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Critical Dynamics near a Hard Mode Instability*

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Scaling hypothesis and a renormalization group procedure are formulated in the vicinity of the bifurcation point, where the behaviour is governed by inhomogeneous fluctuations. The working of the general ideas is illustrated in a model system in which the number of components of the complex order parameter field goes to infinity.

I. Introduction and Summary

Similarities and differences between phase transitions and instabilities occuring far away from thermodynamic equilibrium have extensively been discussed in the literature (see [1, 2] and references therein). In this paper we are interested in hard mode instabilities leading to homogeneous limit cycles from this point of view. It is assumed that the transition is of similar type as a second order phase transition, i.e. the order parameter changes continuously at the bifurcation point (normal Hopf bifurcation). We consider continuously extended systems containing inhomogeneous fluctuations, and our purpose is to study the behaviour in the vicinity of the bifurcation point where a region analogous to the critical region of second order phase transitions exists.

At the phenomenological level we formulate a scaling hypothesis for the correlation and the response of the slowly relaxing unstable mode which is a generalization of the dynamical scaling hypothesis [3, 4] near ordinary critical points. It is then shown how a renormalization group transformation can be defined to substantiate this scaling hypothesis and whose properties also in other respects resemble those of the dynamic renormalization group (for recent reviews see [5, 6]) near ordinary critical points. An additional feature is that the condition of criticality now yields, besides the critical value of the control parameter, the fluctuation correction to the frequency of the limit cycle at the bifurcation point, too.

Besides discussing such general ideas our aim in this paper is to demonstrate their working in a model system. In searching for a suitable model we recall that in ordinary critical phenomena the limit when the number of components of the order parameter field goes to infinity [7] has provided a useful theoretical framework for general investigations [8-16]. An analogous situation is expected in the present case, too. For the construction of such a model it is a basic fact that a wide class of hard mode instabilities has been pointed out to be describable by a TDGL type equation for a complex order parameter field [17-19, 2] which is more general than the usual one in the sense that its parameters are also complex numbers. Thus e.g. Kuramoto and Tsuzuki [17] have found that an adiabatic elim-

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ination of the stable modes in the Brusselator results in such a generalized TDGL equation for the slowly relaxing critical mode. The effects of noise in this equation have also been considered [20, 21] and dynamic renormalization group calculation has been carried out by Hentschel [20] for the case when the dimensionality of the system is close to four. His formulation leads to a scaling behaviour which is similar to that at tricritical points.

We shall consider the m-component version of the afore-mentioned model which corresponds to a situation where m modes become simultaneously unstable at the bifurcation point. The simplifying features arising in the limit $m \to \infty$ make explicit solutions possible. Thus in the postbifurcational region an "equation of state" will be deduced which gives both the amplitude and the frequency of the limit cycle. The correlation as well as the response functions will be determined both in the pre-bifurcational and in the post-bifurcational regions. It will be shown that the results fit in with the general predictions of the scaling hypothesis.

The renormalization group transformation becomes also tractable in the large-m case and the transformation of the parameters in an invariant subset of the parameter space can be followed in a global way. Moreover the non-linear scaling fields [22] can also be determined. We shall illustrate that at the bifurcation point a stable finite fixed point can be achieved by means of our procedure. The connection between the renormalization group procedure and the form of the scaling hypothesis will also be demonstrated.

The paper is organized as follows: The scaling hypothesis and the suggested renormalization group procedure is introduced in Sect. II. Section III contains the explicit solution of the generalized TDGL model in the limit $m \to \infty$, while Sect. IV is devoted to the application of the renormalization group method. Some details of the renormalization group calculation are presented in the Appendix.

II. Scaling Hypothesis

We are going to study a normal Hopf bifurcation: for control parameter values $\lambda < \lambda_c$ the system has a homogeneous steady state while for $\lambda > \lambda_c$ a homogeneous limit cycle with frequency $\omega_{lc}(\lambda)$ is approached asymptotically. The amplitude of the limit cycle is considered to be the order parameter which sets in continuously when λ goes through its critical value. Let $\phi_k(t)$ denote the slow mode dominating the behaviour of the system around the bifurcation point (k and t denote wave number and time, respectively). The instability occurs at $k=0$. We define a correlation function by

$$C(k, t) = \langle \bar{\phi}_k(t)\, \phi_k(0) \rangle, \qquad k \neq 0, \tag{2.1}$$

where, and in the following, bracket denotes average taken in the asymptotic state of the system (reached for $t \to \infty$) and bar denotes complex conjugation.

Contrary to equilibrium transitions or more generally speaking to soft mode instabilities, in the vicinity of a hard mode transition point the imaginary part of the frequency of the slow mode does not vanish, thus a new characteristic quantity enters the theory. To account for the new features we generalize the dynamical scaling hypothesis [3, 4] postulating the following form for the correlation function (2.1) near the bifurcation point:

$$C(k, t) = \exp[i\omega_0(\lambda)\, t]\, k^{-2+\eta}\, \tilde{C}(k\xi, k^z t), \tag{2.2}$$

where $\xi \propto |\lambda - \lambda_c|^{-\nu}$ is the correlation length in the asymptotic state, η and z stand for the critical exponent of the equal time correlation function at $\lambda = \lambda_c$, and the dynamical critical exponent, respectively. The hypothesis includes the following properties for the function $\omega_0(\lambda)$: real, independent of k and t and equal to the frequency of the limit cycle at the bifurcation point, that is $\omega_0(\lambda_c) = \omega_{lc}(\lambda_c)$. If such an $\omega_0(\lambda)$ exists it is not unique since it is determined by (2.2) only up to an additive term proportional to ξ^{-z}.

Depending on the analytic properties of $\omega_0(\lambda)$ and its relation to $\omega_{lc}(\lambda)$ we can distinguish the following cases:

Case A: $\omega_0(\lambda)$ does not have any of the special features listed under cases B-D.

