Z. Phys. B - Condensed Matter 49, 157–160 (1982)

Condensed Matter
Zeitschrift für Physik B

On the Construction of Stable and Unstable Manifolds of Two-Dimensional Invertible Maps

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Received August 25, 1982

An equation is proposed for describing stable and unstable manifolds for a wide class of two-dimensional invertible maps. Several branches of the stable and unstable manifolds of the dissipative map $x_{n+1}=1-a|x_n|+bz_n$, $z_{n+1}=x_n$ are constructed explicitly. The limiting case when the strange attractor disappears is discussed.

The concepts of strange attractors, stable and unstable manifolds (separatrices) are essential in understanding chaotic behaviour [1]. Most of our present day knowledge of these objects comes, however, from computer simulations. Methods and models making analytic calculations possible are, therefore, extremely useful. Here, we shall propose an equation for the stable and unstable manifolds of two-dimensional invertible maps and give a solution by means of a method which, by specifying a piece of the manifold, generates further and further parts of it. As an illustrative example, a simple model is chosen wherein analytic calculations can be performed.

We consider the class of two-dimensional maps described by the following recursion relations

$$\begin{aligned} x_{n+1}&=f(x_n)+bz_n,\\ z_{n+1}&=x_n, \end{aligned} \tag{1}$$

where f stands for a single-humped symmetric function, and $|b|$, the modulus of the Jacobian, is assumed to be smaller than unity.

The particular choice of $f=1-ax^2$ corresponds to the well-known Hénon map [2], the stable and unstable manifolds of which have been calculated with the aid of numerical methods [3–5]. If we take, on the other hand, $f=1-a|x|$, (1) becomes equivalent to a map introduced by Lozi. His computer simulation [6] indicates that the strange attractor in this case is simpler than that found in the Hénon model since it seems to be the product of pieces of straight lines by a Cantor set. The mathematical properties of this map have been investigated by Misiurewicz [7], who proved that for a certain range of parameters, including the point $a=1.7$, $b=0.5$ considered in [6], the map has indeed a strange attractor. As we shall see, the fact that f is a piecewise linear function makes analytic calculations possible for the separatrices of this model.

To begin with, we deduce an equation for the stable and unstable (not necessarily differentiable) manifolds of map (1). Let us assume that the manifold in question is described by an equation $x=f^*(z)$ on the x, z plane, where f^* is a continuous function. It can be, of course, multi-valued at least in certain ranges of z. Applying (1) to the points of the manifold, we obtain new points with coordinates $x'=f(x)+bf^{*-1}(x)$ and $z'=x$, where f^{*-1} denotes the inverse of f^*. The appearance of the inverse of a multi-valued function might create some difficulties but, as it will be seen later, they can be overcome. Note, that f^{*-1} itself is, in general, a multi-valued function. Since stable and unstable manifolds are invariant under (1), $x'=f^*(z')$ holds and we find a sort of fixed point equations for f^*, namely

$$f^*(z)=f(z)+bf^{*-1}(z). \tag{2}$$

Equation (2) with $f=1-ax^2$, motivated by an iterative procedure, has been considered by Bridges and Rowlands [8]. They used it to give an approximate expression for the form of the Hénon attractor. A strange attractor is, however, related to one of the unstable manifolds only, being the closure of it [3]. We emphasize here that (2) should have several solutions corresponding to different separatrices, and

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furthermore, its validity is not restricted to the chaotic regime.

It can be useful to see another version of Eq. (2) as well. Let us consider the inverted map of (1), which after a change of variables $x \to -z$, $z \to -x$ can be written in the same form as (1) since f is symmetric. It reads

$$x_{n+1} = \tilde{f}(x_n) + \tilde{b} z_n, \qquad z_{n+1} = x_n, \tag{3}$$

where

$$\tilde{f} = \frac{1}{b} f, \quad \tilde{b} = \frac{1}{b}. \tag{4}$$

We can now formally repeat the argument above to find an equation

$$\tilde{f}^*(z) = \tilde{f}(z) + \tilde{b} \tilde{f}^{*-1}(z). \tag{5}$$

Since the separatrices of the inverted map and those of the original one are the same, just stable separatrices become unstable ones and vice versa, the solutions described by $\tilde{f}^*$ correspond to those of (2). In an actual calculation one can choose the more convenient equation to work with.

In order to illustrate a possible method for solving (2) or (5) we turn to a particular map where an analytic calculation can be carried out. We take

$$f(x) = c(1 - a|x|), \tag{6}$$

i.e. we consider essentially the same model as in [6] and [7] but introduce a factor c for later purposes.

