Supplemental Material for:
Probabilistic concepts in a changing climate: a snapshot attractor picture

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I. COMPARISON WITH THE SINGLE-REALIZATION PICTURE

It is important to illustrate the contrast between the amount of information one can extract from the single-realization and the snapshot pictures in cases with shifting parameters. Fig. SM1 shows the stroboscopically generated points of a single trajectory [i.e., a

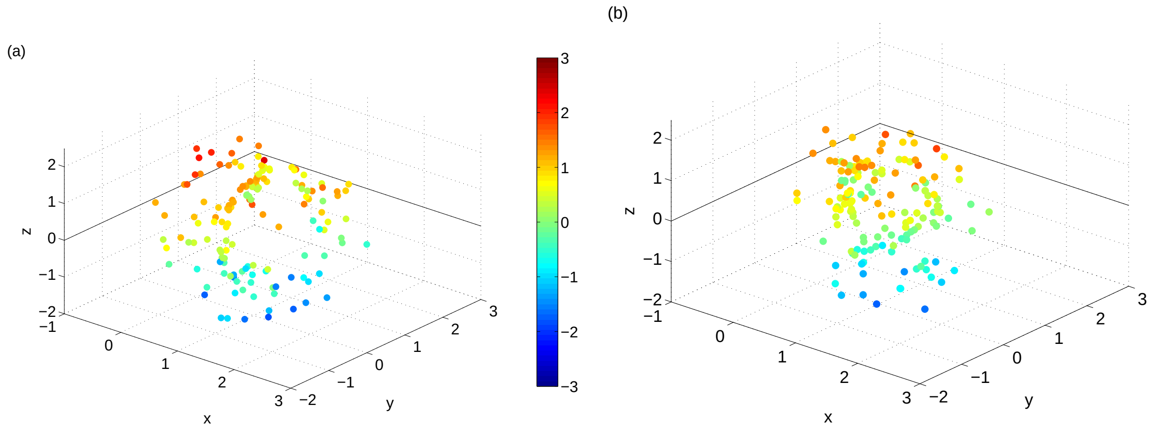

FIG. SM1. Midwinter (a) and midsummer (b) points of a 151 years long trajectory started on the chaotic attractor of the $F_0 = 9.5$ stationary climate with $(x_0 \approx 0.55, y_0 \approx 0.47, z_0 \approx 1.29)$ at $t = 99.75$ years. The z-component is color-coded monotonically in the spectrum.

single realization of the dynamics (1)-(3)] initiated on the chaotic attractor of the $F_0 = 9.5$ stationary climate at midsummer in year 99. From $t = t_{\rm st} = 100$ years on, the ramp in (3) is active, so the trajectory is subject to a continuous shift of the temperature contrast parameter. Storing the values of the single trajectory at the midwinter and the midsummer time instants of each year, the sets of points shown in panels (a) and (b), respectively, of Fig. SM1 are obtained. Note that such data sets are the ones corresponding to the traditional treatment of the climate and of its change, see e.g. Stocker et al. (2013). In spite of their conceptual difference, it is worth comparing these single-realization data sets with the results of the snapshot approach, the attractors of Fig. 2. The difference is striking: not only that no structure is traced out in the single-realization sets, in addition, when overlaying the corresponding plots, one observes that some points of Fig. SM1(a) lie outside the corresponding snapshot attractor of Fig. 2a. There are so few points (151 in number) that no reliable statistics can be based on them. (The situation would be even worse if we carried out the investigation in an earlier year.) Moreover, there is practically no point of the single-realization set with its z-coordinate being close to zero, and hence no analog of the section on the (x, y) plane of Fig. 1 can be generated, in strong contrast with the case of a stationary climate. Note that the selected points (one from every year) of a single trajectory can be considered to be the union over all different years of the sets consisting of a single point each, with this point being on the snapshot attractor of the particular year. It is thus not a surprise that the patterns of the snapshot attractors become washed out by forming a union over all snapshot attractors via a single trajectory.