Case B: We can choose

$$\omega_0(\lambda) = \omega_{lc}(\lambda), \qquad \text{for } \lambda > \lambda_c. \tag{2.3}$$

Case C: There is at least one particular $\omega_0(\lambda)$ that is analytic at λ_c.

Case D: Both requirements under B and C can be satisfied simultaneously with the same $\omega_0(\lambda)$.

Let us now discuss how it is possible to define a renormalization group (RG) procedure supporting the scaling hypothesis introduced above. In general starting with the original slow variables, $\phi_k(t)$, one will not find any finite stable fixed point after repeating the transformation since an extra relevant scaling field appears due to the presence of an enlarged parameter space in systems exhibiting limit cycle behaviour as compared to that of occuring in case of ordinary critical phenomena. The appearance of an extra relevant scaling field was pointed out first by Hentschel [20] in a generalized TDGL model with complex parameters near four dimensions. In order to handle this situation we suggest the following method: We change variables by the transformation $\phi_k(t) \to \phi_k(t) \exp(-i\omega t)$, ω real, and then at a particular choice $\omega = \Omega(\lambda)$ it will be possible to eliminate the extra relevant scaling field in the whole critical region. At the same time the requirement of a finite fixed point determines λ_c and the frequency of the limit cycle at λ_c. Applying the usual RG arguments one obtains for the correlation function (2.1) of the original field variables a form like (2.2) with $\Omega(\lambda)$ as $\omega_0(\lambda)$. In addition, since the RG transformation is expected to be analytic, Ω will be also analytic around λ_c, obeying the requirement of "*case C*". If "*case D*" can not be fulfilled it does not exclude the possibility that for one choice ω_0 obeys "*case B*" while for an other one ω_0 obeys "*case C*". This possibility arises because on the basis of the RG procedure one expects that

$$\Omega(\lambda) - \omega_{lc}(\lambda) = B\xi^{-z}, \quad \text{for } \lambda > \lambda_c, \tag{2.4}$$

where B is a constant.

In Sect. IV we shall illustrate the working of this RG procedure on the model obtained when the number of components of ϕ goes to infinity.

Finally a remark is in order on the response functions. Knowing the equation of motion of the slow mode one can formally introduce an external field coupled to $\phi_k(t)$ and define a response function, $G(k, t)$. In general, the fluctuation-dissipation theorem is not valid in such systems thus an independent scaling hypothesis is to be formulated for this function as follows

$$G(k, t) = k^\rho \exp[-i\omega_0(\lambda) t]\, \tilde{G}(k\xi, k^z t), \tag{2.5}$$

where ξ, z and $\omega_0(\lambda)$ has been defined in (2.2) and ρ represents the critical exponent of the response function.

III. The m-Component Model. Solution for $m \to \infty$

We generalize the TDGL model with complex parameters [17–21, 2] for m-component complex fields: $\phi_1, \phi_2, \dots \phi_m$, assuming isotropy in the component space. The general form of the equation of motion to be studied is the following in coordinate representation:

$$\dot{\phi}_j(x, t) = -\Gamma(-a\nabla^2 + r(|\phi|^2))\,\phi_j + \zeta_j(x, t), \tag{3.1}$$

where a is a complex parameter,

$$|\phi|^2 = (1/2) \sum_{l=1}^{m} |\phi_l|^2, \tag{3.2}$$

(the factor 1/2 has been introduced for convenience) and the function $r(|\phi|^2)$ is expressed as a power series

$$r(|\phi|^2) = \sum_{\alpha=1}^{\infty} u_{2\alpha} (2|\phi|^2)^{\alpha-1}. \tag{3.3}$$

The coefficients $u_{2\alpha}$ are complex. We shall use the notation for complex numbers z: $\operatorname{Re} z = z^{(1)}$ and $\operatorname{Im} z = z^{(2)}$. It is assumed for the real part of u_2 that

$$u_2^{(1)} = \lambda_0 - \lambda, \tag{3.4}$$

and $u_2^{(2)}$ and all the other $u_{2\alpha}$–s are considered to be independent of the control parameter. λ_0 is the critical value of the control parameter which would be obtained by a linear approximation of (3.1) and $\Gamma u_2^{(2)}$ represents the frequency of the limit cycle at λ_0 in the same approximation. To keep terms of powers up to infinity in (3.3) is required by the RG treatment (see Sect. IV).

The complex noise ζ is assumed to be a Gaussian white noise with zero mean value and correlation functions as

$$\langle \bar{\zeta}_j(x, t)\, \zeta_{j'}(x', t') \rangle = 4\Gamma\, \delta(x - x')\, \delta(t - t')\, \delta_{jj'}, \tag{3.5}$$

$$\langle \zeta\zeta \rangle = \langle \bar{\zeta}\bar{\zeta} \rangle = 0, \tag{3.6}$$

where Γ is a real constant, the same as in (3.1), where it was separated from the other parameters for convenience.

We will be interested in the many component limit ($m \to \infty$) which, similarly as in the theory of ordinary critical phenomena [8–16], will provide a simple but non-trivial model. In order to find terms which are of the same order of magnitude for $m \to \infty$ in (3.1) $u_{2\alpha}$ is assumed to be of order $m^{1-\alpha}$. For the dimensionality of the system $2 < d < 4$ will be assumed.

