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Quantization of Hénon's Map with Dissipation

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Hénon's map with dissipation is suspended to the nonlinearly kicked damped harmonic oscillator and then quantized. The ensuing master equation between two subsequent kicks is solved exactly in the representation by the Wigner distribution, resulting in a quantized version of Hénon's dissipative map. The semi-classical limit of the map is studied. The leading quantum corrections are shown to be associated with dissipation and can be formulated as a classical map with classical stochastic perturbations. The next-to-leading quantum corrections, arising from the nonlinearity of the kicks, are similar as in the area conserving map and cannot be described within the framwork of classical statistics. The Wigner distribution in the steady state is investigated in the limit of strong dissipation, where Hénon's map is reduced to the logistic map. The insensitivity of the main results against details of the quantization procedure is demonstrated by comparing with the results of a different phenomenological quantization procedure.

1. Introduction

The analysis of chaos in dissipative dynamical systems has been greatly advanced during the last decade by the study of one- and two-dimensional discrete return maps (cf. e.g. [1, 2]). Among these the 1-dimensional logistic map (cf. [3]) and a 2-dimensional dissipative map first introduced by Hénon [4] have been particularly important, since they have been found to capture many generic features of return maps in realistic physical systems.

More recently, the study of quantum systems whose classical counterparts exhibit chaotic dynamics has also attracted a great deal of interest. Again the study of quantum systems described by discrete-time maps has been found to be fruitful [5–7]. While so far most studies of quantum maps have been concerned with quantum versions of classical 2-dimensional area conserving maps [5, 6] very recently the study of quantized two-dimensional dissipative, i.e. area contracting, maps has also been started [7]. However, the effects of quantization on dissipative maps have so far been considered in detail only for a particularly simple but somewhat artificial example – the two-dimensional Kaplan-Yorke map [8]. It is therefore of great interest to study also quantized versions of more realistic 2-dimensional maps, most notably the dissipative Hénon map and extensions of it. This is our aim in the present paper. The Hénon map is of particular interest not only because it arises under certain conditions as the most general quadratic map with a constant Jacobian [2], but also because it contains the logistic map as an important special case in the limit of strong dissipation.

At the present time, our main motivation for a study of quantized versions of this map is the desire to elucidate the fundamental limitations which are imposed by quantum theory on chaotic behavior on sufficiently small scales. However, we believe that this study is also of a more direct physical relevance because of the natural appearance of the Hénon map or similar maps in some dissipative quantum systems. For instance, single mode lasers cf. [9, 10] are quantum versions [11] of the Lorenz model [12] which in some respect is similar to the Hénon model [13]. Electron storage rings have also been modelled

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by the Hénon map or similar maps [1, 2]. Quantum effects and dissipation in these systems are important and intrinsically related: E.g., in lasers dissipation occurs essentially due to spontaneous emission into the modes of the vacuum, while quantum effects arise primarily from spontaneous emission into the laser mode. In electron storage rings dissipation occurs due to spontaneous emission of synchrotron radiation which gives rise to fluctuations of the electron beam by a statistical back reaction effect. In the latter case these effects are usually described classically [14] but at sufficiently large energies the condition of validity for the classical description [14] seems to be violated, and the fluctuations due to synchrotron radiation must then be treated quantum mechanically. A quantized version of the Hénon map may, therefore, serve as a reasonable phenomenological model which captures the essential features of dissipation and quantum fluctuations in systems with chaotic behavior in the classical limit.

The manuscript is organized as follows. In Sect. 2 the quantization of the Hénon map is prepared by first performing a suspension of the classical map to a continuous two-dimensional but non-autonomous flow. This flow can be physically interpreted as a linear damped harmonic oscillator which is kicked periodically by a force depending on the amplitude of the oscillator [2]. In Sect. 3 the suspended system is quantized, which leads to a master equation governing the statistical operator of the kicked oscillator. In Sect. 4 a quantized form of the original map is obtained by exactly solving the master equation between two subsequent kicks. In Sect. 5 semi-classical limits of the map are studied. In the appendix a different phenomenological quantization of the Hénon map is performed and the dependence of the results on changes in the quantization is exhibited.

2. Kicked Oscillator with Damping and Hénon's Map – Classical Description

We shortly summarize how the time-continuous motion of a kicked damped harmonic oscillator leads to the discrete dynamics of a dissipative Hénon map in the classical description [2]. For later convenience, we represent the equation of motion of the kicked oscillator in the form

$$\begin{aligned} \dot{x} &= -\gamma x + p, \\ \dot{p} &= -\gamma p - \omega^2 x + \sum_{n=-\infty}^{\infty} \delta(t - n\tau)\bar{f}(x), \end{aligned} \tag{2.1}$$

where γ denotes the damping constant and ω is the eigenfrequency. The position dependent amplitude of the kicks which are repeated with period τ is given by an arbitrary function $\bar{f}(x)$. The conservative part of this motion is associated with the time-dependent Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 x^2) - \sum_{n=-\infty}^{\infty} \delta(t - n\tau)\bar{g}(x), \tag{2.2}$$

where $\bar{g}(x)$ stands for the integral of $\bar{f}(x)$: $\bar{g}'(x) = \bar{f}(x)$.

