1.

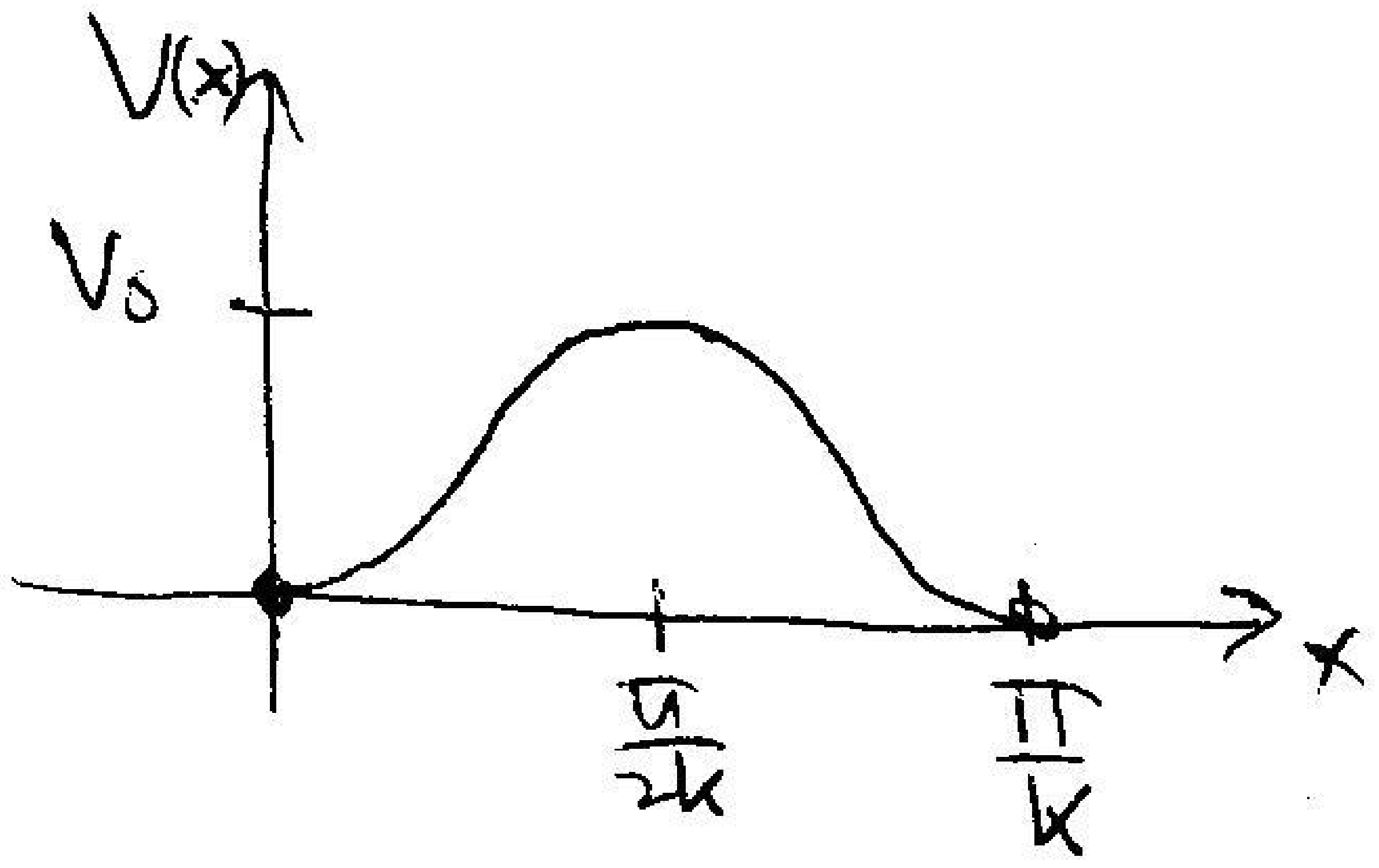
$$V(x) = V_0 \sin^2(Kx) = V_0 \frac{1-\cos(2Kx)}{2}$$

$$\boxed{\begin{aligned}\cos(2\varphi) &= \cos^2\varphi - \sin^2\varphi = \\ &= (1-\sin^2\varphi) - \sin^2\varphi = \\ &= 1 - 2\sin^2\varphi \\ \Rightarrow \sin^2\varphi &= \frac{1-\cos(2\varphi)}{2}\end{aligned}}$$

A cos „hullámhossza”: $\frac{2\pi}{2K} = \frac{\pi}{K}$, vagyis éppen az értelmezési tartomány végére fejlődik le 1 periódus.

$\Rightarrow$

HA $V_0 > 0$
és $K > 0$



Ennyi tájékozódást érdemes megtenni, azonban ez elvileg nem kéne a feladat megoldásához.

$$V'(x) = V_0 \cdot 2\sin(Kx)\cdot\cos(Kx)\cdot K = V_0 \cdot K \cdot \sin(2Kx)$$

$$V'(x)\big|_{x=x^*} = V_0 \cdot K \cdot \sin(2Kx^*) = 0$$

$$\Rightarrow \sin(2Kx^*) = 0$$

$2Kx_1^* = 2\pi n_1,\ n_1 \in \mathbb{Z}$

$x_1^* = \frac{2\pi n_1}{2K} = \frac{\pi}{K} n_1$

Mivel $x \in [0, \frac{\pi}{K})$, csak az $n_1 = 0$ lesz jó:

$x_1^* = 0$

$2Kx_2^* = \pi + 2\pi n_2,\ n_2 \in \mathbb{Z}$

$x_2^* = \frac{\pi + 2\pi n_2}{2K} =$

$= \frac{\pi}{2K} + \frac{\pi}{K} n_2$

Csak $n_2 = 0$ jó:

$x_2^* = \frac{\pi}{2K}$

$$V''(x) = V_0 K \cos(2Kx)\cdot 2K = 2V_0 K^2 \cos(2Kx)$$

$$x_1^*: \quad V''(x)\big|_{x=x_1^*} = 2V_0k^2 \underbrace{\cos(2k\cdot 0)}_{=1} = 2V_0k^2$$

$2V_0k^2 > 0$ akkor és csak akkor, ha $V_0 > 0$. Ebben az esetben lesz x_1^* stabil.

Ha x_1^* stabil, akkor a kis rezgések körfrekvenciája formálisan:

$$\omega_1 = \sqrt{\frac{1}{m} V''(x)\big|_{x=x_1^*}} = \sqrt{\frac{2V_0k^2}{m}} = |k|\sqrt{\frac{2V_0}{m}}$$

De x_1^* az értelmezési tartomány szélén van, és a kis rezgések valójában kilépnek a potenciál értelmezési tartományából. Ha az értelmezési tartományt kiterjesztjük, csak ebben az esetben valósulnak meg ténylegesen a kis rezgések.

$$x_2^*: V''(x)\big|_{x=x_2^*} = 2V_0k^2 \underbrace{\cos\left(2k\cdot\frac{\pi}{2k}\right)}_{=-1} = -2V_0k^2$$

$-2V_0k^2 > 0$ akkor és csak akkor, ha $V_0 < 0$. Ekkor x_2^* stabil.