Solution for $\lambda < \lambda_c$

Since m is large and $|\phi|^2$ is a sum of m terms, the relative fluctuations of $|\phi|^2$ are small, thus $r(|\phi|^2)$ in (3.1) can be replaced by $r(N)$, where N denotes the average value of $|\phi|^2$ in the stationary state. Thus we arrive at a linear equation of motion which in terms of the Fourier components $\phi_{j,k}$ and $\zeta_{j,k}$ reads

$$\dot{\phi}_{j,k}(t) = -\alpha_k\, \phi_{j,k}(t) + \zeta_{j,k}(t), \tag{3.7}$$

with

$$\alpha_k = \Gamma(a k^2 + r(N)), \tag{3.8}$$

where N is to be calculated self-consistently.
In order to determine the stationary distribution we use the path probability introduced by Onsager and Machlup [23–25, 2]. The Lagrangian associated with equation (3.7) is given by

$$L(\dot{\phi}, \phi) = (4\Gamma)^{-1} \sum_{k,j} |\dot{\phi}_{j,k} + \alpha_k \phi_{j,k}|^2. \tag{3.9}$$

Graham showed [26] that for a linear process of the stochastic variable q, the conditional probability $P(q, t\,|\,q^0, 0)$ can be expressed as $\exp\left(-\int_0^t L d\tau\right)$, where the Lagrangian, $L(\dot{q}, q)$ is to be integrated along the most probable path with boundary conditions $q(\tau = t) = q$, $q(\tau = 0) = q^0$. Applying this method for the process related to (3.9) we get

$$P(\{\phi_{j,k}\}, t\,|\,\{\phi^0_{j,k}\}, 0)$$
$$\propto \exp\{-\sum_{k,j} \alpha_k^{(1)} |\phi_{j,k} - \phi^0_{j,k} e^{-\alpha_k t}|^2 / [2\Gamma(1 - e^{-2\alpha_k^{(1)} t})]\}. \tag{3.10}$$

The stationary distribution is generated by the limit $t \to \infty$:

$$P_{st}(\{\phi_{j,k}\}) \propto \exp\{-\sum_{k,j} \alpha_k^{(1)} |\phi_{j,k}|^2 / (2\Gamma)\}. \tag{3.11}$$

This shows that the stationary distribution is completely determined by the real part of $r(N)$. As a consequence the equal time correlation function in the steady is obtained as

$$\langle |\phi_{j,k}|^2 \rangle = 2(k^2 + r^{(1)}(N))^{-1}, \qquad a^{(1)} = 1. \tag{3.12}$$

Hence the self-consistency equation

$$N = (1/2) \sum_{k,j} \langle |\phi_{jk}|^2 \rangle = m \sum_k (k^2 + r^{(1)}(N))^{-1}$$
$$= m \int_0^{\Lambda} (k^2 + r^{(1)}(N))^{-1} d^d k / (2\pi)^d \tag{3.13}$$

is found, where Λ is the cut-off in the wave number space. At the critical point the relaxation rate of $\phi_{j,0}$ vanishes. Consequently if we denote $r(|\phi|^2)$ and N at the bifurcation point by $r_c(|\phi|^2)$ and N_c, respectively, it follows from (3.7) and (3.8) that $r_c^{(1)}(N_c) = 0$ should be fulfilled. This makes straightforward to calculate N_c from (3.13):

$$N_c = m K_d \Lambda^{d-2} / (d-2), \tag{3.14}$$

where $K_d (2\pi)^d$ is the area of the d-dimensional unit sphere. The condition $r_c^{(1)}(N_c) = 0$ determines the critical value of the control parameter as

$$\lambda_c = \lambda_0 + \sum_{\alpha=2}^{\infty} u_{2\alpha}^{(1)} (2N_c)^{\alpha-1}, \tag{3.15}$$

while the frequency of the limit cycle at λ_c is given by

$$\omega_{lc}(\lambda_c) = \Gamma r_c^{(2)}(N_c) = \Gamma u_2^{(2)} + \Gamma \sum_{\alpha=2}^{\infty} u_{2\alpha}^{(2)} (2N_c)^{\alpha-1}. \tag{3.16}$$

Note the deviations as compared to the results obtained from the linearized version of Eq. (3.1), i.e. λ_0 and $\Gamma u_2^{(2)}$, respectively.
Subtracting (3.14) from (3.13) one finds in the vicinity of the bifurcation point ($r^{(1)}(N) \ll \Lambda^2$) for $2 < d < 4$:

$$N - N_c = -[r^{(1)}(N)/A]^{d/2-1}, \tag{3.17}$$

where

$$A^{1-d/2} = m K_d \int_0^{\infty} x^{d-3} (1 + x^2)^{-1} dx. \tag{3.18}$$

Let us introduce the quantity ξ by

$$\xi \equiv [r^{(1)}(N)/A]^{-1/2}, \tag{3.19}$$

which can be interpreted as the correlation length in the steady state (see (3.12)). After similar steps as in the case of critical statics of the large-n system [9, 10] one gets the solution of the self-consistency Eq. (3.13) for λ close to λ_c

$$N - N_c = (\lambda - \lambda_c) / \dot{r}^{(1)}(N_c), \tag{3.20}$$

where the notation

$$\dot{r}(|\phi|^2) = dr(|\phi|^2) / d|\phi|^2 \tag{3.21}$$

has been introduced. From (3.17), (3.19) and (3.20) $\nu = 1/(d-2)$ is found. (Compare it with the spherical model result, see [12].)
By means of (3.10) and (3.11) the following expression is obtained for the correlation function in the stationary state

$$C(k, t) = \langle \bar{\phi}_{j,k}(t)\, \phi_{j,k}(0) \rangle$$
$$= \frac{2}{k^2 + r^{(1)}(N)} \begin{cases} \exp(-\bar{\alpha}_k t), & t > 0, \\ \exp(\alpha_k t), & t < 0. \end{cases} \tag{3.22}$$

Introducing formally a complex external field $h_j(t)$ in the equation of motion (3.1) as an additive term Γh_j on the right hand side, one finds for the response function

$$G(k, t) = \Gamma \exp(-\alpha_k t), \qquad t > 0 \tag{3.23}$$

and $G(k,t)\equiv 0$ for $t<0$. It is easy to check that the fluctuation-dissipation theorem is not fulfilled by (3.22) and (3.23) as expected.
From Eqs. (3.22) and (3.8), (3.19) it is obvious that the scaling hypothesis (2.2) is valid in the large-m case with $\omega_0(\lambda)=\Gamma r^{(2)}(N)$ for $\lambda<\lambda_c$.