Our aim is now to derive a stroboscopic map relating position and momentum immediately after two subsequent kicks. The equation of motion can be easily solved between kicks, and one obtains the relation

$$\begin{aligned} \tilde{x}_{n+1} &= ECx_n + ESp_n/\omega, \\ \tilde{p}_{n+1} &= -\omega ESx_n + ECp_n \end{aligned} \tag{2.3}$$

with

$$E = \exp(-\gamma\tau), \qquad C = \cos(\omega\tau), \qquad S = \sin(\omega\tau) \tag{2.4}$$

which connects the state variables after the n^{th} kick, denoted by x_n, p_n, with those just before the $(n+1)^{\text{th}}$ one, denoted by $\tilde{x}_{n+1}$, $\tilde{p}_{n+1}$. As there is no discontinuity in the position but one in the momentum the stroboscopic map is given by

$$\begin{aligned} x_{n+1} &= \tilde{x}_{n+1}, \\ p_{n+1} &= \tilde{p}_{n+1} + \bar{f}(x_{n+1}). \end{aligned} \tag{2.5}$$

By introducing a new variable y through

$$y = \frac{S}{\omega}\left(-p + \frac{C\omega}{s}x + \bar{f}(x)\right) \tag{2.6}$$

the stroboscopic map can be rewritten as

$$\begin{aligned} x_{n+1} &= 2ECx_n + ES\bar{f}(x_n)/\omega - Ey_n, \\ y_{n+1} &= Ex_n. \end{aligned} \tag{2.7}$$

This is a map of the form

$$\begin{aligned} x_{n+1} &= x^0_{n+1}(x_n, y_n) \equiv f(x_n) - Ey_n, \\ y_{n+1} &= y^0_{n+1}(x_n, y_n) \equiv Ex_n, \end{aligned} \tag{2.8}$$

where the relation

$$\bar{f}(x) = \frac{\omega}{ES}(f(x) - 2ECx) \tag{2.9}$$

has been used. Equation (2.8) is of the same general type as Hénon's map. The continuous flow described by (2.1), therefore, induces stroboscopic maps of Hénon's type, and is, thus, a suspension of such maps. The connection between the extra variable y

and the momentum could be scaled with some arbitrary power of E leading to corresponding changes in (2.8), too. The present choice, however, turns out to be most convenient when describing the semi-classical results obtained in the limit of strong dissipation. Note that the Jacobian, E^2, can take only positive values for all maps with the suspension (2.1). A typical choice for f can be

$$f(x)=c(1-a|x|^z) \tag{2.10}$$

with positive parameters a, c and z. For $z=2$ Hénon's map is recovered. It is worth noting that if the kick amplitude $\bar{f}$ is of order unity, in the strongly dissipative limit $\gamma\tau\to 0$, $E\to 0$, the map (2.7) describes a rapid relaxation to a fixed point. However, it is possible to overcome the effect of strong damping and to obtain a nontrivial dynamics also in this case if the amplitude $\bar{f}(x)$ is scaled with E^{-1} as described by (2.9).

For the sake of simplicity we shall henceforth choose τ according to $\omega\tau=2\pi(k+\frac{1}{4})$ with k integer so that

$$C=0, \qquad S=1. \tag{2.11}$$

There is still a complete freedom in the choice of k. Thus, for sufficiently large k very small values of E can be obtained with the restriction (2.11), too. On the other hand, by letting the damping constant γ go to zero E increases up to unity. The value of the Jacobian of the model, therefore, may vary between 0 and 1.

Finally, we note that there are infinitely many continuous systems leading to the same stroboscopic map since the latter contains merely a part of the information of the continuous dynamics. Thus, for example, maps of Hénon's type can also be obtained from the kicked free motion. Consequently, many unequivalent suspensions exist in general. Quantization, defined for continuous classical flows, requires the choice of a particular suspension of the original map. The non-uniqueness of suspensions, therefore, corresponds to a non-uniqueness of quantizations. Here we shall use the suspension Eq. (2.1).

3. Quantization of the Kicked Oscillator with Damping

Since there is damping acting on the kicked oscillator (2.1), we have to apply the quantum theory of dissipative systems. Damping occurs because a system interacts with another very large system, called the heat bath. The complete quantum mechanical description of this coupled system contains much more information than we need. Therefore, the relevant quantities are those of the original system, which remain after the heat bath variables have been eliminated by taking the trace of the total density matrix over the subspace of the heat bath variables. Thus, one obtains a reduced density matrix ρ which gives the complete quantum mechanical description of the subsystem of interest, i.e. the kicked oscillator. The precise form of the density matrix may depend on the intensive parameters of the heat bath, like its temperature, and on the type of the coupling between the subsystem and its surroundings.

It is convenient to use creation and annihilation operators for the oscillator:

$$a=\frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x}+i\hat{p}), \qquad a^+=\frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x}-i\hat{p}), \tag{3.1}$$

where $\hat{x}$ and $\hat{p}$ stand for the position and momentum operators, respectively. The Hamiltonian of the conservative part of motion is the quantized version of (2.2):

$$\hat{H}=\hbar\omega a^+a-\sum_{n=-\infty}^{\infty}\delta(t-n\tau)\,\bar{g}\left(\sqrt{\frac{\hbar}{2\omega}}(a+a^+)\right). \tag{3.2}$$

The interaction of a harmonic oscillator with a heat bath has been extensively studied [9, 15, 16]. If the coupling of the oscillator to the heat reservoir is chosen to be

$$\hat{H}_{\text{int}}=\sum_i (aR_i^+ + a^+R_i) \tag{3.3}$$

where R_i are bath operators, one obtains under the usual assumptions of weak coupling and a Markovian reservoir [15] the master equation for the reduced density matrix ρ in the form*

$$\frac{\partial\rho}{\partial t}=-\frac{i}{\hbar}[\hat{H},\rho]+\gamma\{(1+\bar{n})([a\rho,a^+] + [a,\rho a^+])+\bar{n}([a^+\rho,a]+[a^+,\rho a])\}. \tag{3.4}$$

The quantity $\bar{n}$ denotes here the average thermal quantum number of the free harmonic oscillator at the temperature T of the heat bath:

$$\bar{n}=(\exp(\hbar\omega/k_BT)-1)^{-1}. \tag{3.5}$$

We shall suppose that the frequency of kicks is rather low: $\omega\tau\gg 1$ (i.e., $k\gg 1$). Then, as a first approximation the master equation (3.4) can be kept with $\hat{H}$ as given by the time-dependent operator (3.2).