Since f^* (*or* $\tilde{f}^*$) is a multi-valued function one cannot hope to find the complete solution at once. A single branch of it, however, can be specified, which then generates further branches. We start our calculation with a branch going through one of the fixed points. Assuming $ca > 1 - b$, the map (1) with (6) possesses two fixed points given by

$$H_+ : (x_+^* = z_+^* = c/(1 - b + ca)), \qquad H_- : (x_-^* = z_-^* = c/(1 - b - ca)). \tag{7}$$

Function f is piecewise linear in our case, therefore we look for an equation of a single branch of f^* in the following linear form

$$x = x_\pm^* + \lambda_\pm (z - x_\pm^*). \tag{8}$$

Substituting (8) into (2) we find four solutions with

$$\lambda_\pm^u = \mp[ca + \sqrt{(ca)^2 + 4b}]/2, \tag{9}$$

$$\lambda_\pm^s = \mp[ca - \sqrt{(ca)^2 + 4b}]/2. \tag{10}$$

The directions of the straight lines going through a fixed point are those of the eigenvectors of the linearized transformation, as it is expected. Moreover, since an eigenvector of (1) associated with an eigenvalue λ can be written as $(\lambda, 1)$ the quantities found in (9), (10) are just the eigenvalues belonging to the fixed points H_+ and H_-. The superscripts s and u refer to stable and unstable solutions, respectively. Indeed $|\lambda_\pm^u| > 1$, $|\lambda_\pm^s| < 1$ in the region $ac > 1 - b$. (Equation (5) provides, of course, the same straight lines as solutions.)

Let us investigate first the branches of the unstable manifolds (Fig. 1a). The coordinates of their intersection points with the x-axis are given by

$$c_\pm^* = (1 - \lambda_\pm^u) x_\pm^*. \tag{11}$$

Considering now the inverse of (8) with λ_+^u and λ_-^u, respectively (see Fig. 1b) and comparing them with (2), it becomes clear that the line going through H_- can be a solution for $z \leqq 0$ only, while the other one for $0 \leqq z \leqq c_+^*$ only. At the same time one observes that both solutions can be extented along the interval where the corresponding inverse is defined. The extension is a piece of a straight line again (Fig. 1c), the parameters of which are specified uniquely by (2), (6), (8) and (9). After this step both

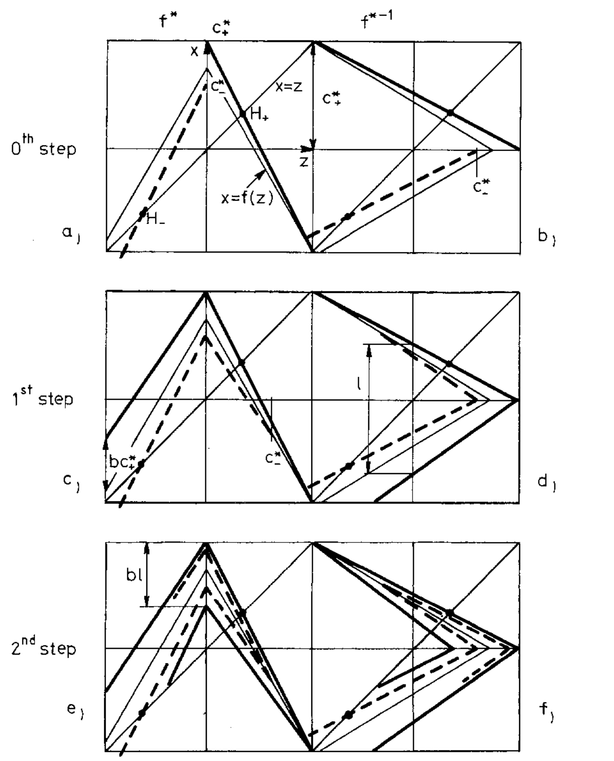

Fig. 1. The first steps in constructing solutions of equation (2) with f as given by (6). Branches of the unstable manifolds associated with H_+ and H_- are marked by full and dotted fat lines, respectively ($a = 1.7$, $b = 0.5$, $c = 1$)

solutions look like asymmetric roofs. If we consider now the inverse of any of these functions (Fig. 1d) we recognize that a new branch has appeared which might generate new branches in f^* so it should be taken into consideration. The simplest way to do it is to add the difference between the two branches of the inverse function multiplied by b to the previous solution. The new solutions consist of two roofs (Fig. 1e). Their inverse functions, however, contain two new branches not considered so far (Fig. 1f), therefore the whole procedure is to be repeated again and again. After each step one can give the analytic expressions for the lines generated. Due to the fact that $b<1$, the new lines come close to the previous ones, and the neighbouring lines become soon indistinquishable on a plot. This way of construction illustrates the stretching and folding property of the map, and the complex structure of the unstable manifolds.