Solution for $\lambda>\lambda_c$. The Frequency of the Limit Cycle

We shall see that in the post-bifurcational region a stationary distribution in the limit $t\to\infty$ exists for the fields ψ_j defined as

$$\phi_j(x,t)=\psi_j(x,t)\exp(-i\omega_{lc}t), \quad j=1,2,\ldots m, \tag{3.24}$$

where ω_{lc} denotes the frequency of the limit cycle. We start by assuming the existence of this stationary distribution and the consistency of this assumption will be shown a posteriori. The order parameter of the system is the amplitude of the limit cycle, in general a complex m-component vector. However, we can always choose the order parameter to point in the direction of the $j=1$ axis and to be real by making use of the isotropy of the system in the component space and the gauge invariance of the equation of motion (3.1), respectively. We again formally introduce a constant external complex field, h, coupled now to the $j=1$ component. Then the equation of motion for ψ_j is as follows

$$\dot{\psi}_j=-\Gamma(-a\nabla^2+r(|\psi|^2)-i\omega_{lc}/\Gamma)\psi_j+\Gamma h\delta_{j,1}+\zeta_j \tag{3.25}$$

We separate the order parameter Ψ by writing

$$\psi_j(x,t)=\psi'_j(x,t)+\Psi\delta_{j,1}, \quad \langle\psi'_j\rangle=0. \tag{3.26}$$

It will turn out (see (3.35)) that Ψ is of order $m^{1/2}$, therefore when calculating $|\psi|^2$ defined like in (3.2), the term $\Psi(\psi'_1+\bar{\psi}'_1)$ can be neglected as compared to terms of order m, and thus we can use the approximate equality

$$|\psi|^2=|\psi'|^2+\Psi^2/2. \tag{3.27}$$

Let N' denote the average value of $|\psi'|^2$ in the asymptotic state. Then it follows from (3.2), (3.24) and (3.27) that the average value of $|\phi|^2$ is given as

$$N=N'+\Psi^2/2. \tag{3.28}$$

Finally we use the fact, that $|\psi'|^2$ can be replaced by N' in the large-m limit. After these steps we arrive at an equation of motion for components $j\geqq 2$ the Fourier transform of which is of the same form as (3.7) with α_k replaced by

$$\alpha'_k=\Gamma(ak^2+r(N)-i\omega_{lc}/\Gamma), \tag{3.29}$$

where N is defined by (3.28).
Therefore it follows from (3.11) that the stationary distribution of ψ'_j is fully determined by $r^{(1)}(N)$, and that the equal-time correlation function of $\psi'_{j,k}$ is given by (3.12). As a consequence of it we obtain in the vicinity of the bifurcation point that

$$N'=N_c-[r^{(1)}(N)/A]^{d/2-1} \tag{3.30}$$

with A defined by (3.18).
Furthermore from the equation of motion of ψ'_1 the following condition is found for a stationary solution

$$h/\Psi=r(N'+\Psi^2/2)-i\omega_{lc}/\Gamma. \tag{3.31}$$

It can be considered as a complex "equation of state". Its real part determines the order parameter, Ψ, while the imaginary part yields the frequency of the limit cycle. From the real part of (3.31):

$$r^{(1)}(N'+\Psi^2/2)=h^{(1)}/\Psi. \tag{3.32}$$

Note the similarity between (3.32) and the expression of the transverse susceptibility of ordinary critical phenomena $\chi_T^{-1}=H/M$ (M: magnetization, H: external magnetic field).
Since h has been formally introduced, the relevant solution of (3.31) corresponds to $h=0$. Then from (3.30) and (3.31)

$$r^{(1)}(N_c+\Psi^2/2)=0. \tag{3.33}$$

Let us define x by the requirement

$$r^{(1)}(x)=0 \tag{3.34}$$

and suppose x to be unique. The order parameter can be expressed as

$$\Psi=\pm(2(x-N_c))^{1/2}. \tag{3.35}$$

In order to find an approximate expression for x near the bifurcation point, we expand the function $r^{(1)}(y)$ around $y=N_c$ and take it at $y=x$. Assuming $x-N_c$ to be small we get

$$x=N_c-r^{(1)}(N_c)/\dot{r}^{(1)}(N_c), \tag{3.36}$$

where $\dot{r}$ has been defined in (3.21). Since $r^{(1)}(N_c)=\lambda_c-\lambda$, which follows from (3.3), (3.4) and (3.15) the expression (3.35) yields

$$\Psi\propto(\lambda-\lambda_c)^{1/2}$$

for $\lambda>\lambda_c$ and $\Psi=0$ for $\lambda<\lambda_c$.

From the imaginary part of (3.31) we obtain for $h=0$

$$\omega_{lc} = \Gamma r^{(2)}(x). \tag{3.37}$$

Using (3.36) $\Gamma r^{(2)}(x)$ can be expressed near the bifurcation point as

$$\begin{aligned}\Gamma r^{(2)}(x) &= \Gamma r^{(2)}(N_c) - \Gamma \kappa r^{(1)}(N_c)\\ &= \Gamma r^{(2)}(N_c) + \Gamma \kappa(\lambda - \lambda_c),\end{aligned} \tag{3.38}$$

where

$$\kappa \equiv \dot{r}^{(2)}(N_c)/\dot{r}^{(1)}(N_c), \tag{3.39}$$

i.e. the frequency of the limit cycle is a linear function of the control parameter.