* The validity of this form of the master equation is restricted to weak damping $\gamma\ll\omega$ and to temperatures $T\gg\hbar_\gamma/k_B$. For recent work on the low temperature regime $T<\hbar_\gamma/k_B$ cf. [17]

Similarly as in the classical description, the evolution of the density matrix between two kicks is given by that of the unperturbed oscillator. The effect of kicks then results in a jump of the density matrix. Around a kick $t \approx n\tau$, the commutator $[\hat{H}, \rho]$ dominates the right hand side of (3.4). In a vicinity of this point the equation can be integrated leading to

$$\rho(t) = U(t,t')\rho(t')U^+(t,t') \tag{3.6}$$

with the unitary operator

$$U(t,t') = \hat{T}\exp\left(-\frac{i}{\hbar}\int_{t'}^{t} d\tau \hat{H}(\tau)\right), \tag{3.7}$$

where $\hat{T}$ denotes time ordering. For $t \to n\tau + 0$, $t' \to n\tau - 0$ only the δ-function contributes, thus

$$U(n\tau+0, n\tau-0) = \exp\left[\frac{i}{\hbar}\bar{g}\left(\sqrt{\frac{\hbar}{2\omega}}(a+a^+)\right)\right]. \tag{3.8}$$

The change of the density matrix owing to the kicks is, therefore,

$$\rho(n\tau+0) = \exp\left[\frac{i}{\hbar}\bar{g}\left(\sqrt{\frac{\hbar}{2\omega}}(a+a^+)\right)\right] \cdot \rho(n\tau-0)\exp\left[-\frac{i}{\hbar}\bar{g}\left(\sqrt{\frac{\hbar}{2\omega}}(a+a^+)\right)\right]. \tag{3.9}$$

It is often convenient to use a representation of the density matrix in terms of a quasi-probability density. Among them the Wigner function turns out to be most useful. The Wigner function $W(x,p,t)$ is defined as [18, 9, 15]

$$W(x,p,t) = \int \frac{dq}{2\pi\hbar}\exp\left(-i\frac{pq}{\hbar}\right)\left\langle x+\frac{q}{2}\middle|\rho(t)\middle|x-\frac{q}{2}\right\rangle. \tag{3.10}$$

By means of the general correspondence between the density matrix and creation and annihilation operators on one side, and the Wigner function and complex numbers on the other side, one finds a c-number equation describing the time evolution of the Wigner function [19] which for $W(x,p,t)$ can be written as:

$$\frac{\partial}{\partial t}W(x,p,t) = -\frac{\partial}{\partial x}(-\gamma x+p)W - \frac{\partial}{\partial p}(-\gamma p - \omega^2 x)W + \hbar\frac{\gamma}{\omega}\left(\bar{n}+\frac{1}{2}\right)\left(\frac{\partial^2}{\partial x^2}+\omega^2\frac{\partial^2}{\partial p^2}\right)W. \tag{3.11}$$

Note that this equation is of the form of a Fokker-Planck equation with the classical equations of motion as drift terms and with a diagonal diffusion matrix of intensity $2\hbar\gamma(\bar{n}+1/2)/\omega$. Different functional forms of the coupling to the reservoir may influence the type of the diffusion matrix similarly to the classical case. This is yet another source of non-uniqueness in the quantization of maps, which is special to dissipative maps (cf. Appendix).
Since the amplitude of the kick depends only on the position, it is particularly easy to describe the effect of a kick in coordinate representation. From (3.9) and (3.10) one finds

$$W(x,p,n\tau+0) = \int \frac{dq}{2\pi\hbar}\exp\left[-i\frac{pq}{\hbar}+\frac{i}{\hbar}\left(\bar{g}\left(x+\frac{q}{2}\right) - \bar{g}\left(x-\frac{q}{2}\right)\right)\right]\left\langle x+\frac{q}{2}\middle|\rho(n\tau-0)\middle|x-\frac{q}{2}\right\rangle. \tag{3.12}$$

In order to obtain an explicit solution it is useful to treat the generating function

$$\phi(\xi,\eta,t) = \int dx\,dp\exp[i(\xi x+\eta p)]W(x,p,t). \tag{3.13}$$

The dynamical equation for ϕ is a first order differential equation:

$$\frac{\partial}{\partial t}\phi + (\gamma\xi+\omega^2\eta)\frac{\partial}{\partial\xi}\phi + (\gamma\eta-\xi)\frac{\partial}{\partial\eta}\phi + \hbar\gamma\left(\bar{n}+\frac{1}{2}\right)\left(\frac{\xi^2}{\omega}+\omega\eta^2\right)\phi = 0 \tag{3.14}$$

which can be solved by the method of characteristics. Thus, between kicks, one obtains

$$\phi(\xi,\eta,t) = \exp\left[\frac{\hbar}{2\omega}\left(\bar{n}+\frac{1}{2}\right)(\xi^2+\omega^2\eta^2)(e^{-2\gamma t}-1)\right] \cdot \phi_0(e^{-\gamma t}(\xi\cos\omega t - \omega\eta\sin\omega t), \omega^{-1}e^{-\gamma t}(\xi\sin\omega t+\omega\eta\cos\omega t)), \tag{3.15}$$

where $\phi_0(u,v)$ denotes the initial distribution. Using (3.12), (3.13) the effect of kicks can be expressed as

$$\Phi(\xi,\eta,n\tau+0) = \int dx\exp\left[i\xi x+\frac{i}{\hbar}\left(\bar{g}\left(x+\frac{\eta\hbar}{2}\right) - \bar{g}\left(x-\frac{\eta\hbar}{2}\right)\right)\right]\left\langle x+\frac{\eta\hbar}{2}\middle|\rho(n\tau-0)\middle|x-\frac{\eta\hbar}{2}\right\rangle. \tag{3.16}$$

From the definition of the characteristic function and (3.10) it follows that

$$\phi(\xi,\eta,n\tau+0) = \int \frac{d\xi' dx}{2\pi}\exp\left[i(\xi-\xi')x + \frac{i}{\hbar}\left(\bar{g}\left(x+\frac{\eta\hbar}{2}\right) - \bar{g}\left(x-\frac{\eta\hbar}{2}\right)\right)\right]\phi(\xi',\eta,n\tau-0). \tag{3.17}$$

This shows that a kick induces a nonlocal transformation of the characteristic function.