Figure 2 shows the unstable manifolds W_+^u and W_-^u associated with H_+ and H_-, respectively, as obtained after five steps at parameter values $a=1.7$, $b=0{,}5$, $c=1$. It can be seen that further steps would not change the picture due to the finite thickness of the drawing facility. Figure 2a should be compared with the form of the strange attractor found in a computer simulation at the same values of parameters [6]. (Note that in [6] $y=zb$ has been used rather than z.) If one superposes the two pictures of Fig. 2a and b it becomes clear how complicated the winding of the two curves is, since they do not intersect.

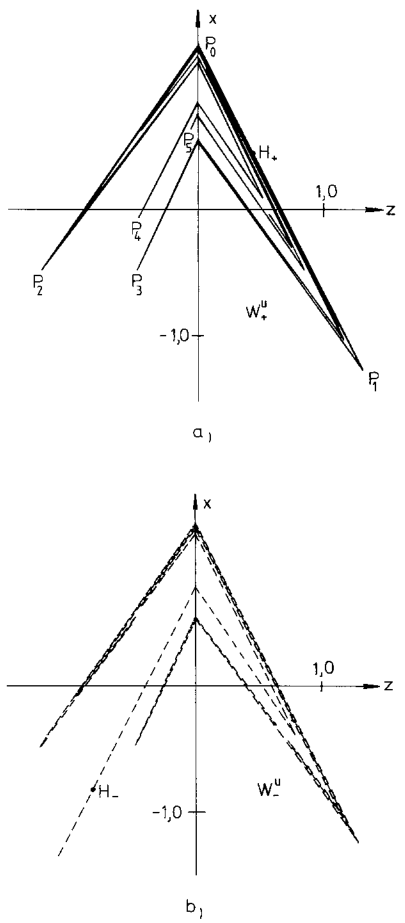

Fig. 2. The unstable manifolds W_+^u (a) and W_-^u (b) as obtained after five steps of construction. The parameters are as on Fig. 1

As for the calculation of the stable manifolds W_+^s, W_-^s, we found it convenient to turn to equation (5). The two lines described by (8) and (10) run, after the transformation $x\to -z$, $z\to -x$, in the system of coordinates in a similar way as those defined by (8) and (9) in the original one. Therefore, it becomes possible to apply the procedure sketched above. Even formulas can be taken over by replacing b and c (see (4), (6)) by $1/b$. (This is why we kept the parameter c free in the calculation.) The construction goes along the same lines as that for $W_\pm^u$. The only modification is the fact that the pieces of straight lines do not remain confined within a finite region, since now $\tilde{b}>1$. Figure 3 shows W_+^s and W_-^s as obtained after five steps. The contour of W_+^u is given as well in order to see the difference in size, and the region where homoclinic points can be found. Note the qualitative similarity among Fig. 2 and 3 and the corresponding pictures in the case of the Hénon map [3, 4].

The method used above makes it possible to follow the modification of the invariant manifolds as the parameters are changed. Let us set $c=1$, fix b and start with a value of a at which the form of the separatrices is similar to those shown on Fig. 2 and 3. Increasing now a, a qualitative change is found, namely the points $P_3, P_4, P_5, \ldots$, (P_n stands for the n-th image of P_0: $(c_+^*, 0)$ (see Fig. 2a)) come close to each other and at same time to H_-. At a critical value a_c the series $\{P_n\}$ converges to the fixed point H_- (numerical studies suggest that just above a_c no finite attractor exists). In other words, at a_c the points P_n lie on the stable manifold W_-^s in accordance with the general belief that the destruction of strange attractors is due to the appearance of transverse heteroclinic points [3]. Knowing the equation of the lines of W_-^s and the coordinates of points P_n, it is easy to formulate the condition for P_n lying on W_-^s. Considering for example, P_1: $(1-ac_+^*, c_+^*)$, it will be a point of line (8) with λ_-^s (see Fig. 3) if

$$(1-\lambda_+^u)(a+\lambda_-^s)x_+^* + (1-\lambda_-^s)x_-^* - 1 = 0. \tag{12}$$

Using (7), (9), (10) with $c=1$, one finds after some algebra a surprisingly simple result