It is easy to calculate by means of (3.22), (3.23) and (3.29) the correlation and response functions of $\psi'_{j,k}(j \geqq 2)$ and to deduce from them the correlation and response functions of the original field variables $\phi_{j,k}(j \geqq 2)$ in the asymptotic state of the system. We give here as an example the correlation function:

$$C(k, t) = \exp(i\omega_{lc} t)\, 2k^{-2} \exp(-\Gamma \bar{a} k^2 t), \qquad t > 0, \tag{3.40}$$

where ω_{lc} is determined by (3.37).

Scaling Functions

First we note that the definition of x by (3.34) can be extended also in the pre-bifurcational region and x is expressed in terms of the parameters of the model in the same way here as for $\lambda > \lambda_c$. Consequently, close to the bifurcation point relation (3.36) remains valid also for $\lambda < \lambda_c$ ($r^{(1)}(N_c)$ is positive in this region).

Thus we can define a frequency

$$\omega_0(\lambda) = \Gamma r^{(2)}(x), \tag{3.41}$$

which in the post-bifurcational region coincides with the frequency of the limit cycle (see (3.37)). Note that (3.38) is valid both above and below λ_c and shows that (3.41) is analytic at λ_c.

Expanding $r^{(1)}(N)$, $r^{(2)}(N)$, $r^{(2)}(x)$ around N_c and using (3.36) one finds in the critical region

$$r^{(2)}(N) - r^{(2)}(x) = \kappa r^{(1)}(N), \tag{3.42}$$

with κ given by (3.39). Substituting it into (3.8) and (3.22) we obtain a scaling form (2.2) with $\omega_0(\lambda)$ as defined in (3.41). Moreover using (3.19) the scaling function can be cast into the following form

$$\begin{aligned}\tilde{C}(k\xi, k^2 t) &= 2(1 + A(k\xi)^{-2})^{-1}\\ &\cdot \exp[-\Gamma(\bar{a} + (1 - i\kappa) A(k\xi)^{-2}) k^2 t], \qquad \lambda < \lambda_c.\end{aligned} \tag{3.43}$$

As for the critical exponents $\eta = 0$, $z = 2$. Comparing (3.37), (3.40), (3.41), (3.43) one can see that in the large-m limit the scaling hypothesis (2.2) can be realized in its most stringent form ("*case D*"). A similar statement is valid for the response function (with $\rho = 0$). Though by this reason (3.41) is the most attractive choice for $\omega_0(\lambda)$ it is worth noting that this is not the only possibility. As mentioned at the end of the previous subsection $\omega_0(\lambda) = \Gamma r^{(2)}(N)$ can be taken for $\lambda < \lambda_c$. With this choice one can at best achieve a scaling form corresponding to "*case B*" or "*case C*" depending on what is taken as $\omega_0(\lambda)$ for $\lambda > \lambda_c$.

IV. RG Procedure

The RG transformation is defined by eliminating the field variables $\phi_{jk}(t)$ with large wave numbers, i.e. with k values between Λ/b and Λ and by an appropriate rescaling [27, 5, 6] ($b > 1$: parameter of the RG, Λ: cut-off). For a more complete definition see the Appendix.

After performing the gauge transformation $\phi_j \to \phi_j \exp(-i\omega t)$ in (3.1) we arrive at a similar equation

$$\dot{\phi}_j(x, t) = -\Gamma(-a\nabla^2 + s(|\phi|^2))\, \phi_j + \zeta_j(x, t), \tag{4.1}$$

where

$$s(|\phi|^2) = r(|\phi|^2) - i\omega/\Gamma. \tag{4.2}$$

Applying the RG transformation to such an equation of motion a great number of new parameters are generated, because the vertices become random variables (see [15]). It turns out, however, that the parameters Γ, a_2 and $\mu = (u_2 - i\omega/\Gamma, u_4, \ldots)$ specified by the form of the starting equation of motion (4.1), transform among themselves in the large-m limit. In order to illustrate the general ideas introduced in Sect. II it will be sufficient to consider these parameters only, since the other ones are expected to be irrelevant in the RG sense.

We relegate the details of the calculation to the Appendix and give here only the resulting recursion relations

$$s'(|\phi|^2) = b^2 s(b^{2-d} Q + N_c), \tag{4.3}$$

$$a' = b^{-\eta} a, \tag{4.4}$$

$$\Gamma' = b^{-2+\eta+z}\, \Gamma, \tag{4.5}$$

where

$$\begin{aligned}Q &= |\phi|^2 - N_c + m \int_{\Lambda}^{\Lambda b} [(q^2 + s^{(1)}(|\phi|^2))^{-1} - q^{-2}]\\ &\cdot d^d q/(2\pi)^d.\end{aligned} \tag{4.6}$$

From (4.4) and (4.5) $\eta = 0$, $z = 2$ follows.

One can see that $\mu^{(1)}=(u_2^{(1)}, u_4^{(1)}, \ldots)$ forms an additional subset the elements of which transform among themselves. Since the stationary distribution is specified by $r^{(1)}$ (see (3.11)) we call these parameters *steady state parameters*. In addition the recursion relation of $s^{(1)}$ coincides with that of the spherical model studied extensively in the literature [9–11]. Thus the steady state scaling fields can be determinated by taking over the method applied there. Let us consider the inverse functions of $s^{(1)}(|\phi|^2)$, and $s'^{(1)}(|\phi|^2)$:

$$|\phi|^2=f(s^{(1)})=f'(s'^{(1)}), \tag{4.7}$$

where f' denotes the transformed quantity. From (4.3)

$$b^{2-d}Q+N_c=f(s'^{(1)}/b^2) \tag{4.8}$$

is obtained.

