems with a finite n.
Let us shortly compare the scaling fields and scaling variables resulting from static and dynamic calculations, respectively. As it has been demonstrated the manifolds $\{g_{1,\beta}\}$ and $\{g^{(G)}_{1,\beta}\}$ are subsets of the parameter space the elements of which transform among themselves and are of purely static character. The scaling fields $g_{1,\beta}$ and $g^{(G)}_{1,\beta}$ correspond to the scaling fields obtained by a direct static calculation [9] in the non-trivial and the trivial fixed point representation, respectively.
As for the scaling variables those associated with $g_{1,\beta}$ turn out to be $t^{\beta+1}$ with dimension $2+2\beta$ in the static calculation [3] where the function $t(\phi^2)$ corresponds to $Y_{0,1}(\phi^2,0)$ in our notation, while the result of the dynamic calculation is $D_{1,\beta}=Y_{1,0}Y^\beta_{0,1}$ with dimension $4+c+2\beta$. Furthermore it can be shown that the scaling variable associated to $g^{(G)}_{1,\beta}$ in a static calculation is $\tau^{\beta+1}$ with dimension $(d-2)(\beta+1)$, where the function τ corresponds to $\tau_2(0,z_2)$, whereas the dynamic calculation gives $D^{(G)}_{1,\beta}=\tau_1\tau_2^\beta$ with dimension $d+c+(d-2)\beta$. Thus we conclude that two types of scaling variables can be associated with the static scaling fields when the dynamics of the system is also considered.
As it has been derived in *I* by a direct calculation and here within a more general framework the deviation of Y' from its fixed function can be expressed for large b at T_c as the product of the scaling field with the leading exponent and the corresponding eigenvector of the linearized transformation. This can be traced back to the fact that there is only one scaling field which belongs to the largest exponent. Similar behaviour is expected also when the number of components of the order parameter is finite.

Appendix: The Evaluation of (2.4) in the Large-n Limit

After separating the fields according to (3.5) the "action density" (3.1) reads

$$A=A_0+\sum_{j=1}^{n}\{-\hat{\tilde{\phi}}_j L\hat{\tilde{\phi}}_j+i\hat{\tilde{\phi}}_j(\dot{\hat{\phi}}_j-aL\Delta\hat{\phi}_j)\}$$
$$+Y(\phi^2+\hat{\phi}^2,\varphi+\hat{\varphi}),$$
$$A_0=\sum_{j=1}^{n}\{-\tilde{\phi}_j L\tilde{\phi}_j+i\tilde{\phi}_j(\dot{\phi}_j-aL\Delta\phi_j)\}, \tag{A.1}$$

where

$$\varphi\equiv i\sum_{j=1}^{n}\tilde{\phi}_j L\phi_j+(n/2)\Gamma V^{-1}\sum_{k<\Lambda/b}k^c \tag{A.2}$$

$$\hat{\varphi}\equiv i\sum_{j=1}^{n}\hat{\tilde{\phi}}_j L\hat{\phi}_j+(n/2)\Gamma V^{-1}\sum_{\Lambda/b<k<\Lambda/b}k^c. \tag{A.3}$$

Here we have used the fact that terms like $\phi\hat{\phi}, \phi\hat{\tilde{\phi}}, \dots$ etc. are to be neglected in the large-n limit, as it has been shown in Sect. IV of I.
According to (3.6) Y is expanded in terms of $\hat{\varphi}-\langle\hat{\varphi}\rangle_b$. In leading order

$$Y(\phi^2+\hat{\phi}^2,\varphi+\hat{\varphi})=Y(\phi^2+\hat{\phi}^2,\varphi+\langle\hat{\varphi}\rangle_b)$$
$$+Y_{0,1}(\phi^2+\hat{\phi}^2,\varphi+\langle\hat{\varphi}\rangle_b)[\hat{\varphi}-\langle\hat{\varphi}\rangle_b],$$

with $Y_{0,1}$ defined by (3.7). Thus W, (2.1), becomes a simple Gaussian distribution in $\hat{\tilde{\phi}}$ and different averages can easily be calculated. In terms of Fourier components we obtain that

$$\langle\hat{\phi}_{j,k,\omega}\hat{\tilde{\phi}}_{j-k,-\omega}\rangle_b=i\langle|\hat{\phi}_{j,k,\omega}|^2 g(k,\omega)\rangle_b/(2\Gamma k^c), \tag{A.4}$$

where $\Lambda/b<k<\Lambda$, and

$$g(k,\omega)\equiv i\omega+\Gamma k^c(ak^2+Y_{0,1}(\phi^2+\hat{\phi}^2,\varphi+\langle\hat{\varphi}\rangle_b)). \tag{A.5}$$