$$a_c = 2 - b/2. \tag{13}$$

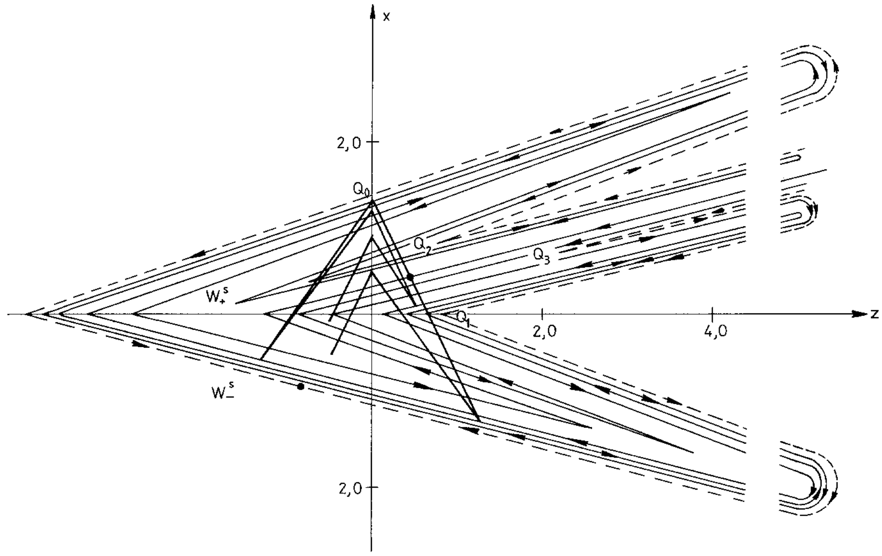

Fig. 3. The stable manifolds W_+^s and W_-^s as obtained after five steps of construction. The contour of W_+^u is given, too. The parameters are as on Fig. 1

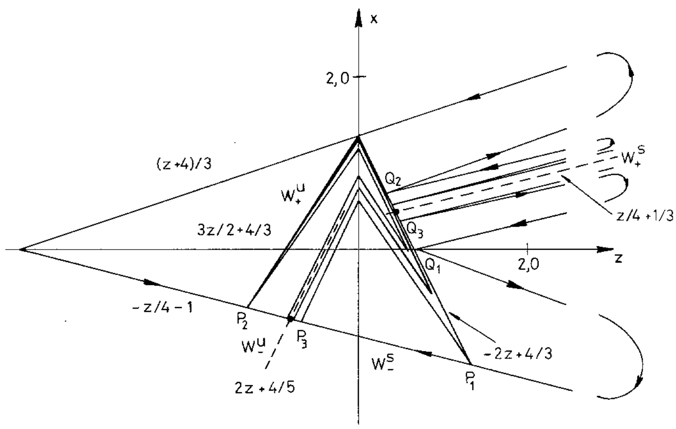

Fig. 4. The most important lines of the stable manifolds W_-^s and those of the unstable one W_+^u at the critical situation $a_c = 1.75$, $b = 0.5$, $c = 1$. Equations for some of these lines are also indicated

The same condition has been conjectured in [7]. A geometric interpretation can be associated with (13) by noticing that $|\lambda_\pm^u| = 2$ at a_c, independently of the value of b, as it follows from (9). This means that in the critical case the slope of the lines of the unstable manifolds at H_- and H_+ are plus and minus two, respectively, giving a nice generalization of the criterion valid for the one-dimensional ($b=0$) case [9]. The situation for $a_c = 1.75$, $b = 0.5$ is shown on Fig. 4. Note that as $a \to a_c$ the points Q_n (the n-th image of Q_0 in the inverted map, see Fig. 3) come close to W_+^u, and finally at a_c they reach the unstable manifold of H_+.

Due to the qualitative similarity between the Hénon map and the model studied here one expects a similar situation at the critical value a_c of the Hénon model, e.g. points analogous to P_n ($n = 1, 2 \ldots$) should lie on a smooth curve.

Finally, we mention that the method described here for constructing separatrices works for other ranges of parameter, too, furthermore, it is suspected to be applicable not only for a piecewise linear f in (1) but for more general cases as well, after an appropriate single valued (perhaps numerical) solution of (2) or (5) has been found.

The author is indebted to Prof. P. Szépfalusy for several illuminating discussions on the subject and, for calling his attention to ref. [8]. Helpful conversations with P. Gnädig, G. Györgyi, Z. Rácz and L. Sasvári are also acknowledged. Thanks are due to G. Szabó for helping in some numerical simulations.

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