Ha x_2^* stabil:

$$\omega_2 = \sqrt{\frac{1}{m} V''(x)\big|_{x=x_2^*}} = \sqrt{-\frac{2V_0k^2}{m}} = |k|\cdot\sqrt{-\frac{2V_0}{m}}$$

2.

$$V'(x) = V_0\left(-\frac{a}{(x+a)^2} + \frac{a}{(x-a)^2}\right)$$

$$V'(x)\big|_{x=x^*} = V_0 a\left(-\frac{1}{(x^*+a)^2} + \frac{1}{(x^*-a)^2}\right) = 0$$

$$\Rightarrow \quad -\frac{1}{(x^*+a)^2} + \frac{1}{(x^*-a)^2} = 0$$

$$(x^*-a)^2 = (x^*+a)^2$$

$$x^*-a = x^*+a$$

$$-a = a \ ↯$$

$$x^*-a = -(x^*+a)$$

$$x^* = 0 \longleftarrow 2x^* = 0$$

$$V''(x) = V_0\left(2\frac{a}{(x+a)^3} - 2\frac{a}{(x+a)^3}\right) = 2V_0 a\left(\frac{1}{(x+a)^3} - \frac{1}{(x-a)^3}\right)$$

$$V''(x)\Big|_{x=x^*} = 2V_0 a\left(\frac{1}{(0+a)^3} - \frac{1}{(0-a)^3}\right) = 2V_0 a\left(\frac{1}{a^3} + \frac{1}{a^3}\right) =$$

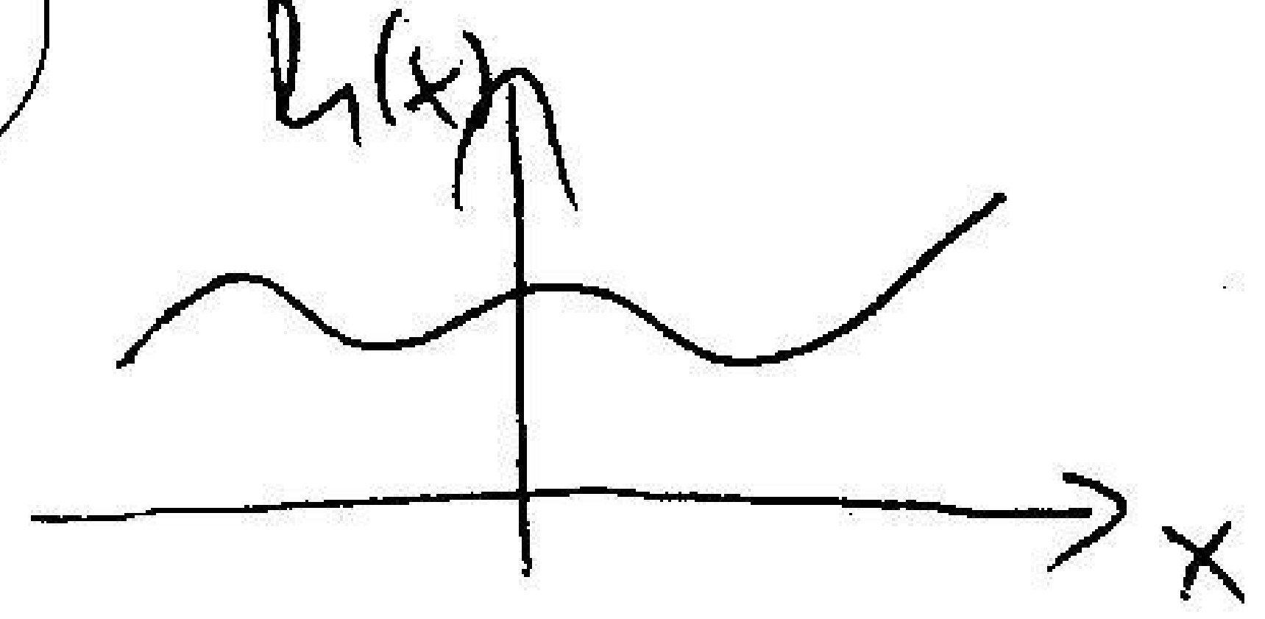
$$= 4V_0\frac{a}{a^3} = 4\frac{V_0}{a^2} > 0 \Rightarrow$$ x^* körül [illegible] végez.

$$\swarrow$$

$$\omega = \sqrt{\frac{1}{m}V''(x)\Big|_{x=x^*}} = \sqrt{\frac{1}{m}4\frac{V_0}{a^2}} = \quad 2\frac{1}{a}\sqrt{\frac{V_0}{m}}$$

EMB/GY2/04

③



1. x, y
2. $h(x)$
3. x
4. $x \qquad \dot{x}$

$$y = h(x) \qquad \dot{y} = \frac{dh(x)}{dx} \cdot \dot{x} \equiv h'(x) \cdot \dot{x}$$

5. $V = mgy = mgh(x)$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left(\dot{x}^2 + h'(x)^2 \dot{x}^2 \right) = \frac{1}{2} m \left(1 + h'(x)^2 \right) \dot{x}^2$$

6. $\frac{\partial L}{\partial x} = -mgh'(x) + \frac{1}{2} m \cdot \cancel{2} h'(x) \cdot h''(x) \cdot \dot{x}^2$

$$\frac{\partial L}{\partial \dot{x}} = m \left(1 + h'(x)^2 \right) \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = m \cdot 2 h'(x) \cdot h''(x) \cdot \dot{x} \cdot \dot{x} + m \left(1 + h'(x)^2 \right) \ddot{x}$$

$\longrightarrow$

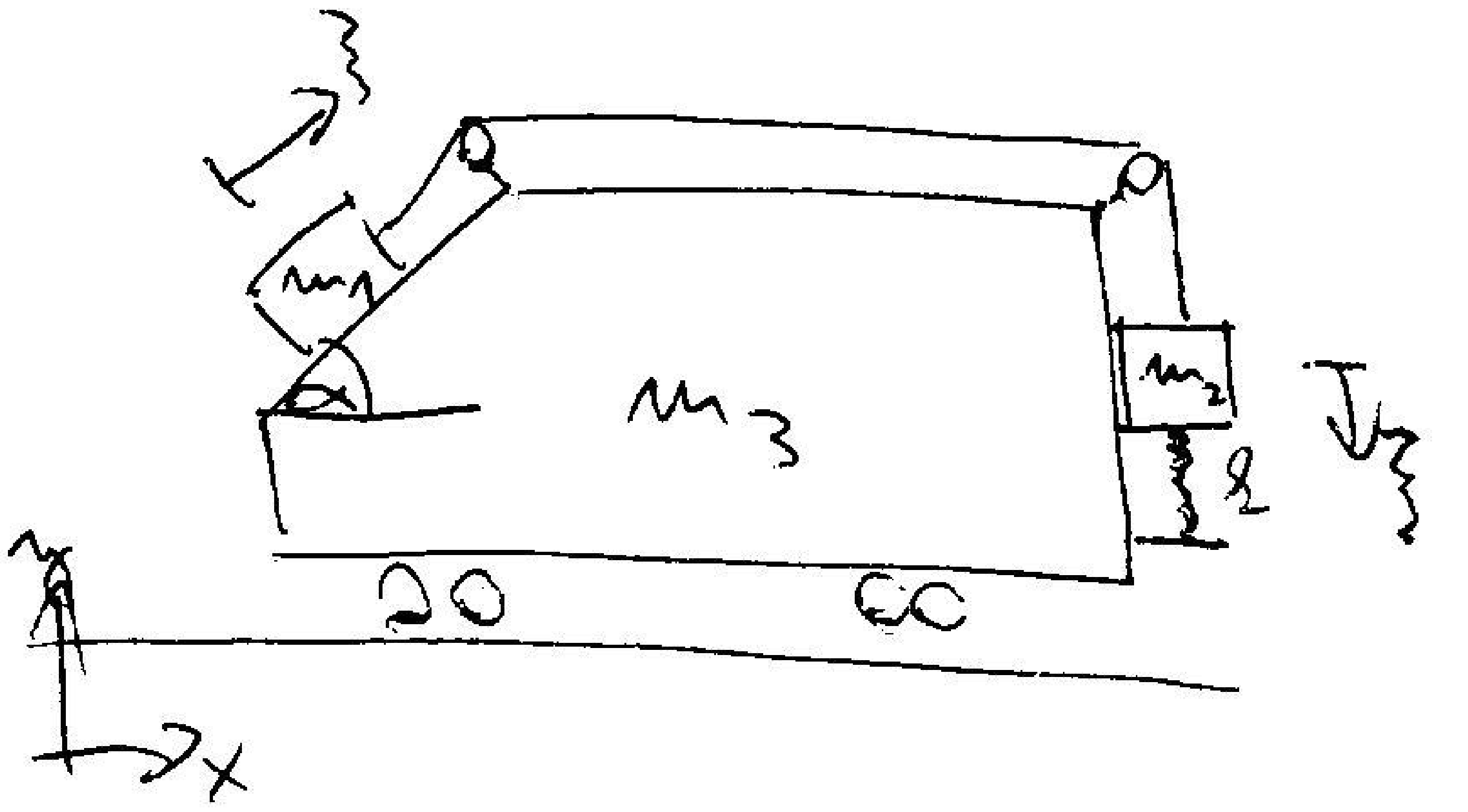
$$2m h'(x) h''(x) \dot{x}^2 + m (1 + h'(x)^2) \ddot{x} =$$

$$= -mgh'(x) + m h'(x) h''(x) \dot{x}^2$$

$$m \ddot{x} = \frac{1}{1 + h'(x)^2} \cdot \left(-mgh'(x) - m h'(x) h''(x) \dot{x}^2 \right)$$

A sebességtől függő tag azért jelenik meg, mert $h''(x) \neq 0$ esetén a gyorsulásnak van a lejtő lokális érintőjére merőleges komponense is. Ez egyfajta centripetális gyorsulásnak tekinthető, ez tartja az előírt $h(x)$ pályán a tömegpontot.