II. DEPENDENCE OF SINGLE-REALIZATION TEMPORAL STATISTICS ON THE TIME INTERVAL USED

As pointed out in Sec. 5, a 30-year time interval contains too much information on the past, and the time evolution of a trajectory in this interval is too specific for the particular realization, lacking hints on the ensemble behavior. Relevance for the given time instant

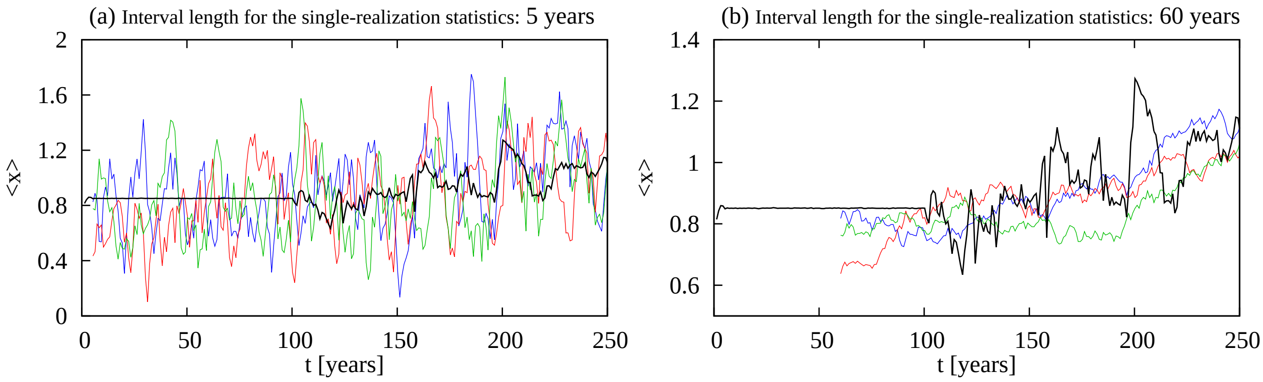

FIG. SM2. : E averages (black lines) and three different SRT averages (blue, red and green lines) taken over 5 (a) and 60 (b) years, as a function of time. Both panels exhibit midwinter x averages. Note the different scale in the panels. The numerical ensemble is the one of Fig. 5.

could be improved by taking, for example, 5-year intervals instead of 30-year ones. Fig. SM2(a) shows very clearly that, in this case, the individual realizations produce almost random signals which, therefore, do not contain practically any information on the ensemble behavior. This could be improved by increasing the considered time interval to, for example, 60 years, as shown in Fig. SM2(b). In this case, however, the past contributes to the statistics with even larger weight, making it less sensitive to even coarse structures occurring in the time evolution of the natural distribution of the snapshot attractor. These structures can be indicated faithfully by the E statistics only. These examples demonstrate that SRT averages can only lead to inadequate results from the point of view of instantaneous probabilities.

III. INTERVAL-WISE-TAKEN ENSEMBLE STATISTICS

In climate science, one is often interested in the weather of a time interval (e.g. a season in the year, or a few decades). If the dynamics were completely uncorrelated in time, then one could easily integrate over the instantaneous natural distributions of the time interval of interest. The subsequent time instants may, however, be strongly correlated by means of the continuous time evolution of the particular trajectories. In order to take these correlations into account, in such situations we propose to first evaluate the quantity of interest on the time interval of interest along single realizations, and then calculate the statistics of this quantity over the ensemble of the realizations (representing the probabilities faithfully). We call such statistics interval-wise-taken E (ensemble) statistics.

As an example for interval-wise-taken E statistics, a climatologically relevant idea is to take the temporal average over the past one year of a variable along individual trajectories, and calculate some statistics afterwards. In particular, we show in Fig. SM3(a) an E average (in black), similar to the one in Fig. 5(b) but obtained from the yearly averaged x (i.e., x averaged over the period running from the last midwinter time instant to the current midwinter one, having the average value ordered to the current midwinter) instead of its current, instantaneous midwinter value. The black line (i.e., an interval-wise-taken E average) changes less than the black line of Fig. 5(b). In addition, we also evaluate the 30-year SRT average of the yearly averaged x (which coincides with the 30-year continuously-averaged x along a single realization) and plot the results in Fig. SM3(a) in different colors. A comparison with the black line shows that the SRT averages are still not capable of resolving fine structures, do not follow faithfully all coarse structures, and indicate false changes very often.

One may ask what happens if we take a longer (but still continuous) time interval for averaging before calculating any statistics. To investigate this, we now take a 30-year time interval for averaging, beginning 30 years before the current midwinter time instant. That is, we take the traditional 30-year SRT average, and consider it to be a member of the ensemble over which we calculate statistics afterwards. Such a member is the line plotted in Fig. SM3(b) in magenta color, which is exactly the same as the green line of Fig. SM3(a). We now take the E average over 10^6 such lines and plot it in Fig. SM3(b) in black color. This is an interval-wise-taken E average. In our particular case, the magenta line, standing