It follows from a result obtained by *Ma* for a recursion like the real part of (4.3) that the non-linear scaling fields g_α associated to the non-trivial fixed point are generated by the series [11, 12]

$$|\phi|^2=-\sum_{\alpha=1}^{\infty}\alpha(g_\alpha+a_\alpha^*)(s^{(1)})^{\alpha-1}, \tag{4.9}$$

where $a_\alpha^*=mK_d(-1)^\alpha \quad \Lambda^{d-2\alpha}/[\alpha(d-2\alpha)]$. The exponent of g_α is

$$y_\alpha=d-2\alpha, \qquad \alpha=1,2,\ldots. \tag{4.10}$$

Let us turn to the imaginary part of s. Expanding the right hand side of (4.3) in a Taylor series around N_c and considering the ratio of the real and imaginary parts in the limit $b\to\infty$, we find at the fixed point

$$s^{*(2)}(|\phi|^2)=\kappa s^{*(1)}(|\phi|^2), \tag{4.11}$$

where κ has been introduced in (3.39) and $s^{*(1)}(|\phi|^2)$ is determined by the equation

$$|\phi|^2=N_c-m\int_{\Lambda}^{\infty}[(q^2+s^{*(1)}(|\phi|^2))^{-1}-q^{-2}]\,d^dq(2\pi)^{-d}. \tag{4.12}$$

We have used the fact that a finite fixed point can be achieved only if $s(N_c)=0$ at $\lambda=\lambda_c$.

Since κ contains the original parameters, Eq. (4.11) exhibits a non-universal behaviour in the model.

In order to find the nonlinear scaling fields generated by $s^{(2)}$ we substitute (4.7) and (4.8) into (4.3):

$$s'^{(2)}(f'(s'^{(1)}))=b^2 s^{(2)}(f(s'^{(1)}/b^2)). \tag{4.13}$$

This relation indicates that the Taylor coefficients c_α defined by

$$s^{(2)}(f(z))=\sum_{\beta=0}^{\infty}c_\beta z^\beta$$

are scaling fields. Using (4.7) and (4.9) we obtain the equation determining them

$$s^{(2)}\left(-\sum_{\alpha=1}^{\infty}\alpha(g_\alpha+a_\alpha^*)(s^{(1)})^{\alpha-1}\right)=\sum_{\beta=0}^{\infty}c_\beta(s^{(1)})^\beta. \tag{4.14}$$

The corresponding exponents are

$$y_\beta=2-2\beta, \qquad \beta=0,1,2,\ldots. \tag{4.15}$$

It is seen that there are two relevant scaling fields c_0 and g_1 with exponents 2 and $d-2$, respectively. A scaling field like g_1 appears also in the spherical model (g_1 will be related to $\lambda_c-\lambda$), thus the extra scaling field mentioned in Sect. II is c_0. In addition, besides a and Γ, a new marginal scaling field, c_1, is present in the large-m case.

Finally we give explicit expressions for the most important scaling fields. Let x denote the special value of $|\phi|^2$ where $r^{(1)}(x)=s^{(1)}(x)=0$ at a given λ (see (3.34)). It immediately follows from (4.9) that

$$g_1=N_c-x, \tag{4.16}$$

$$g_2=\frac{1}{2}\left(\frac{mK_d\Lambda^{d-4}}{4-d}-\frac{1}{\dot{r}^{(1)}(x)}\right). \tag{4.17}$$

Using (3.36), in the vicinity of λ_c (4.16) takes the form

$$g_1=(\lambda_c-\lambda)/\dot{r}^{(1)}(N_c).$$

Substituting g_1, g_2 into (4.14) and using (4.2) one obtains

$$c_0=r^{(2)}(x)-\omega/\Gamma, \tag{4.18}$$

$$c_1=\dot{r}^{(2)}(x)/\dot{r}^{(1)}(x). \tag{4.19}$$

At $\lambda=\lambda_c$ g_1 must vanish thus from (4.16), (4.19) and (3.39) it follows that at the bifurcation point $c_1=\kappa$. The presence of this marginal scaling field explains why the fixed point (4.11) is non-universal. Note also the non-universal form of the scaling functions: they depend not only on a_2 but also on κ (see for example (3.43)).

The results obtained for a general $r(|\phi|^2)$ can be cast into explicit forms if we start with

$$r(|\phi|^2)=u_2+2u_4|\phi|^2, \tag{4.20}$$

where $u_4^{(1)}$ is assumed to be positive in order to ensure the stability of the asymptotic state. Namely the scaling fields g_α are as follows: $g_1=(\lambda_c-\lambda)/(2u_4^{(1)})$, $g_2=mK_d\Lambda^{d-4}/(2(4-d))-(4u_4^{(1)})^{-1}$, $g_\alpha=-a_\alpha^*$ for $\alpha>2$. Only two of the scaling fields c_β will be non-vanishing: $c_1=u_4^{(2)}/u_4^{(1)}$ and $c_0=u_2^{(2)}-u_2^{(1)}c_1-\omega/\Gamma$.

In accordance with the general scheme introduced in Sect. II the requirement $c_0=0$ fixes the value of the parameter ω of the gauge transformation. Denoting it by $\Omega(\lambda)$ we obtain from (4.18) that

$$\Omega(\lambda)=\Gamma r^{(2)}(x). \tag{4.21}$$

In the particular case when $r(|\phi|^2)$ is given by (4.20) $\Omega(\lambda)$ reads

$$\Omega(\lambda)=\Gamma[u_2^{(2)}-u_2^{(1)}(\lambda)\,u_4^{(2)}/u_4^{(1)}]. \tag{4.22}$$

After eliminating c_0 only one relevant scaling field g_1 is left which is related to the correlation length characterizing the asymptotic state of the system. Furthermore the RG is well-behaved near the finite fixed point and hence the scaling forms (2.2) and (2.5) follow.

The expression (4.21) of $\Omega(\lambda)$ is exactly the same as that of ω_0 defined in Sect. III (see (3.41)) so the properties found there also apply for $\Omega(\lambda)$. One expects on general grounds that $\Omega(\lambda)$ is analytic at λ_c, which in our case is explicitely shown by the expression (3.38) valid near the bifurcation point. Thus the RG analysis of the large-m model demonstrates that the RG procedure introduced in Sect. II can in general lead to "*case C*" of the scaling hypothesis. It is a specific feature of the large-m system, however, that for $\lambda>\lambda_c$ $\Omega(\lambda)$ coincides with the frequency of the limit cycle, ω_{lc} (see (3.37)) and consequently even "*case D*" is actually realized.