④.



1., $x_1, y_1, x_2, y_2, x_3, y_3$

2., • a 1-es test a lejtőn mozog
• a 2-es test nem leng ki
• y_3 = állandó
• kötél

3., x_3, ξ

4., $x_1 = x_3 + c_1 + \xi\cos\alpha \qquad \dot{x}_1 = \dot{x}_3 + \dot{\xi}\cos\alpha$

konstans

$y_1 = y_3 + d_1 + \xi\sin\alpha = \qquad \dot{y}_1 = \dot{\xi}\sin\alpha$

$= d_3 + d_1 + \xi\sin\alpha$

$x_2 = x_3 + c_2 \qquad \dot{x}_2 = \dot{x}_3$

konstans

$y_2 = y_3 + d_2 - \xi = \qquad \dot{y}_2 = -\dot{\xi}$

$= d_3 + d_2 - \xi$

$x_3 = x_3 \qquad \dot{x}_3 = \dot{x}_3$

$y_3 = d_3 \qquad \dot{y}_3 = 0$

→ konstans

5.) $L = T - V$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2) =$$

$$= \frac{1}{2}m_1\left(\dot{x}_3^2 + 2\dot{x}_3\dot{\xi}\cos\alpha + \underbrace{\dot{\xi}^2\cos^2\alpha + \dot{\xi}^2\sin^2\alpha}_{\dot{\xi}^2}\right) + \frac{1}{2}m_2(\dot{x}_3^2 + \dot{\xi}^2) +$$

$$+ \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}_3^2 + m_1\dot{x}_3\dot{\xi}\cos\alpha + \frac{1}{2}(m_1 + m_2)\dot{\xi}^2$$

$$V = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + \frac{1}{2}k\xi^2 =$$

$$= m_1 g d_3 + m_1 g d_1 + m_2 g d_3 + m_2 g d_2 + m_3 g d_3 +$$

Ez mind konstans!

$$+ m_1 g \xi \sin\alpha \quad - m_2 g \xi + \frac{1}{2}k\xi^2$$

6.) $\dfrac{\partial L}{\partial x_3} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = (m_1 + m_2 + m_3)\ddot{x}_3 + m_1\ddot{\xi}\cos\alpha$$

$$\rightarrow \quad (m_1 + m_2 + m_3)\ddot{x}_3 + m\ddot{\xi}\cos\alpha = 0$$

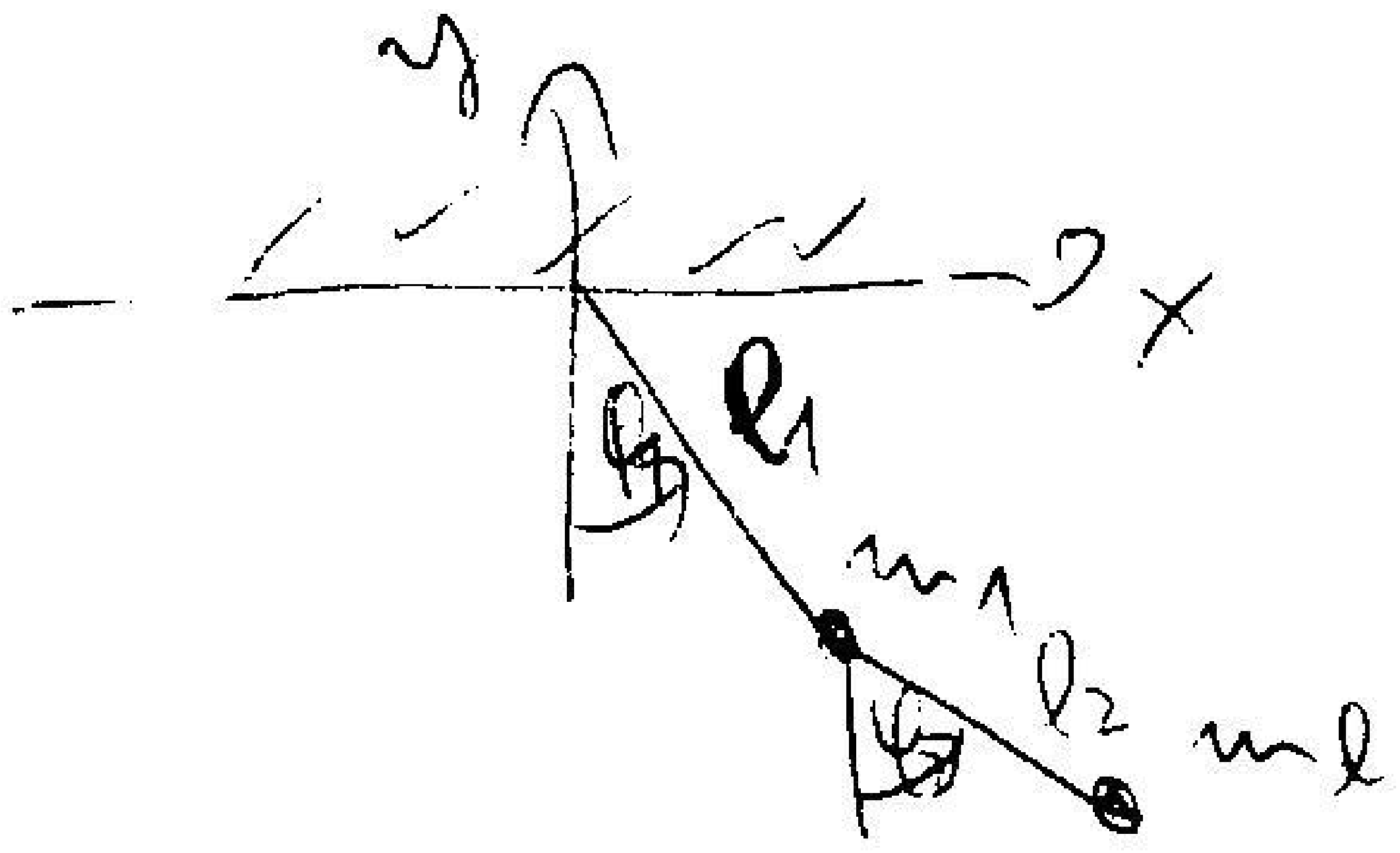
$$\frac{\partial L}{\partial \xi} = -m_1 g \sin\alpha + m_2 g - k\xi$$

$$\frac{\partial L}{\partial \dot{\xi}} = m_1\dot{x}_3\cos\alpha + (m_1 + m_2)\dot{\xi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = m_1\ddot{x}_3\cos\alpha + (m_1 + m_2)\ddot{\xi}$$

$$\rightarrow \quad -m_1 g \sin\alpha + m_2 g - k\xi = m_1\ddot{x}_3\cos\alpha + (m_1 + m_2)\ddot{\xi}$$