By means of (3.15) and (3.17) the characteristic function, and, consequently, also the Wigner function, can be constructed at any time, i.e. the quantum mechanical description of the kicked oscillator is complete.

4. The Quantum Map

Like in the classical case, the stroboscopic map is taken immediately after the kicks. Our aim is, therefore, to relate the Wigner function after the $(n+1)^{\text{th}}$ kick, W_{n+1}, to that after the n^{th} kick, W_n.
We start by deriving the recursion relation for the characteristic function. The periodicity is chosen in such a way that (2.11) is fulfilled. Then, it follows from (3.15) that the characteristic function, $\tilde{\phi}_{n+1}$ just before the $(n+1)^{\text{th}}$ kick reads

$$\tilde{\phi}_{n+1}(\xi,\eta)=\exp\left[\frac{-\hbar}{2\omega}\left(\bar{n}+\frac{1}{2}\right)(1-E^2)(\xi^2+\omega^2\eta^2)\right]$$
$$\cdot\,\phi_n(-E\omega\eta, E\omega^{-1}\xi). \tag{4.1}$$

By means of (3.17) the complete map can be written as

$$\phi_{n+1}(\xi,\eta)=\int\frac{\mathrm{d}\xi'\,\mathrm{d}x}{2\pi}\,\phi_n(-E\omega\eta, E\omega^{-1}\xi')$$
$$\cdot\exp[i((\xi-\xi')x+\eta\bar{f}(x))]\exp\left[-\frac{\hbar}{2\omega}\left(\bar{n}+\frac{1}{2}\right)(1-E^2)\right.$$
$$\left.\cdot(\xi'^2+\omega^2\eta^2)\right]\exp\left[\frac{i}{\hbar}\left(\bar{g}\left(x+\frac{\eta\hbar}{2}\right)-\bar{g}\left(x-\frac{\eta\hbar}{2}\right)\right.\right.$$
$$\left.\left.-\eta\hbar\bar{f}(x)\right)\right], \tag{4.2}$$

where it should be recalled that $\bar{f}(x)=\bar{g}'(x)$. One may easily interpret the different exponential factors under the integral on the right hand side of (4.2). The first one is the kernel of the classical map according to the deterministic dynamics. The second factor represents the quantum correction due to dissipation, while the last factor denotes a quantum corretion caused by the non-linearity of the kicks. In the typical case of a smooth kick amplitude $\bar{f}(x)$ it can also be seen that for small values of $\hbar$ the third factor is of order $\hbar^2$ in the exponent and, therefore, negligible compared to the second factor, i.e. the dissipation, if present, dominates the semiclassical limit (cf. also next section).
After having found the map for the characteristic function one can easily transform it to the Wigner function itself. Taking the Fourier transform of (4.2) we find that the map is nonlocal and of the form

$$W_{n+1}(x,p)=\int\mathrm{d}x_n\,\mathrm{d}p_n\bar{K}(x,p,x_n,p_n)\,W_n(x_n,p_n). \tag{4.3}$$

The kernel can be expressed as

$$\bar{K}(x,p;x_n,p_n)=\int\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(2\pi)^2}\exp[-i\xi(x-x^0_{n+1}(x_n,p_n))$$
$$-i\eta(p-\bar{p}^0_{n+1}(x_n,p_n,x))]\exp\left[-\frac{\hbar Q}{2\omega}(\xi^2+\omega^2\eta^2)\right]$$
$$\cdot\exp[iG(x,\eta,\hbar)], \tag{4.4}$$

where we introduced the abbreviations

$$Q=\left(\bar{n}+\frac{1}{2}\right)(1-E^2)=\frac{1-E^2}{2}\coth\left(\frac{1}{2}\frac{\hbar\omega}{k_B T}\right)$$
$$G(x,\eta,\hbar)=\frac{1}{\hbar}\left(\bar{g}\left(x+\frac{\eta\hbar}{2}\right)-\bar{g}\left(x-\frac{\eta\hbar}{2}\right)-\eta\hbar\bar{g}'(x)\right). \tag{4.5}$$

Furthermore, $x^0_{n+1}(x_n,p_n)$ denotes the classical value of the x variable after the $(n+1)^{\text{th}}$ kick expressed in terms of x_n, p_n, in particular with the choice (2.11): $x^0_{n+1}=E\omega^{-1}p_n$, while $\bar{p}^0_{n+1}(x_n,p_n,x)$ is defined as

$$\bar{p}^0_{n+1}(x_n,p_n,x)=-\omega E x_n+\bar{f}(x) \tag{4.6}$$

which differs from the classical recursion only by the fact that the argument of $\bar{f}$ is not x^0_{n+1} but rather the variable x of the Wigner function W_{n+1}. The first factor in the integrand of (4.4) is again the classical part which would give $\bar{K}(x,p;x_n,p_n)=\delta(x-x^0_{n+1})\,\delta(p-p^0_{n+1})$, where p^0_{n+1} denotes the classical momentum after the $(n+1)^{\text{th}}$ kick (cf. (2.3)–(2.5)). For finite $\hbar$ the function is no longer localized at the classical values but there is a broadening due to dissipation (second factor) and non-linearity (third factor). The integration over ξ can always be performed in (4.4) leading to

$$\bar{K}(x,p;x_n,p_n)=\left(\frac{\omega}{2\pi\hbar Q}\right)^{1/2}$$
$$\cdot\exp\left[-\frac{\omega}{2\hbar Q}(x-x^0_{n+1}(x_n,p_n))^2\right]$$
$$\cdot\int\frac{\mathrm{d}\eta}{2\pi}\exp\left[-i\eta(p-\bar{p}^0_{n+1}(x_n,p_n,x))\right.$$
$$\left.-\frac{\hbar Q\omega}{2}\eta^2\right]\exp[iG(x,\eta,\hbar)]. \tag{4.7}$$