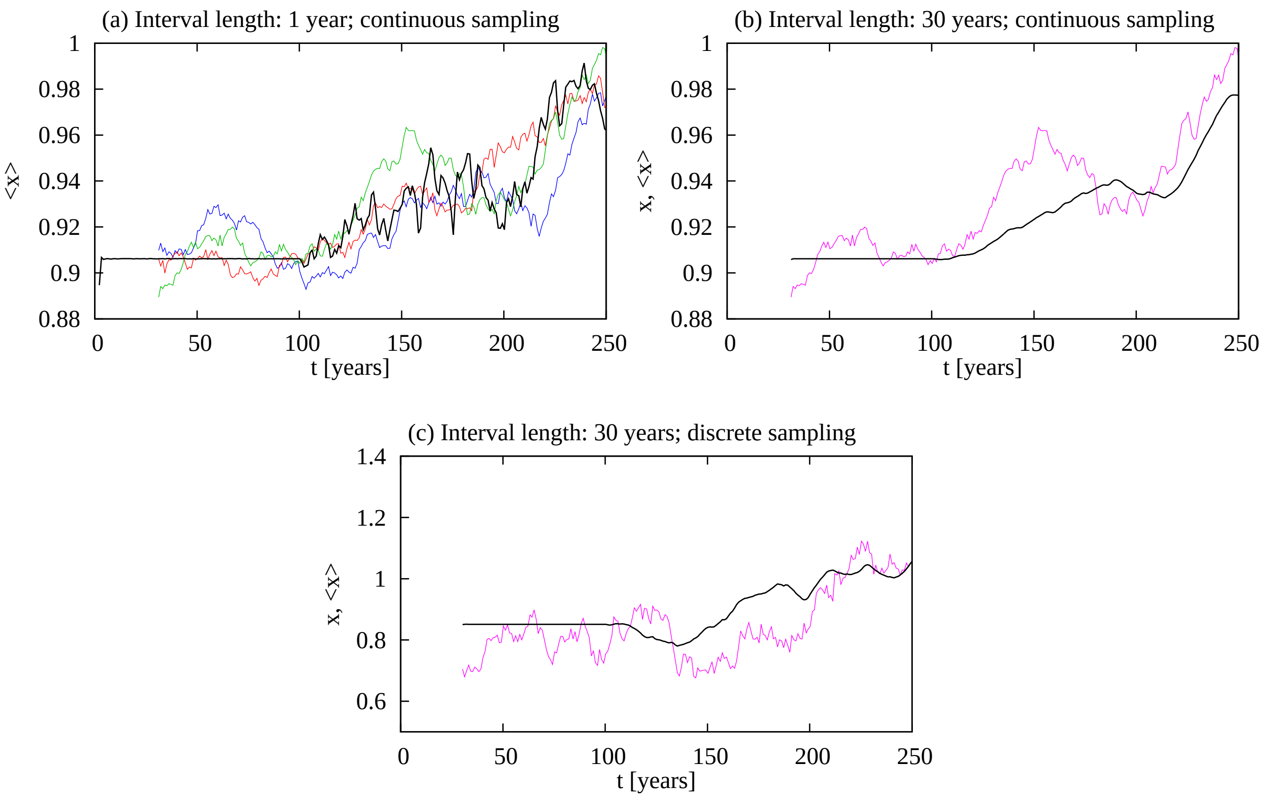

FIG. SM3. E averages (black lines), three different 30-year SRT averages (blue, red and green line) and single-realization values (magenta) as a function of time. The black line in panel (a) [(b)] is based on x values averaged continuously over the last one year [30 years], ending at midwinter. The black line in panel (c) is calculated from the average of the midwinter x values of the last 30 years. Note the different scale in this panel. The numerical ensemble is the one of Fig. 5. Three realizations are taken to illustrate individual 30-year SRT averages in panel (a). The green one is also included in panel (b) but in magenta color, expressing its similar role (i.e., being an ad hoc ensemble member) in this panel to that of the magenta line of Fig. 5(a). Panel (c) also contains an ensemble member in magenta color.

for one of the ensemble members, follows the black E average line somewhat closer than what is seen in Fig. 5(a). The difference is, however, still too large to consider the magenta line (a single realization case) as a useful approximant to the black one (an ensemble result in the snapshot picture).

For completeness, we also show an example for the application of a discrete sampling of the trajectory values from the 30-year interval of the past. Particularly, we take the midwinter value of the variable x in each of the last 30 years (including the current year) along a single realization, and calculate the average of these values. Note that this coincides with the 30-year SRT average investigated in Section 5 and marked by colored lines in Fig. 5(b). Now we take the ensemble average of 10^6 such individual lines to obtain an interval-wise-taken E average. This is plotted in Fig. SM3(c) in black color as a function of time. In addition, one ensemble member is also marked by a magenta line (which happens to be the same line as the green one of Fig. 5(b). By comparing the magenta and the black lines both in panels (b) and (c) one can conclude that an ensemble member is again not very informative from the point of view of the full ensemble behavior.