⑤



1., x_1, y_1, x_2, y_2

2., 2 db fonál

3., φ_1, φ_2

4., $x_1 = l_1 \sin\varphi_1$

$y_1 = -l_1 \cos\varphi_1$

$x_2 = x_1 + l_2 \sin\varphi_2 =$

$= l_1 \sin\varphi_1 + l_2 \sin\varphi_2$

$y_2 = y_1 - l_2 \cos\varphi_2 =$

$= -l_1 \cos\varphi_1 - l_2 \cos\varphi_2$

$\dot{x}_1 = l_1 \cos\varphi_1 \dot{\varphi}_1$

$\dot{y}_1 = l_1 \sin\varphi_1 \dot{\varphi}_1$

$\dot{x}_2 = l_1 \cos\varphi_1 \dot{\varphi}_1 + l_2 \cos\varphi_2 \dot{\varphi}_2$

$\dot{y}_2 = l_1 \sin\varphi_1 \dot{\varphi}_1 + l_2 \sin\varphi_2 \dot{\varphi}_2$

5., $L = T - V$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) =$$

$$= \frac{1}{2} m_1 (l_1^2 \cos^2\varphi_1 \dot{\varphi}_1^2 + l_1^2 \sin^2\varphi_1 \dot{\varphi}_1^2) + \Rightarrow$$

$$+\frac{1}{2}m_2\Big(\underline{l_1^2\cos^2\varphi_1\dot\varphi_1^2}+2l_1\cos\varphi_1\dot\varphi_1\, l_2\cos\varphi_2\dot\varphi_2+\underline{l_2^2\cos^2\varphi_2\dot\varphi_2^2}+$$

$$+\underline{l_1^2\sin^2\varphi_1\dot\varphi_1^2}+2l_1\sin\varphi_1\dot\varphi_1\, l_2\sin\varphi_2\dot\varphi_2+\underline{l_2^2\sin^2\varphi_2\dot\varphi_2^2}\Big)=$$

$$=\frac{1}{2}m_1l_1^2\dot\varphi_1^2+\frac{1}{2}m_2\Big[l_1^2\dot\varphi_1^2+l_2^2\dot\varphi_2^2+2l_1l_2\underbrace{(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)}_{\cos(\varphi_2-\varphi_1)}\dot\varphi_1\dot\varphi_2\Big]=$$

$$=\frac{1}{2}(m_1+m_2)l_1^2\dot\varphi_1^2+\frac{1}{2}m_2l_2^2\dot\varphi_2^2+m_2l_1l_2\cos(\varphi_2-\varphi_1)\dot\varphi_1\dot\varphi_2$$

$$V=m_1gy_1+m_2gy_2=\quad -m_1gl_1\cos\varphi_1+m_2g\overset{(-l_1\cos\varphi_1-l_2\cos\varphi_2)}{\text{[illegible]}}=$$

$$=-(m_1+m_2)gl_1\cos\varphi_1-m_2gl_2\cos\varphi_2$$

G.1 $$\frac{\partial L}{\partial\varphi_1}=m_2l_1l_2\dot\varphi_1\dot\varphi_2\cdot\underbrace{(-\sin(\varphi_2-\varphi_1))\cdot(-1)}-(m_1+m_2)gl_1\sin\varphi_1$$

„láncszabály" $\qquad \frac{\partial}{\partial\varphi_1}(\varphi_2-\varphi_1)=-1$

$$\frac{\partial L}{\partial\dot\varphi_1}=(m_1+m_2)l_1^2\dot\varphi_1+m_2l_1l_2\cos(\varphi_2-\varphi_1)\dot\varphi_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi_1}\right)=(m_1+m_2)l_1^2\ddot\varphi_1+m_2l_1l_2\frac{d}{dt}\big(\cos(\varphi_2-\varphi_1)\big)\dot\varphi_2+$$

$$+m_2l_1l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2=(m_1+m_2)l_1^2\ddot\varphi_1+$$

$$+m_2l_1l_2\big(-\sin(\varphi_2-\varphi_1)\big)\cdot(\dot\varphi_2-\dot\varphi_1)\cdot\dot\varphi_2+m_2l_1l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2$$

$$\rightarrow\quad \cancel{m_2l_1l_2\dot\varphi_1\dot\varphi_2\sin(\varphi_2-\varphi_1)}-(m_1+m_2)gl_1\sin\varphi_1=$$

$$\varphi\quad =(m_1+m_2)l_1^2\ddot\varphi_1-m_2l_1l_2\sin(\varphi_2-\varphi_1)\cdot(\dot\varphi_2^2-\cancel{\dot\varphi_1\dot\varphi_2})+m_2l_1l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2$$

$$\frac{\partial L}{\partial \varphi_2} = m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 (-\sin(\varphi_2-\varphi_1)) \cdot 1 - m_2 g l_2 \sin\varphi_2$$

$$\frac{\partial L}{\partial \dot\varphi_2} = \qquad m_2 l_2^2 \dot\varphi_2 + m_2 l_1 l_2 \cos(\varphi_2-\varphi_1)\dot\varphi_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right) = m_2 l_2^2 \ddot\varphi_2 + m_2 l_1 l_2 \cdot (-\sin(\varphi_2-\varphi_1)) \cdot (\dot\varphi_2 - \dot\varphi_1) \cdot \dot\varphi_1 +$$

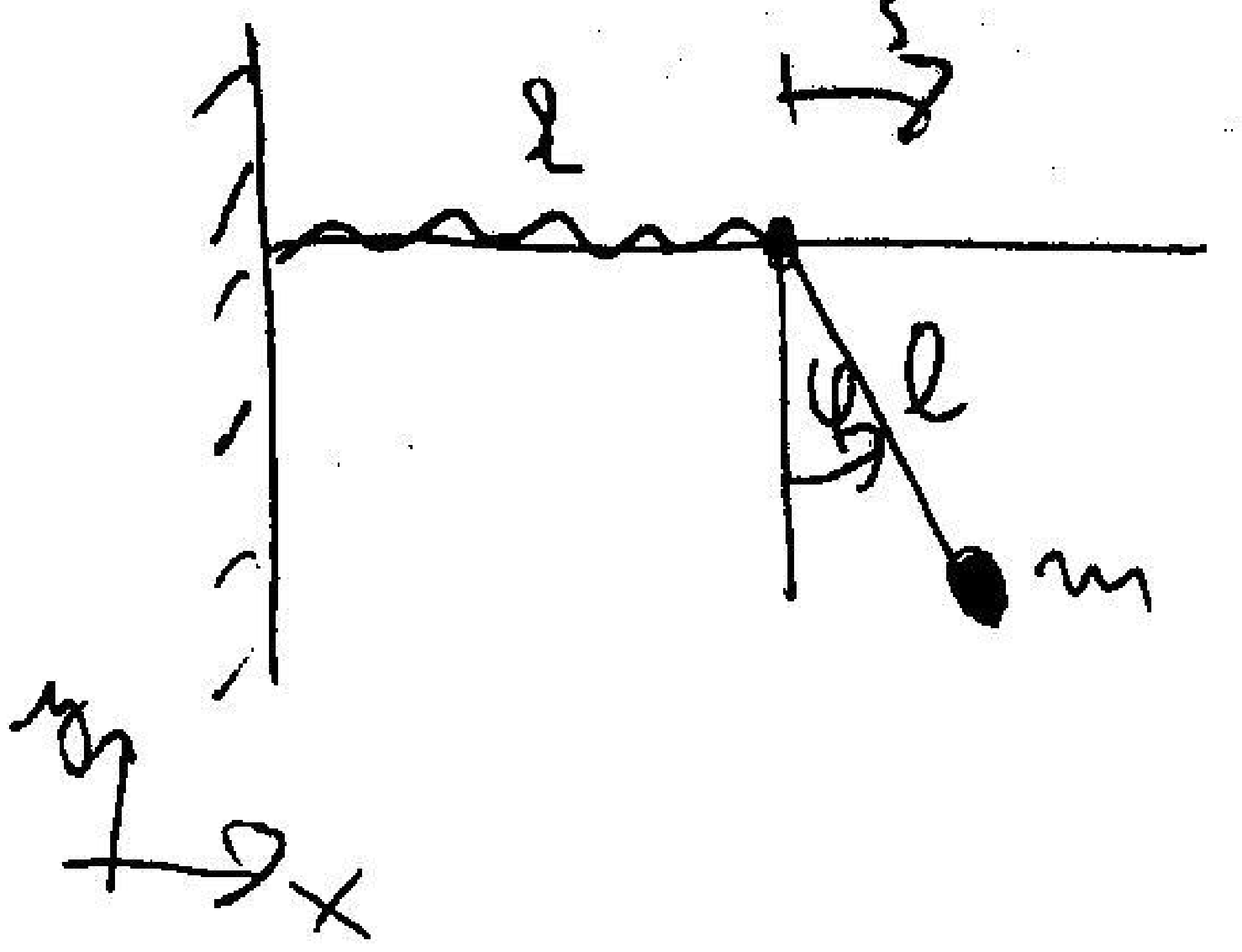
$$+ m_2 l_1 l_2 \cos(\varphi_2-\varphi_1)\ddot\varphi_1$$