Appendix. The RG in the Large-*m* Limit

In order to describe the dynamic renormalization group it is convenient to use the response field formalism [28–32] and then the transformation is to be carried out on the path probability functional $W=\exp\mathscr{I}$. For the action associated to Eq. (4.1) we obtain in the large-m limit:

$$\mathscr{I}=\mathscr{I}_0+\int dt\int d^d x \cdot\left[\sum_{j=1}^{m}(1/2)(-\Gamma\tilde\phi_j\phi_j+\Gamma K)\,s(|\phi|^2)+\text{c.c.}\right], \tag{A.1}$$

where c.c. denotes complex conjugation, $\tilde\phi_j(x,t)$ represents the m-component complex response field, furthermore

$$\mathscr{I}_0=\int dt\int d^d x \cdot\left[\sum_{j=1}^{m}\{\Gamma|\tilde\phi_j|^2-(1/2)(\tilde\phi_j(\dot\phi_j-a\Gamma\nabla^2\phi_j)+\text{c.c.})\}\right] \tag{A.2}$$

and

$$K=K_d\int_0^\Lambda k^{d-1}\,dk. \tag{A.3}$$

When calculating averages by means of the path probability $W\{\tilde\phi,\phi\}$, integration is to be performed over $\phi_j^{(1)}$, $\phi_j^{(2)}$ and $i\tilde\phi_j^{(1)}$ and $i\tilde\phi_j^{(2)}$.

Note that the dependence on $\tilde\phi_j$ in $\mathscr{I}-\mathscr{I}_0$ appears only through the combination

$$\varphi(x,t)\equiv(\Gamma/2)\sum_{j=1}^{m}(-\tilde\phi_j\phi_j+K). \tag{A.4}$$

The RG transformation is defined by integrating the path probability over field variables with wave numbers in the shell $\Lambda/b<k<\Lambda$ and by a rescaling of the remaining variables. The new action is determined by the equation:

$$\exp\mathscr{I}'=\int\prod_{j,\,\Lambda/b<k<\Lambda,\,\omega} d\phi_{j,k,\omega}^{(1)}\,d\phi_{j,k,\omega}^{(2)}\,d(i\tilde\phi_{j,k,\omega}^{(1)})\,d(i\tilde\phi_{j,k,\omega}^{(2)}) \exp\mathscr{I}\Big|_{\substack{\tilde\phi(x,t)\to b^{1-\eta/2-d/2}\tilde\phi(x/b,\,t/b^z)\\ \phi(x,t)\to b^{-1+\eta/2-d/2}\phi(x/b,\,t/b^z)}}. \tag{A.5}$$

Here the quantities with subscripts k, ω stand for the Fourier components of the field variables.

Before turning to the calculation let us discuss first the structure of the parameter space. If we start with (A.1), after the RG transformation an infinite number of new couplings arise in the new action, which are non-local in space and time. We shall see below, however, that a sufficiently broad parameter space is kept in the large-m limit if the following action is considered

$$\mathscr{I}=\mathscr{I}_0+\int dt\int d^d x\,Y(|\phi|^2,\varphi,\bar\varphi), \tag{A.6}$$

where Y denotes a real valued function, φ is defined by (A.4) and φ and $\bar\varphi$ are considered as independent variables. Causality [31, 16] requires that

$$Y(|\phi|^2,0,0)=\text{constant}, \tag{A.7}$$

where the constant will be chosen to be zero. The derivatives of Y, namely

$$Y_{1,0}\equiv\partial Y/\partial|\phi|^2,\qquad Y_{0,1}\equiv\partial Y/\partial\varphi,$$
$$Y_{1,1}\equiv\partial^2 Y/(\partial|\phi|^2\,\partial\varphi) \tag{A.8}$$

will play an important role in what follows.

The form of (A.6) remains unaltered after the RG transformation, indicating that the parameters specified by Y transform among themselves, i.e. they form an invariant subspace of the full parameter space. Using the expression (A.6) means that we treat only that part of the action which contains coupling local in space and time.

The parameters specified by Y can be further divided into different groups. A comparison between the results of the RG applied directly on the equation of motion and those of the present formulation leads to a similar conclusion as in [16]. Parameters specified by $Y_{0,1}(|\phi|^2, 0, 0)$ give the averages of the random vertices arising in the equation of motion, while the complementer set of parameters are related to the second or higher order cumulants of the random vertices. (Note that $Y_{1,0}(|\phi|^2, 0, 0) \equiv 0$ due to (A.7)). It will be demonstrated that the group of parameters specified by $Y_{0,1}(|\phi|^2, 0, 0)$ is itself also an invariant subset within the parameter space specified by $Y(|\phi|^2, 0, 0)$. An even smaller subgroup of the parameters is defined by the real part of $Y_{0,1}(|\phi|^2, 0, 0)$, i.e. by $Y_{0,1}^{(1)}(|\phi|^2, 0, 0)$. These parameters will be called steady state ones and they transform again among themselves.

Finally it is to be noted that for the action decribed by (A.1)

$$Y_{0,1}(|\phi|^2, 0, 0) = s(|\phi|^2) = r(|\phi|^2) - i\omega/\Gamma$$

and the recursions discussed in Sect. IV are given for this subsection of the parameter space.