Equations (4.3), (4.7) give the complete quantum map for the kicked oscillator with damping.
The quantized version of the map (2.8) is obtained from the above results by eliminating the momentum p in favour of the variable y introduced by (2.6). Denoting the Wigner function in the x, y representation by $W(x,y)$ we find

$$W_{n+1}(x,y)=\int\mathrm{d}x_n\,\mathrm{d}y_n K(x,y;x_n,y_n)\,W_n(x_n,y_n) \tag{4.8}$$

with the kernel

$$K(x,y;x_n,y_n)=\left(\frac{\omega^3}{2\pi\hbar Q}\right)^{1/2}$$
$$\cdot\exp\left[-\frac{\omega}{2\hbar Q}(x-x^0_{n+1}(x_n,y_n))^2\right]$$
$$\cdot\int\frac{d\eta}{2\pi}\exp\left[i\eta\omega(y-y^0_{n+1}(x_n,y_n))\right.$$
$$\left.-\frac{\hbar Q\omega}{2}\eta^2\right]\exp[iG(x,\eta,\hbar)]. \tag{4.9}$$

The quantities x^0_{n+1}, y^0_{n+1} are here the classical values after the $(n+1)^{\text{th}}$ step expressed in terms of x_n, y_n (see (2.8)). Note that the quantum map depends on the three additional parameters $\hbar$, ω, Q beyond those already contained in the classical map. The appearance of the frequency ω is caused by the special choice of the classical suspension. The parameter Q is due to finite temperature effects. Even more parameters like S or C of Eq. (2.4) would have appeared had we not chosen τ in such a way that C vanishes.

5. Semiclassical Limits

The kernel (4.9) cannot be evaluated exactly, in general. In the limit of small $\hbar$, however, explicit expressions can be found. For smooth kick amplitudes, i.e. for a smooth function $\bar{f}$, (and, consequently, also for a smooth f, cf. (2.9)), the quantity G defined by (4.5) can be expanded as

$$G(x,\eta,\hbar)=\hbar^2\frac{\eta^3\omega}{4E}\sum_{j=0}^{\infty}f^{(2j+2)}(x)\frac{1}{(2j+3)!}\left(\frac{\eta\hbar}{2}\right)^{2j}, \tag{5.1}$$

where $f^{(j)}$ denotes the j^{th} derivative of f. For small $\hbar$ we can approximate G by

$$G(x,\eta,k)\approx\hbar^2\frac{f^{(2)}(x)\omega}{24E}\eta^3. \tag{5.2}$$

The integral over η can then be performed in (4.9) and we obtain

$$K(x,y;x_n,y_n)=\left(\frac{\omega}{2\pi\hbar Q}\right)^{1/2}\left(\frac{8E\omega^2}{\hbar^2|f^{(2)}(x)|}\right)^{1/3}$$
$$\cdot\exp\left[-\frac{\omega}{2\hbar Q}(x-x^0_{n+1}(x_n,y_n))^2\right]$$
$$\cdot\exp\left[\frac{16}{3\hbar}\frac{Q^3E^2\omega}{(f^{(2)}(x))^2}+\frac{4QE\omega}{\hbar f^{(2)}(x)}(y-y^0_{n+1}(x_n,y_n))\right]$$
$$\cdot Ai\left[\left((y-y^0_{n+1}(x_n,y_n))+\frac{2Q^2E}{f^{(2)}(x)}\right)\left(\frac{8E\omega^2}{\hbar^2 f^{(2)}(x)}\right)^{1/3}\right], \tag{5.3}$$

where $Ai(z)$ stands for the Airy function [20]. Even though $\hbar$ was assumed to be small, we did not assume $\hbar\omega\ll k_BT$. Therefore, Q has not been expanded in powers of $\hbar$.

Equation (5.3) represents a semiclassical approximation for arbitrary functions f. However, it follows from (5.1) that for polynomial kick amplitudes of degree less than four (5.3) is the exact quantum mechanical expression. This class contains two important cases: that of the original Hénon model [4] and that of a cubic map related to the Duffing oscillator [21]. (Another exact result can be obtained for the piecewise linear map introduced by Lozi [22]. Then, of course, the expansion (5.1) is not valid.)

The conservative limit can be easily taken in (5.3). For $E\to 1$ $(Q\to 0)$ one finds

$$K(x,y;x_n,y_n)=\delta(x-x^0_{n+1}(x_n,y_n))\left(\frac{8\omega^2}{\hbar^2|f^{(2)}(x)|}\right)^{1/3}$$
$$\cdot Ai\left[(y-y^0_{n+1}(x_n,y_n))\left(\frac{8\omega^2}{\hbar^2 f^{(2)}(x)}\right)^{1/3}\right]. \tag{5.4}$$

This result is qualitatively similar to that obtained by Berry et al. [6]. Note, however, that by applying their expression to maps of Hénon's type a kernel different from (5.4) is obtained because they used another classical suspension. The limits $\hbar\to 0$, of course, coincide. A comparison of (5.3) and (5.4) shows that dissipation causes a broadening of the kernel in the x direction.

Another, more phenomenological, semiclassical approximation can be performed in the dissipative case by neglecting $G(x,\eta,\hbar)$ entirely (a first order calculation in $\hbar$). Then, only Gaussian integrals remain in (4.9) and we find

$$K(x,y;x_n,y_n)=\frac{\omega}{2\pi\hbar Q}\exp\left[-\frac{\omega}{2\hbar Q}\{(x-x^0_{n+1}(x_n,y_n))^2\right.$$
$$\left.+(y-y^0_{n+1}(x_n,y_n))^2\}\right], \tag{5.5}$$

which is of the same form as the classical transition probability in a classical noisy system. On this semiclassical level employing the Wigner function in the description, the quantum effects in the dissipative map, therefore, appear as effective classical noise. The same result has been found in other dissipative systems [7, 11]. We note that a similar interpretation of Eq. (5.3) in terms of classical noise is not possible because of the presence of the Airy function, which assumes negative values in some regions of its domain. As a consequence of (5.5) the quantum map becomes equivalent to a noisy recursion $(x_n, y_n\to x_{n+1}, y_{n+1})$ which can be written in the form

$$x_{n+1} = x^0_{n+1}(x_n, y_n) + \xi^x_n,$$
$$y_{n+1} = y^0_{n+1}(x_n, y_n) + \xi^y_n. \tag{5.6}$$