$$\Rightarrow \cancel{-\dot\varphi_1\dot\varphi_2 m_2 l_1 l_2 \sin(\varphi_2-\varphi_1)} - m_2 g l_2 \sin\varphi_2 =$$

$$= m_2 l_2^2 \ddot\varphi_2 - m_2 l_1 l_2 \sin(\varphi_2-\varphi_1) \cdot (\cancel{\dot\varphi_1\dot\varphi_2} - \dot\varphi_1^2) +$$

$$+ m_2 l_1 l_2 \cos(\varphi_2-\varphi_1)\ddot\varphi_1$$

⑥



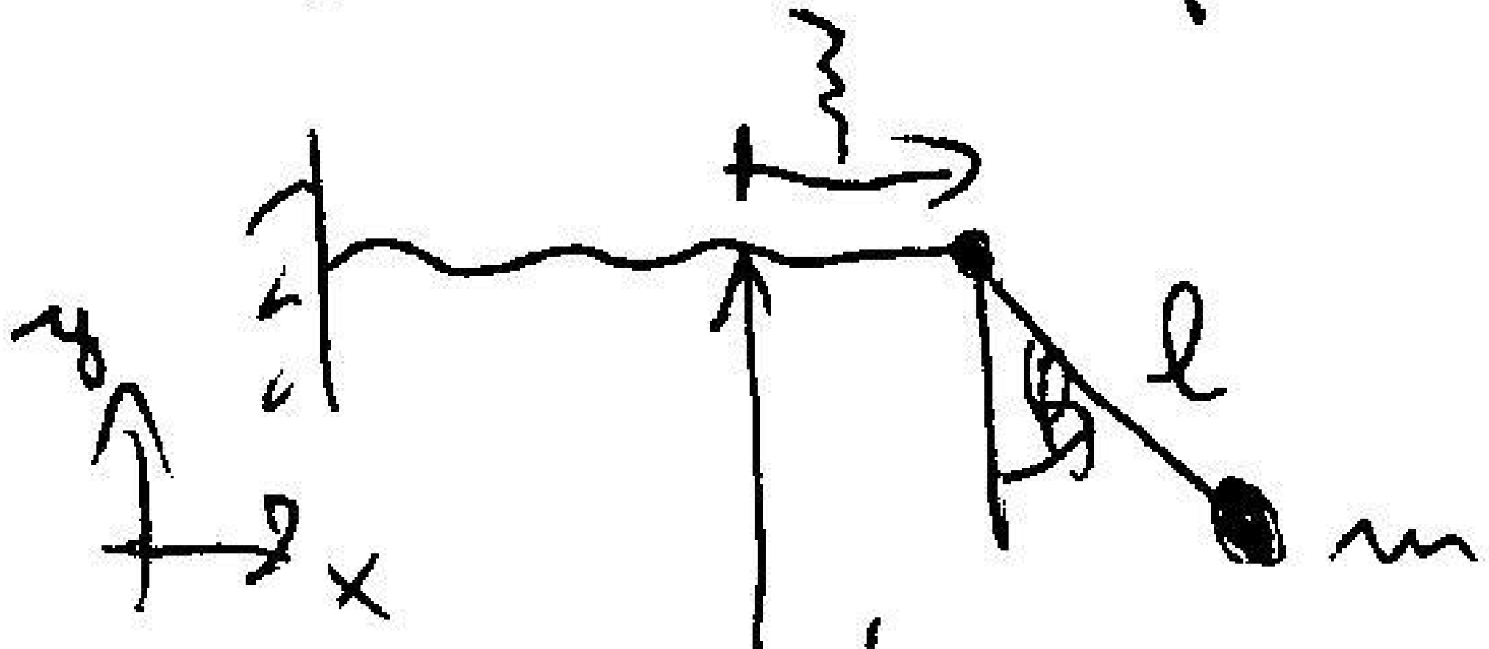
1., x, y (az inga végpontjára)

2., –

3., ξ, φ → a rugó nyugalmi helyzetétől nézve → a rugó végpontjánál nézve

4., $x = c + \xi + l\sin\varphi$

$y = d - l\cos\varphi$



a rugó nyugalmi helyzete, ennek a koordinátái „c" és „d"

$\dot{x} = \dot{\xi} + l\cos\varphi\,\dot{\varphi}$

$\dot{y} = l\sin\varphi\,\dot{\varphi}$

5., $L = T - V$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) =$

$= \frac{1}{2}m(\dot{\xi}^2 + 2\dot{\xi}\,l\cos\varphi\,\dot{\varphi} +$

$+ l^2\cos^2\varphi\,\dot{\varphi}^2 + l^2\sin^2\varphi\,\dot{\varphi}^2) =$

$= \frac{1}{2}m(\dot{\xi}^2 + 2l\cos\varphi\,\dot{\varphi}\,\dot{\xi} + l^2\dot{\varphi}^2)$

$V = mgy + \frac{1}{2}k\xi^2 = mgd - mgl\cos\varphi + \frac{1}{2}k\xi^2$

6., $\frac{\partial L}{\partial \xi} = -k\xi$

$\frac{\partial L}{\partial \dot{\xi}} = m\dot{\xi} + ml\cos\varphi\,\dot{\varphi}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = m\ddot{\xi} - ml\,\overbrace{\sin\varphi\cdot\dot{\varphi}}\cdot\dot{\varphi} + ml\cos\varphi\cdot\ddot{\varphi}$

$\rightarrow \quad -k\xi = m\ddot{\xi} - ml\sin\varphi\,\dot{\varphi}^2 + ml\cos\varphi\,\ddot{\varphi}$