After integrating over $\hat{\tilde{\phi}}$ the following new term arises in the exponent of W

$$-\sum_{\Lambda/b<k<\Lambda,\omega}|\hat{\phi}_{j,k,\omega}|^2|g(k,\omega)|^2/(4\Gamma k^c),$$

apart from an unimportant constant. Then the exponent is expanded in powers of $\hat{\phi}^2-\langle\hat{\phi}^2\rangle_b$ and according to (3.6) one may keep only the leading term. A lot of new terms appear but several of them cancel if

$$\langle\hat{\varphi}\rangle_b=(n/2)\Gamma V^{-1}\sum_{\Lambda/b<k<\Lambda}k^c$$
$$-V^{-1}\sum_{j,\Lambda/b<k<\Lambda,\omega}(1/2)\langle|\hat{\phi}_{jk\omega}|^2\rangle_b$$
$$\cdot\Gamma k^c(ak^2+Y_{0,1}(\phi^2+\langle\hat{\phi}^2\rangle_b,\varphi+\langle\hat{\varphi}\rangle_b)) \tag{A.6}$$

holds. We shall verify this equality only later but use it at this point already to simplify the calculation. Finally we obtain in the exponent:

$$A'=A_0+Y-Y_{1,0}\langle\hat{\phi}^2\rangle_b$$
$$-Y_{0,1}\langle\varphi-(n/2)\Gamma V^{-1}\sum_{\Lambda/b<k<\Lambda}k^c\rangle_b$$
$$-V^{-1}\sum_{j,\Lambda/b<k<\Lambda,\omega}(1/2)|\hat{\phi}_{j,k\omega}|^2$$
$$\cdot\{[\omega^2+\Gamma^2k^{2c}(ak^2+Y_{0\,1})^2]/(2\Gamma k^c)-Y_{1,0}\}, \tag{A.7}$$

where the functions Y, $Y_{1,0}$, $Y_{0,1}$ have in their arguments $\phi^2+\langle\hat{\phi}^2\rangle_b$ and $\varphi+\langle\hat{\varphi}\rangle_b$. It has been used here that ϕ^2 and φ can be considered as nearly constant quantities since their deviation from the mean is small.
According to (A.7)

$$\langle|\hat{\phi}_{j,k,\omega}|^2\rangle_b=2\Gamma k^c(\omega^2+\Gamma^2k^{2c}(ak^2+Y_{0,1})^2$$
$$-2Y_{1,0}\Gamma k^c)^{-1}. \tag{A.8}$$

In the large-n limit

$$\langle\hat{\phi}^2\rangle_b=(1/2)V^{-1}\sum_{j,\Lambda/b<k<\Lambda,\omega}\langle|\hat{\phi}_{j,k,\omega}|^2\rangle_b. \tag{A.9}$$

The expansion in terms of $\hat{\phi}^2-\langle\hat{\phi}^2\rangle_b$ applied to (A.4) gives in leading order

$$\langle\hat{\phi}_{j,k,\omega}\hat{\tilde{\phi}}_{j,-k,-\omega}\rangle_b$$
$$=i\langle|\hat{\phi}_{jk,\omega}|^2\rangle_b g(k,\omega)/(2\Gamma k^c), \tag{A.10}$$

where g is defined by (A.5) but now with $\langle\hat{\phi}^2\rangle_b$ replacing $\hat{\phi}^2$. It follows from (A.3), (A.8) and (A.10) that (A.6) is a correct assumption. The integration over $\hat{\phi}$ yields a new term in the exponent of W and finally the action associated with the small wave number components can be written as $\int d^dx\,dt(A_0+w)$ where

$$w=Y-Y_{1,0}\langle\hat{\phi}^2\rangle_b$$
$$-Y_{0,1}\langle\varphi-(n/2)\Gamma\int_{\Lambda/b}^{\Lambda}dk\,K_d k^{c+d-1}\rangle_b$$
$$+(n/4\pi)\int_{\Lambda/b}^{\Lambda}dk\,K_d k^{d-1}\int_{-\infty}^{\infty}d\omega\ln[n/2\langle|\hat{\phi}_{j,k,\omega}|^2\rangle_b] \tag{A.11}$$

with the same arguments of Y and of its derivatives as in (A.7). Differentiating w one obtains

$$\partial w/\partial\phi^2=Y_{1,0}(\phi^2+\langle\hat{\phi}^2\rangle_b,\varphi+\langle\hat{\varphi}\rangle_b), \tag{A.12}$$
$$\partial w/\partial\varphi=Y_{0,1}(\phi^2+\langle\hat{\phi}^2\rangle_b,\varphi+\langle\hat{\varphi}\rangle_b). \tag{A.13}$$

After introducing new scales according to (2.4) from (A.6), (A.8), (A.9) and (A.12), (A.13) we recover the transformation rule (3.9)–(3.12) where a and Γ are chosen to be unity.

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