The noise forces $\xi^{x(y)}_n$ are found from (5.5) to be independent Gaussian noise terms with mean zero and correlation functions

$$\langle \xi^{x(y)}_n \xi^{x(y)}_m \rangle = \frac{\hbar Q}{\omega} \delta_{n,m} = \frac{\hbar(1-E^2)}{2\omega} \coth\left(\frac{1}{2}\frac{\hbar\omega}{k_B T}\right) \delta_{n,m}. \tag{5.7}$$

Note that for $\hbar \to 0$, T fixed, they represent thermal fluctuations with intensity $k_B T \omega^{-2}(1-E^2)$ which are associated with the dissipation by the classical fluctuation-dissipation theorem. The independence of the random noises ξ^x, ξ^y is a consequence of the special choice of the period τ of the kicks which led to (2.11). This illustrates that noisy recursions giving the semiclassical description of a dissipative quantum map are, in general, cross-correlated.
In the strongly dissipative limit, $E \ll 1$, of (5.6) y^0_{n+1} and $E y_n$ are negligible. The first equation describes then a one-dimensional map with noise:

$$x_{n+1} = f(x_n) + \xi^x_n, \qquad \langle (\xi^x_n)^2 \rangle = \frac{\hbar}{2\omega} \coth\left(\frac{1}{2}\frac{\hbar\omega}{k_B T}\right). \tag{5.8}$$

Thus, all results known for $1-D$ noisy maps [23] can be applied. The instability arising if trajectories leave the basin of the chaotic attractor and tend to infinity can be avoided by making the kick amplitude bounded for large values of x. In the case of quadratic dependence a possible renormalization is

$$f(x) = c\left(1 - \frac{a x^2}{1 + e x^4}\right) \tag{5.9}$$

with e of order unity. In this way instability can be avoided also at finite dissipation.
In the conservative limit both noise terms of (5.6) disappear and the classical result is recovered, which is consistent with the fact that (5.5) becomes the product of two delta functions in this limit. This shows that for dissipative maps, in contrast to conservative ones, there are two different levels of semiclassical approximations. The higher one ($O(\hbar^2)$) agrees with the usual semiclassical description for conservative maps. At finite dissipation, however, the quantum corrections already show up on a lower level ($O(\hbar)$). Here the system becomes equivalent with a classical noisy map, the noise intensity of which depends also on $\hbar$ and can be described by a noise temperature

$$T_N = \hbar\omega(\bar{n} + 1/2)/k_B. \tag{5.10}$$

The quantum mechanical feature shows up also in the fact that the probability density associated with (5.6) plays now the role of a Wigner function, and, therefore, when calculating average values of mixed products, the specific rules for the Wigner function [18] are to be used.
In the second part of this section we give an approximate calculation for the Wigner distribution around the classical strange attractor. It will be assumed that the function $f(x)$ specifying the classical dynamics has a single maximum and the parameters in (2.9) are chosen in such a way that the recursion generates chaotic motion. First, we recall that the invariant manifolds of a fixed point in the map (2.8) can be written as $x = f^*(y/E)$ where the multivalued function f^* is a solution of the equation [24]:

$$f^* = f - E^2 f^{*-1}. \tag{5.11}$$

f^{*-1} denotes the inverse of f^*. Since a one-piece strange attractor is the closure of the unstable manifold of a fixed point the corresponding solution provides an analytic description of the shape of the strange attractor. For strong dissipation an approximate solution is possible in powers of E^2 and the result may be interpreted as an approximate form for the branched manifold of the system.
The first quantum correction can be taken into consideration by writing

$$y = E f^{*-1}(x) + \hbar^{1/2} u, \tag{5.12}$$

where u measures the deviation from the closest branch of the branched manifold at a given order in E. In the presence of quantum fluctuations x is close to but not exactly equal to $f^*(x_n)$, therefore, we expand the exponent of the kernel (5.5) up to second in $x_n - f^{*-1}(x)$. We may write

$$x = f^*(x_n) - f^{*\prime}(x_n)(x_n - f^{*-1}(x)) + O((x_n - f^{*-1}(x))^2). \tag{5.13}$$

After inserting this in (5.5) the exponent reads

$$-\frac{\omega}{2\hbar Q}\Big[(f^{*\prime 2}(x_n) + E^2 + Q(\hbar^{1/2}))(x_n - f^{*-1}(x) - O(\hbar^{1/2}))^2 + \hbar \frac{(u f^{*\prime}(x_n) - E^2 u_n)^2 + O(\hbar^{1/2})}{f^{*\prime 2}(x_n) + E^2 + O(\hbar^{1/2})}\Big]. \tag{5.14}$$

Note that for vanishing $f^{*\prime}(x_n)$ the u-dependence disappears. This is related to the fact that the variable u defined by (5.12) describes fluctuations in the y-direction. A more realistic ansatz would assume deviations locally perpendicular to the branches of the strange attractor. Just around the extrema of f^* these would extend in the x-direction. For strong dissipation, however, the strange attractor of (2.8) is practically parallel to the x-axis except in a tiny neighbourhood of the points where $f^{*\prime}(y/E) = 0$. Thus, we may use (5.12) except in a small vicinity of the extremal points of f^*. In the following calculation, therefore, $f^{*\prime}(x_n) \neq 0$ will be assumed.