φ

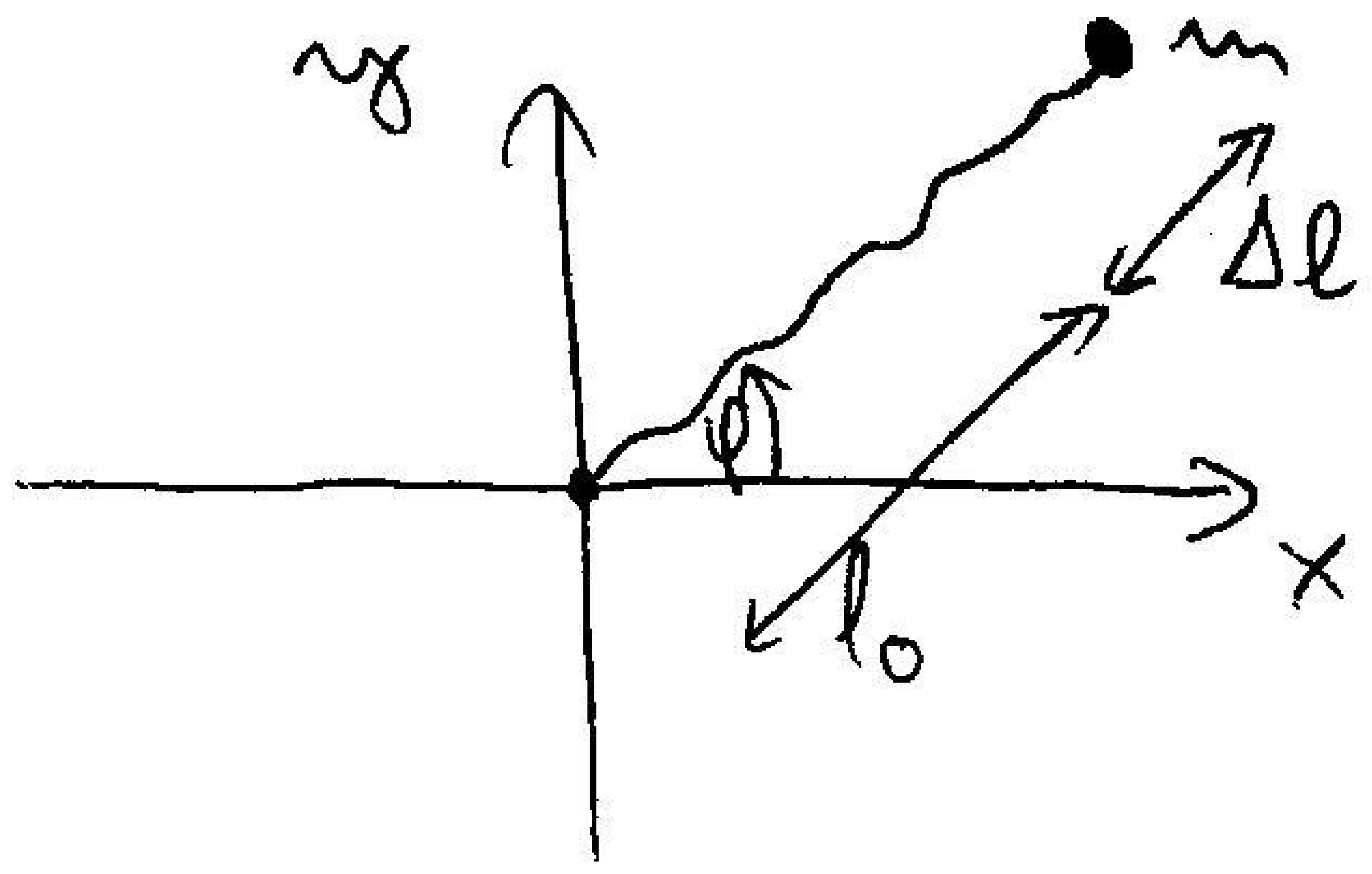
$$\frac{\partial L}{\partial \varphi} = -ml\sin\varphi\,\dot\varphi\,\dot\xi - mgl\sin\varphi$$

$$\frac{\partial L}{\partial \dot\varphi} = ml\cos\varphi\,\dot\xi + ml^2\dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) = -ml\sin\varphi\cdot\dot\varphi\cdot\dot\xi + ml\cos\varphi\,\ddot\xi + ml^2\ddot\varphi$$

$$\Rightarrow \quad \cancel{-ml\sin\varphi\dot\varphi\dot\xi} - \cancel{m}gl\sin\varphi =$$

$$= \cancel{-ml\sin\varphi\dot\varphi\dot\xi} + \cancel{m}l\cos\varphi\,\ddot\xi + \cancel{m}l^2\ddot\varphi$$



1. x, y
2. —
3. φ, Δl
4. $x=(l_0+\Delta l)\cos\varphi$

$y=(l_0+\Delta l)\sin\varphi$

$\dot x=\dot{\Delta l}\cos\varphi-(l_0+\Delta l)\sin\varphi\,\dot\varphi$

$\dot y=\dot{\Delta l}\sin\varphi+(l_0+\Delta l)\cos\varphi\,\dot\varphi$

5. $V=\frac{1}{2}k(\Delta l)^2$

$$T=\frac{1}{2}m(\dot x^2+\dot y^2)=\frac{1}{2}m\Big(\dot{\Delta l}^2\cos^2\varphi\cancel{-2\dot{\Delta l}\cos\varphi(l_0+\Delta l)}\sin\varphi\,\dot\varphi+$$

$$+(l_0+\Delta l)^2\sin^2\varphi\,\dot\varphi^2+\dot{\Delta l}^2\sin^2\varphi+$$

$$\cancel{+2\dot{\Delta l}\sin\varphi(l_0+\Delta l)\cos\varphi\,\dot\varphi}+(l_0+\Delta l)^2\cos^2\varphi\,\dot\varphi^2\Big)=$$

$$=\frac{1}{2}m\left(\dot{\Delta l}^2+(l_0+\Delta l)^2\dot\varphi^2\right)=\frac{1}{2}m\left(v_r^2+v_\varphi^2\right)$$

így is adódik, ez akkor használható ha a „centrum" áll !

6. $$\frac{\partial L}{\partial\varphi}=0$$

$$\frac{\partial L}{\partial\dot\varphi}=m(l_0+\Delta l)^2\dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right)=2m(l_0+\Delta l)\dot{\Delta l}\dot\varphi+m(l_0+\Delta l)^2\ddot\varphi$$

$$\rightarrow\quad 2\cancel{m(l_0+\Delta l)}\dot{\Delta l}\dot\varphi+\cancel{m}(l_0+\Delta l)^{\cancel{2}}\ddot\varphi=0\quad,\;2\dot{\Delta l}\dot\varphi+(l_0+\Delta l)\ddot\varphi=0$$

avagy:

$$m(l_0+\Delta l)^2\dot\varphi\equiv N=\text{állandó}$$

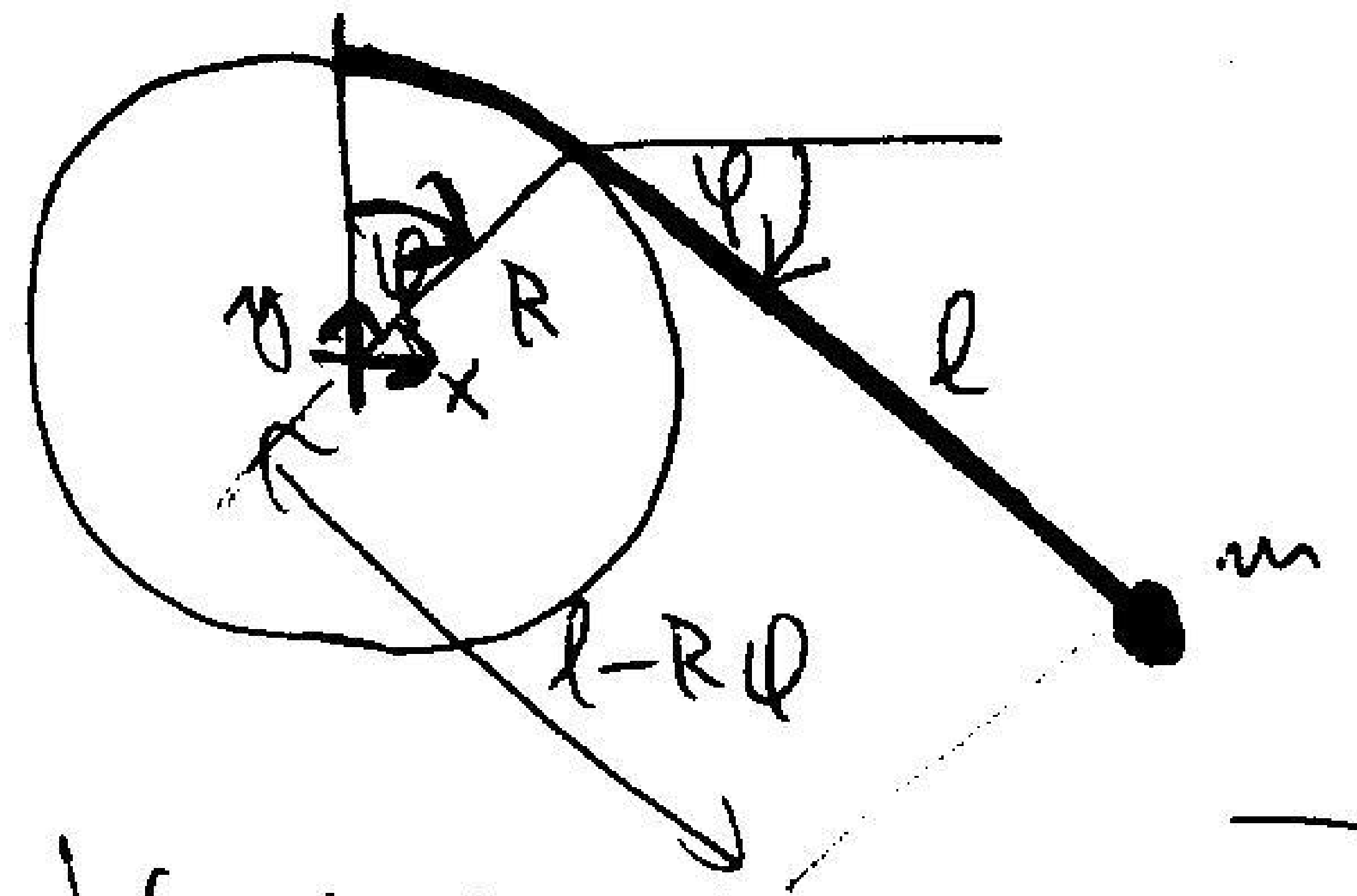
$$\frac{\partial L}{\partial\Delta l}=-k\Delta l+m(l_0+\Delta l)\dot\varphi^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\Delta l}}\right)=m\ddot{\Delta l}$$