In order to perform the multiple integral (A.5) the fields are decomposed into two parts

$$\phi_j \to \phi_j + \hat{\phi}_j, \quad \tilde{\phi}_j \to \tilde{\phi}_j + \hat{\tilde{\phi}}_j, \tag{A.9}$$

where ϕ_j and $\tilde{\phi}_j$ on the right hand side involve only wave numbers smaller than Λ/b, while $\hat{\phi}_j$ and $\hat{\tilde{\phi}}_j$ contain the large wave number components. In the large-m limit cross terms like $\sum \tilde{\phi}_j \hat{\phi}_j$ are negligible as compared to $\sum \hat{\tilde{\phi}}_j \phi_j$. Consequently we can write

$$|\phi|^2 \to |\phi|^2 + |\hat{\phi}|^2, \quad \varphi \to \varphi + \hat{\varphi}. \tag{A.10}$$

Since $|\hat{\phi}|^2(\hat{\varphi})$ is a sum of m terms and m is large the relative deviation of it from $\langle|\hat{\phi}|^2\rangle_b(\langle\hat{\varphi}\rangle_b)$ is small, where $\langle\ldots\rangle_b$ denotes the average over field variables with wave numbers between Λ/b and Λ. Thus $Y(|\phi|^2 + |\hat{\phi}|^2, \varphi + \hat{\varphi}, \bar{\varphi} + \hat{\bar{\varphi}})$ in (A.6) can be replaced by the first few terms of its Taylor series expanded in powes of $\hat{\varphi} - \langle\hat{\varphi}\rangle_b$, $\hat{\bar{\varphi}} - \langle\hat{\bar{\varphi}}\rangle_b$ and $|\hat{\phi}|^2 - \langle|\hat{\phi}|^2\rangle_b$ reducing the multiple integral (A.5) to Gaussian integrations. The calculation is a straightforward generalization of that followed in [16], therefore we shall skip the intermediate steps (the interested reader will find some more details in the Appendix of [16]) and jump directly to the recursions obtained for the quantities $Y_{0,1}$, $Y_{1,0}$ defined by (A.8):

$$Y'_{1,0}(|\phi|^2, \varphi, \bar{\varphi}) = b^4\, Y_{1,0}(b^{2-d} Q + N_c, b^{-d} R, b^{-d} \bar{R}), \tag{A.11}$$

$$Y'_{0,1}(|\phi|^2, \varphi, \bar{\varphi}) = b^2\, Y_{0,1}(b^{2-d} Q + N_c, b^{-d} R, b^{-d} \bar{R}), \tag{A.12}$$

where N_c is given by (3.14) and

$$Q = |\phi|^2 + b^{d-2}(\langle\hat{\phi}^2\rangle_b - N_c) = |\phi|^2 - N_c + m \int_q^{>} (S^{-1} - q^{-2}), \tag{A.13}$$

$$R = \varphi + b^d \langle\hat{\varphi}\rangle_b = \varphi - (m/2) \int_q^{>} \{(q^2 + Y_{0,1}^{\prime(1)}) S^{-1} - 1\}, \tag{A.14}$$

with

$$S = [(q^2 + Y_{0,1}^{\prime(1)})^2 - 2 Y'_{1,0}]^{1/2}, \tag{A.15}$$

and

$$\int_q^{>} \equiv K_d \int_\Lambda^{\Lambda b} dq\, q^{d-1}. \tag{A.16}$$

Since Γ does not transform it have been set equal to unity.

It follows from (A.7) that $Y'_{1,0}(|\phi|^2, 0, 0) \equiv 0$, consequently (see (A.14), (A.15)) $R = 0$ if $\varphi = 0$, thus the function $Y_{0,1}(|\phi|^2, 0, 0)$ specifies an invariant subset of the parameter space as stated above.

Expanding the right hand sides of (A.11) and (A.12) in a Taylor series around $(N_c, 0, 0)$ we find as conditions for the existence of a finite fixed point as follows.

$$Y_{0,1}^{(1)}(N_c, 0, 0) = 0, \quad Y_{0,1}^{(2)}(N_c, 0, 0) = 0$$

should be fulfilled at the bifurcation point. These requirements specify the values of two parameters in Y at the bifurcation point, namely that of the control parameter and that of the parameter of the gauge transformation. The latter one fixes the value of the frequency of the limit cycle at λ_c. Then we find the requirements $Q \to 0$ and $(Y_{1,1}(N_c, 0, 0) R + \text{c.c.}) \to 0$ in the limit $b \to \infty$, which yield for the fixed point expression of functions $Y_{0,1}^{(1)}$, and $Y_{1,0}$ the equations as follows:

$$|\phi|^2 = N_c - m \int_\Lambda^\infty ((S^*)^{-1} - q^{-2})\, d^d q (2\pi)^{-d}, \tag{A.17}$$

$$(1 + i\kappa)\varphi + (1 - i\kappa)\bar{\varphi} = m \int_\Lambda^\infty \{(q^2 + Y_{0,1}^{*(1)})(S^*)^{-1} - 1\}\, d^d q (2\pi)^{-d}, \tag{A.18}$$

where

$$S^* = [(q^2 + Y_{0,1}^{*(1)})^2 - 2 Y^*_{1,0}]^{1/2}, \tag{A.19}$$

and

$$\kappa \equiv Y_{1,1}^{(2)}(N_c, 0, 0) / Y_{1,1}^{(1)}(N_c, 0, 0). \tag{A.20}$$

Comparing the real and imaginary parts of (A.12) for $b \to \infty$ one obtains

$$Y^{*(2)}_{0,1} = \kappa Y^{*(1)}_{0,1}. \tag{A.21}$$

Equations (A.17)–(A.21) determine the fixed function Y^*. It is, however, not a universal expression since κ appears in it. An other interesting property of Y^* is that besides $|\phi|^2$ it depends only on the combination $(1+i\kappa)\varphi + (1-i\kappa)\bar{\varphi}$.
Finally we note that when $\varphi = \bar{\varphi} = 0$ from (A.18), (A.19) $Y_{1,0} = 0$ follows and (A.17) and (A.21) go over to equations (4.12) and (4.11), respectively.

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