In order to find an explicit expression for the distribution in u we take the limit $\hbar \to 0$. Thus,

$$K(x,u;x_n,u_n)=\delta(x_n-f^{*-1}(x))\,|f^{*\prime}(x_n)|^{-1}$$
$$\cdot\,\hbar^{-1/2}\omega^{1/2}(2\pi Q)^{-1/2}(1+E^2/f^{*\prime 2}(x_n))^{-1/2}$$
$$\cdot\exp\left[-\frac{\omega}{2Q}\frac{(u-u_nE^2/f^{*\prime}(x_n))^2}{1+E^2/f^{*\prime 2}(x_n)}\right]. \tag{5.15}$$

The first factor shows that the dynamics in the x-direction is classical. This reflects the fact that the stationary probability distribution along the branches of the strange attractor will not be qualitatively changed by the quantum noise. There is, however, a qualitative change perpendicular to these branches.

We restrict our attention first to the extremely dissipative case. The chaotic attractor is then specified by $f^*(x)=f(x)$ for x in the interval enclosed by the first and second image of the maximum point of $f(x)$ under the map $x_{n+1}=f(x_n)$. By integrating (4.8) with (5.12), (5.15) over y, we find for the reduced Wigner distribution $W(x)=\int \mathrm{d}y\, W(x,y)$:

$$W_{n+1}(x)=\sum_{z\in f^{-1}(x)}\frac{W_n(z)}{|f'(z)|}. \tag{5.16}$$

Equation (5.16) is of the form of the Frobenius-Perron equation [25, 3]. The stable stationary solution of (5.16) is, therefore, the invariant distribution $W_{cl}(x)$ of the one-dimensional map $x_{n+1}=f(x_n)$. In general, one may write

$$W_n(x,y)=\hbar^{-1/2}P_n(u|x)\,W_n(x), \tag{5.17}$$

where $P_n(u|x)$ denotes the conditional quasi-probability density for finding a value u under the condition that the value of x is fixed. Next, we make the ansatz that the distribution perpendicular to the chaotic attractor is a local Gaussian whose squared width $\alpha^2(x)$ is a function of x:

$$P_n(u|x)=(2\pi\alpha_n^2(x))^{-1/2}\exp\left(-\frac{u^2}{2\alpha_n^2(x)}\right). \tag{5.18}$$

By inserting (5.15), (5.17), (5.18) into (4.8) and taking into account the Frobenius-Perron equation for $W_n(x)$ we find that such a solution exists with

$$\alpha_n^2(x)=Q/\omega \tag{5.19}$$

independently of n and x. The joint distribution in the stationary state is, thus, given by

$$W(x,y)=W_{cl}(x)\left(\frac{\omega}{2\pi\hbar Q}\right)^{1/2}\exp\left[-\frac{(y-Ef^{-1}(x))^2}{2\hbar Q\omega^{-1}}\right] \tag{5.20}$$

in qualitative agreement with the Langevin description (5.7).

In a higher order calculation in the dissipation strength E the x-dependence of the broadening in the y-direction is no longer negligible. To illustrate this, we now assume that the function $f(x)$ specifying the classical map is even in its variable $f(-x)=f(x)$. In a next to leading order approximation in E the branched manifold is described by a multi-valued function $f^*=f-Ef^{-1}$ consisting of two branches. Note, however, that in the Gaussian part of the kernel K (5.15) f^* appears always in the form $E^2/f^{*\prime}$ and, therefore, $f^{*\prime}$ can there be replaced in the present approximation by f'. Due to these properties the ansatz (5.18) with (5.17) turns out to be valid. A direct substitution then gives up to order E^2 a variance α^2 independent of n

$$\alpha^2(x)=\frac{Q}{\omega}\left(1+\frac{E^2}{[f'(f^{-1}(x))]^2}\right). \tag{5.21}$$

The approximation breaks down in a tiny region where $|f'(f^{-1}(x))|<E$ as we discussed earlier. In its region of validity (5.21) illustrates that the x-dependence of the variance α^2 is weak in a next to leading order calculation in the dissipation strength E. Note that $\alpha^2(f(y)/E)$ is an even function of y due to the symmetry $f(y)=f(-y)$. Finally, for the stationary distribution around a point x, y_{cl} on the branched manifold approximating the strange attractor we obtain

$$W(x,y)=W_{cl}(x)(2\pi\hbar\alpha^2(x))^{-1/2}\exp\left[-\frac{(y-y_{cl})^2}{2\hbar\alpha^2(x)}\right]. \tag{5.22}$$

W_{cl} here denotes the projection of the classical stationary density on the x axis in the given approximation.

In general, the broadening of the distribution around the branches of the classical strange attractor is specified by a characteristic length, which is of the order of magnitude $l_c=(\hbar Q/\omega)^{1/2}$. Quantum fluctuations wash out the Cantor structure of the strange attractor on the scale l_c and thereby destroy the classical strange attractor. However, for small $\hbar$ a well defined fractal dimension may be found when measuring the asymptotic dynamics of the quantum mechanical mean values on scales larger than l_c and a break-down occurs only on finer scales as in the presence of a weak classical noise [26].

Appendix: Other Quantization Procedures

Once a suspension has been chosen a considerable freedom still exists in the quantization of dissipative maps because of the possibility of choosing different couplings to the heat bath. In order to study the influence of this freedom on the form of the resulting quantum map, we have investigated also the case when, in place of Eq. (3.3), the main oscillator and the heat bath oscillators are bilinearly coupled through their position coordinates and the coupling is weak. In this case the equation governing the time development of the Wigner function associated with the reduced density matrix [27] possesses another form then (3.11). However, in the limit of weak damping ($\gamma \ll \omega$), which is of interest to us here, the stroboscopic map turns out to be very similar to that we described in Sect. 4. In fact, in leading order in γ they become equivalent. This may illustrate the insensitivity of quantum maps with respect to the type of the coupling of the suspended system to the heat bath.