$$\rightarrow\quad m\ddot{\Delta l}=-k\Delta l+m(l_0+\Delta l)\dot\varphi^2$$

Valójában egyfajta centrális potenciálban történő mozgásról van szó.

⑧



1. x, y
2. a kötél érintőirányban „hagyja el" a korongot
3. φ
4. $x = R\sin\varphi + (l-R\varphi)\cos\varphi$

   $y = R\cos\varphi - (l-R\varphi)\sin\varphi$

   érintési pont helye

   a tömegpont helye az érintési ponthoz viszonyítva

   $\dot x = R\cancel{\cos\varphi\,\dot\varphi} - \cancel{R\dot\varphi\cos\varphi} -$
   $-(l-R\varphi)\sin\varphi\,\dot\varphi = -(l-R\varphi)\sin\varphi\,\dot\varphi$

   $\dot y = \cancel{-R\sin\varphi\,\dot\varphi} + \cancel{R\dot\varphi\sin\varphi} -$
   $-(l-R\varphi)\cos\varphi\,\dot\varphi = -(l-R\varphi)\cos\varphi\,\dot\varphi$

5. $V = mgy =$

$= mg(R\cos\varphi - (l-R\varphi)\sin\varphi)$

$T = \frac{1}{2}m(\dot x^2 + \dot y^2) =$

$= \frac{1}{2}m((l-R\varphi)^2\sin^2\varphi\,\dot\varphi^2 +$

$+ (l-R\varphi)^2\cos^2\varphi\,\dot\varphi^2) =$

$= \frac{1}{2}m(l-R\varphi)^2\dot\varphi^2$ ⊛

6. $\frac{\partial L}{\partial\varphi} = mg(R\sin\varphi - \cancel{R\sin\varphi} + (l-R\varphi)\cos\varphi) = mg(l-R\varphi)\cos\varphi -$

$+m(l-R\varphi)\cdot(-R)\dot\varphi^2$ | $-mR\cdot(l-R\varphi)\dot\varphi^2$

$\frac{\partial L}{\partial\dot\varphi} = m(l-R\varphi)^2\dot\varphi$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) = 2m(l-R\varphi)\cdot(-R\dot\varphi)\cdot\dot\varphi + m(l-R\varphi)^2\ddot\varphi$

$\rightarrow \quad \cancel{m}g\cancel{(l-R\varphi)}\cos\varphi = \cancel{-2}\cancel{m}\cancel{(l-R\varphi)}R\dot\varphi^2 + \cancel{m}(l-R\varphi)^{\cancel{2}}\ddot\varphi$

$g\cos\varphi = \cancel{-2}R\dot\varphi^2 + (l-R\varphi)\ddot\varphi$

⊛ Tehát $\underline{v}^2 = \dot x^2 + \dot y^2 = v_\varphi^2$. A tanulság az, hogy
$(v_r \equiv 0)$

a „centrum" <u>pillanatonként</u> állónak tekinthető ebben

az elrendezésben.

9.

A teljes mechanikai energia:

$$E = T + V = \frac{1}{2} m \underline{v}^2 + V(r) = \frac{1}{2} m (v_r^2 + v_\varphi^2) + V(r) =$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r)$$

Az effektív potenciálban történő mozgás mechanikai energiája:

$$E = T + V_{eff} = \frac{1}{2} m \dot{r}^2 + \frac{N^2}{2mr^2} + V(r) = \frac{1}{2} m \dot{r}^2 +$$

$$+ \frac{(mr^2\dot{\varphi})^2}{2mr^2} + V(r) = \frac{1}{2} m \dot{r}^2 + \frac{mr^2\dot{\varphi}^2}{2} + V(r) = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)$$

10.

$$V_{eff}(r) = \frac{N^2}{2mr^2} - \frac{GmM}{r}$$

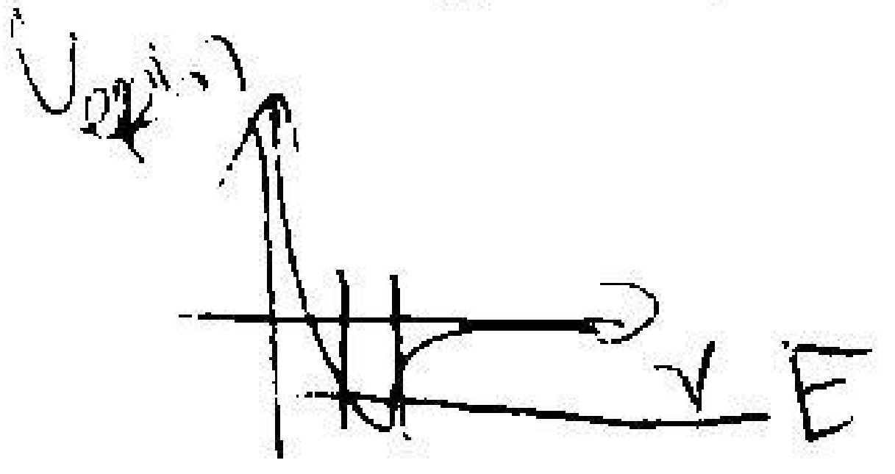
$$V_{eff}(r_{fp}) \overset{!}{=} E$$

$$\rightarrow \quad E r_{fp}^2 + GmM r_{fp} - \frac{N^2}{2m} = 0$$

vagy ∓, ha E<0

$$r_{fp\,1,2} = \frac{-GmM \pm \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} = -\frac{GmM}{2E} \pm \sqrt{\frac{G^2 m^2 M^2}{4E^2} + \frac{N^2}{2mE}}$$

- $E < 0$: $\quad -\frac{GmM}{2E} > 0, \quad \sqrt{\frac{G^2 m^2 M^2}{4E^2} + \underbrace{\frac{N^2}{2mE}}_{<0}} < -\frac{GmM}{2E}$



$\Longrightarrow$ 2 megoldás van

- $\lim\limits_{E \to 0-} E$: $\quad r_{fp\,2} = \frac{-GmM - \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} \xrightarrow[\substack{\downarrow \\ 0-}]{E} +\infty$, ui. mindkét tag külön-külön is így viselkedik (E<0!)

$$\lim_{E \to 0-} r_{fp1} = \lim_{E \to 0-} \frac{-GmM + \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} =$$

l'Hospital-szabály (a nevező és a számláló is 0-hoz tart)

$$= \lim_{E \to 0-} \frac{0 + \frac{1}{2}\left(G^2 m^2 M^2 + 4E \frac{N^2}{2m}\right)^{-\frac{1}{2}} \cdot \frac{4N^2}{2m}}{2} =$$

$$= \frac{1}{4} \frac{1}{GmM} \frac{4N^2}{2m} = \frac{N^2}{2Gm^2M}$$

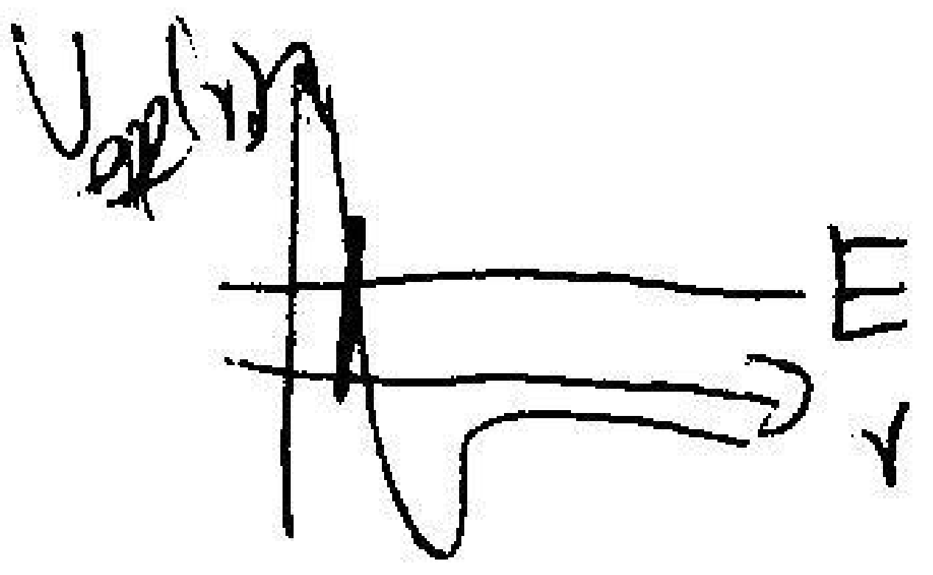
Ez tehát véges marad.

- $E=0$:

$$GmMr_{fp}-\frac{N^2}{2m}>0$$

$$r_{fp}=\frac{N^2}{2Gm^2M}$$ (Megegyezik r_{fp1} határértékével $E\to 0$-esetben.)

- $E>0$:



$$-\frac{GmM}{2E}<0 \quad , \quad \sqrt{\frac{G^2m^2M^2}{4E^2}+\underbrace{\frac{N^2}{2mE}}_{>0}} > \left|-\frac{GmM}{2E}\right|$$

$\Rightarrow$ Csak a pozitív előjeles megoldás lesz >0, az valóban csak egy. 1 megoldás van.

Pályaalakok:

$E<0$: ellipszis, r-ben 2 fordulópont: napközel és naptávol
$E=0$: parabola } r-ben 1 fordulópont: csak napközel
$E>0$: hiperbola }

11.

$$V(r) = \alpha r, \quad \alpha > 0$$

$$V_{eff}(r) = \frac{N^2}{2mr^2} + \alpha r$$

$$V'_{eff}(r) = -\frac{N^2}{mr^3} + \alpha$$

$$V'_{eff}(r)\Big|_{r=r^*} = -\frac{N^2}{mr^{*3}} + \alpha \overset{!}{=} 0$$

$$\rightarrow r^* = \sqrt[3]{\frac{N^2}{m\alpha}}$$

$$\boxed{\Omega^* = \frac{N}{mr^{*2}} = \frac{N}{m} m^{+2/3} \alpha^{+2/3} N^{-4/3} = \frac{\alpha^{2/3}}{m^{1/3} N^{1/3}}}$$

$$V''_{eff}(r) = 3\frac{N^2}{mr^4}$$

$$V''_{eff}(r)\Big|_{r=r^*} = 3 \cdot \frac{N^2}{m} \cdot \left(\frac{m\alpha}{N^2}\right)^{4/3} = 3N^{-2/3} m^{1/3} \alpha^{4/3}$$

$$V''_{eff}(r)\Big|_{r=r^*} > 0$$

$\Rightarrow$ A körpálya stabil.

$$\omega = \sqrt{\frac{3N^{-2/3} m^{1/3} \alpha^{4/3}}{m}} = \sqrt{3} \cdot \frac{\alpha^{2/3}}{|N|^{1/3} m^{1/3}}$$

$$\frac{\omega}{|\Omega^*|} = \sqrt{3} \notin \mathbb{Q}$$

$\Rightarrow$ A kis rezgések pályája nem záródik.

12.

$$V(r) = \alpha r, \quad \alpha > 0$$

$$r^* = R \rightarrow N = ?$$

$$V_{eff}(r) = \frac{N^2}{2mr^2} + \alpha r$$

$$V'_{eff}(r) = -\frac{N^2}{mr^3} + \alpha$$

$$V'_{eff}(r)\Big|_{r=R} = -\frac{N^2}{mR^3} + \alpha \overset{!}{=} 0$$

$$\rightarrow N = \pm\sqrt{\alpha m R^3}$$

Fontos, két körüljárási irányt ír le.

13.

$$\varphi(t)=\Omega t \Rightarrow \dot{\varphi}(t)=\Omega=\text{áll.}$$

$$N=mr^2\dot{\varphi} \Rightarrow r(t)=\sqrt{\frac{N}{m\dot{\varphi}(t)}}=\sqrt{\frac{N}{m\Omega}}=\text{áll.}=r_{eq}$$

$$v=\sqrt{v_r^2+v_\varphi^2}\underset{\uparrow}{=}|v_\varphi|=r|\dot{\varphi}|=\sqrt{\frac{N}{m\Omega}}\cdot|\Omega|=\sqrt{\frac{N\Omega}{m}}$$

$$v_r=\dot{r}=0$$

$$V_{eff}(r)=\frac{N^2}{2mr^2}+\frac{1}{4}\gamma r^4$$

$$\frac{\partial V_{eff}(r)}{\partial r}=-\frac{N^2}{mr^3}+\gamma r^3$$

$$\left.\frac{\partial V_{eff}(r)}{\partial r}\right|_{r=r_{eq}}=-\frac{N^2}{mr_{eq}^3}+\gamma r_{eq}^3=0$$

$$\Rightarrow \gamma=\frac{N^2}{mr_{eq}^6}=\frac{N^2}{m\frac{N^3}{m^3\Omega^3}}=\frac{m^2\Omega^3}{N}$$

$$\frac{\partial^2 V_{eff}(r)}{\partial r^2}=3\frac{N^2}{mr^4}+3\gamma r^2$$

$$\left.\frac{\partial^2 V_{eff}(r)}{\partial r^2}\right|_{r=r_{eq}}=3\frac{N^2}{mr_{eq}^4}+3\gamma r_{eq}^2=3\frac{\cancel{N^2}}{m\frac{\cancel{N^2}}{m^2\Omega^2}}+3\cdot\frac{m^2\Omega^3}{\cancel{N}}\cdot\frac{\cancel{N}}{m\Omega}=$$

$$=6m\Omega^2$$

$$\omega=\sqrt{\frac{1}{m}\left.\frac{\partial^2 V_{eff}(r)}{\partial r^2}\right|_{r=r_{eq}}}=\sqrt{6}|\Omega|$$

(14.)

A szögelfordulást a következő összefüggés kapcsolja össze a radiális mozgással:

$$\dot{\varphi} = \frac{N}{mr^2}$$

$$\varphi(t)=\varphi(t_0)+\int_{t_0}^{t}\frac{N}{mr(t')^2}\,dt'$$

Ha a mozgás korlátos, akkor az $r(t)$ függvénynek van felső korlátja: $r(t)<R \ \forall t$, ahol R egy felső korlát. Ezzel:

$$\left|\varphi(t)-\varphi(t_0)\right|=\left|\int_{t_0}^{t}\frac{N}{mr(t')^2}\,dt'\right| > \left|\int_{t_0}^{t}\frac{N}{mR^2}\,dt'\right|=\frac{|N|}{mR^2}(t-t_0)$$

Az eredményből látható, hogy ha „t”-t kellően nagynak választjuk („eleget várunk”), akkor a szögelfordulás tetszőlegesen naggyá tehető, így a nagysága biztosan átlépi a 2π értéket is.

Az eredményt úgy is megfogalmazhatjuk, hogy ha a tömegpont nem szökik el, akkor biztosan megkerüli az origót, ill. „kering” körülötte.

Ugyanez nem igaz, ha a tömegpont elszökik, azaz az origótól mért távolsága tetszőlegesen naggyá válik: gondoljunk például a Kepler-probléma hiperbolapályájára.