In the following we wish to illustrate also how the results change with the quantization procedure. We employ a phenomenological quantization procedure which has recently been worked out [7]. It corresponds to the choice of a different suspension of the original map. First, we factorize the total map into a part which is locally area preserving and another linear part which is area contracting. The total quantum map is the convolution of the two separate parts. Among several possibilities we choose here for illustration the following factorization of (2.8):

$$x_{n+1} = -E y_{n+\frac{1}{2}}, \quad y_{n+1} = E x_{n+\frac{1}{2}}, \tag{A.1}$$

$$x_{n+\frac{1}{2}} = x_n, \quad y_{n+\frac{1}{2}} = y_n - E^{-1} f(x_n). \tag{A.2}$$

For the conservative part (A.2) one can always find a Hamiltonian [7] which brings it into the form

$$x_{n+\frac{1}{2}} = x_n + \frac{\partial H(x_n, y_{n+\frac{1}{2}})}{\partial y_{n+\frac{1}{2}}},$$
$$y_{n+\frac{1}{2}} = y_n - \frac{\partial H(x_n, y_{n+\frac{1}{2}})}{\partial x_n}. \tag{A.3}$$

By comparing (A.3) and (A.2)

$$H(x_n, y_{n+\frac{1}{2}}) = E^{-1} g(x_n) \tag{A.4}$$

follows, where g denotes the integral of f: $g'(x) = f(x)$. Note that the Hamiltonian is independent of the "momentum" y in the present case. With (A.4) the kernel $K(x_{n+\frac{1}{2}}|x_n)$ transforming a wave function $\psi_n(x_n)$ into $\psi_{n+\frac{1}{2}}(x_{n+\frac{1}{2}})$ reads [7]

$$K(x_{n+\frac{1}{2}}|x_n) = \int \frac{dz}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left(z(x_{n+\frac{1}{2}} - x_n) - \frac{g(x_n)}{E}\right)\right]$$
$$= \delta(x_{n+\frac{1}{2}} - x_n) \exp\left(-\frac{i}{\hbar E} g(x_n)\right). \tag{A.5}$$

From here it is easy to construct the kernel for the Wigner function $W(x, y)$. By means of the definition (3.10) one obtains

$$K_1(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}; x_n, y_n) = \delta(x_{n-\frac{1}{2}} - x_n)$$
$$\cdot \int \frac{d\xi}{2\pi} \exp[-i\xi(y_{n+\frac{1}{2}} - y_n)]$$
$$\cdot \exp\left[-\frac{i}{\hbar E}\left(g\left(x_n + \frac{\xi\hbar}{2}\right) - g\left(x_n - \frac{\xi\hbar}{2}\right)\right)\right]. \tag{A.6}$$

Next, we turn to the dissipative part (A.1) and look for the corresponding kernel. The simplest way to obtain it is to find a quantum master equation with continuous time variable which on a stroboscopic map generates just the dynamics (A.1) for the mean values. Here there are again several possible choices. After the calculation of Sect. 3, however, it is for us most convenient to take the master equation for a damped harmonic oscillator with $\omega = 1$, $\tau = 2\pi$. Thus we obtain for the kernel

$$K_2(x, y; x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$
$$= \int \frac{d\xi\, d\eta}{(2\pi)^2} \exp[-i\xi(x + E y_{n+\frac{1}{2}}) - i\eta(y - E x_{n+\frac{1}{2}})]$$
$$\cdot \exp[-\hbar Q(\xi^2 + \eta^2)/2], \tag{A.7}$$

where Q, defined by (4.5), plays the role of a phenomenological parameter. Of course, (A.7) is a special case of (4.4) for $g = f = 0$.

The kernel for the total map (A.1), (A.2) is then the convolution of (A.6), (A.7):

$$K(x, y; x_n, y_n) = \int dx_{n+\frac{1}{2}} dy_{n+\frac{1}{2}} K_2(x, y; x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$
$$\cdot K_1(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}; x_n, y_n). \tag{A.8}$$

A direct substitution yields

$$K(x, y; x_n, y_n)$$
$$= (2\pi\hbar Q)^{-1/2} \exp\left[-\frac{(y - E x_n)^2}{2\hbar Q}\right]$$
$$\cdot \int \frac{d\xi}{2\pi} \exp[-i\xi(x - f(x_n) + E y_n) - \hbar Q \xi^2/\xi^2/2]$$
$$\cdot \exp\left[\frac{i}{\hbar E}\left(g\left(x_n + \frac{E\xi\hbar}{2}\right) - g\left(x_n - \frac{E\xi\hbar}{2}\right) - E\xi\hbar f(x_n)\right)\right]. \tag{A.9}$$

Equation (A.9) has a similar structure as the kernel (4.9) obtained by quantizing via the suspension (2.1). The quantum noise enters somewhat differently, however, only in the order $\hbar^2$, and the dissipation factor E explicitly appears in the nonlinear function g. Due to the similar structre, however, the most important qualitative properties for $\hbar \to 0$ remain unchanged: There are two different levels of semiclassical approximations. On the higher level the kernel contains an Airy function and cannot be interpreted in terms of a classical conditional probability density. On the lower level, which only exists in the non-conservative case ($E<1$), the quantum map becomes equivalent to a noisy classical map with noise intensity proportional to $\hbar$. Whether the noise terms appear in an additive or multiplicative way and how the noise sources scale with E may, in general, depend on the particular method of quantization.

Finally, we mention that the suspension chosen in the main text generates a natural factorization of the classical map. The dissipative part is there given by the stroboscopic map between two subsequent kicks and the conservative part by the action of a kick itself. This factorization is quite different from the one defined by Eqs. (A.1), (A.2). By transforming (2.3) and (2.5) into the x, y representation with $C=0$ we find

$$x_{n+1}=x_{n+\frac{1}{2}}, \quad y_{n+1}=y_{n+\frac{1}{2}}-E^{-1}f(x_{n+\frac{1}{2}}), \tag{A.10}$$

$$x_{n+\frac{1}{2}}=-Ey_n+f(x_n), \quad y_{n+\frac{1}{2}}=Ex_n+E^{-1}f(x_{n+\frac{1}{2}}) \tag{A.11}$$

which is, indeed, different from (A.1), (A.2). It is satisfactory, therefore, that the most important qualitative properties for $\hbar\to 0$ are not changed by these differences.

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