1.

1. x, y
2. $h(x)$
3. x
4. $x \qquad \dot{x}$

$y = h(x) \qquad \dot{y} = \frac{dh(x)}{dx}\cdot\dot{x} \equiv h'(x)\cdot\dot{x}$

5. $V = mgy = mgh(x)$

$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{x}^2 + h'(x)^2\,\dot{x}^2\right) = \frac{1}{2}m\left(1 + h'(x)^2\right)\dot{x}^2$

6. $\frac{\partial L}{\partial x} = -mgh'(x) + \frac{1}{\cancel{2}}m\cdot\cancel{2}\,h'(x)\cdot h''(x)\cdot\dot{x}^2$

$$\frac{\partial L}{\partial \dot{x}} = m\left(1 + h'(x)^2\right)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\cdot 2h'(x)\cdot h''(x)\cdot\dot{x}\cdot\dot{x} + m\left(1 + h'(x)^2\right)\ddot{x}$$

$\longrightarrow$

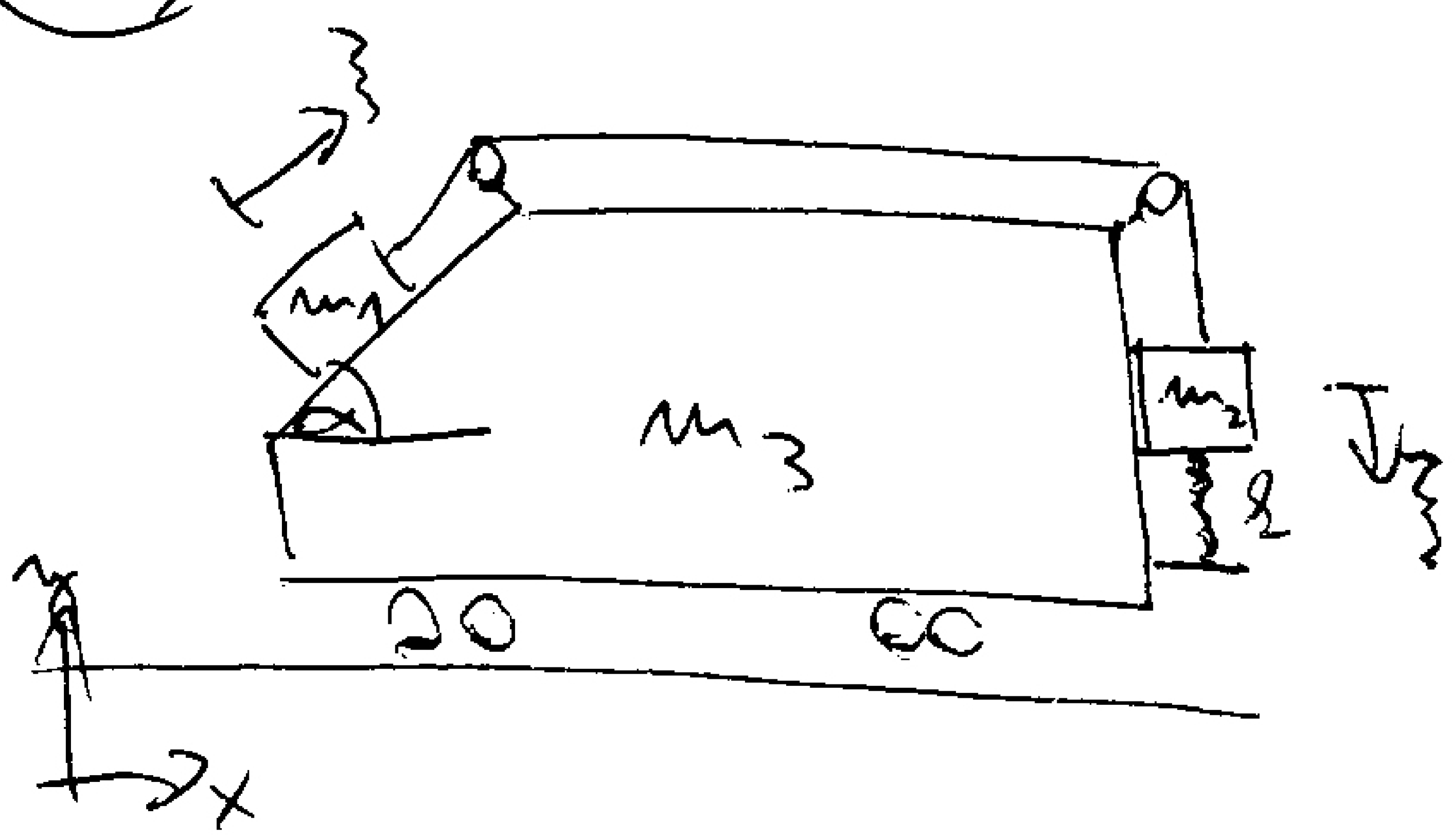
$$2m\,h'(x)\,h''(x)\,\dot{x}^2 + m\left(1 + h'(x)^2\right)\ddot{x} =$$

$$= -mgh'(x) + m\,h'(x)\,h''(x)\,\dot{x}^2$$

$$m\,\ddot{x} = \frac{1}{1 + h'(x)^2}\cdot\left(-mgh'(x) - m\,h'(x)\,h''(x)\,\dot{x}^2\right)$$

A sebességtől függő tag azért jelenik meg, mert $h''(x) \neq 0$ esetén a gyorsulásnak van a lejtő lokális érintőjére merőleges komponense is. Ez egyfajta centripetális gyorsulásnak tekinthető, ez tartja az előírt $h(x)$ pályán a tömegpontot.

2.



1., $x_1, y_1, x_2, y_2, x_3, y_3$

2., • az 1-es test a lejtőn mozog
• a 2-es test nem leng ki
• $y_3 =$ állandó
• kötél

3., x_3, ξ

4., $x_1 = x_3 + c_1 + \xi \cos\alpha$ $\qquad$ $\dot{x}_1 = \dot{x}_3 + \dot{\xi}\cos\alpha$

konstans

$y_1 = y_3 + d_1 + \xi \sin\alpha =$ $\qquad$ $\dot{y}_1 = \dot{\xi}\sin\alpha$

$= d_3 + d_1 + \xi\sin\alpha$

$x_2 = x_3 + c_2$ $\qquad$ $\dot{x}_2 = \dot{x}_3$

konstans

$y_2 = y_3 + d_2 - \xi =$ $\qquad$ $\dot{y}_2 = -\dot{\xi}$

$= d_3 + d_2 - \xi$

$x_3 = x_3$ $\qquad$ $\dot{x}_3 = \dot{x}_3$

$y_3 = d_3$ $\qquad$ $\dot{y}_3 = 0$

konstans

5.) $L = T - V$

$$T = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2} m_3 \left(\dot{x}_3^2 + \dot{y}_3^2\right) =$$

$$= \frac{1}{2} m_1 \Big(\dot{x}_3^2 + 2\dot{x}_3 \dot{\xi} \cos\alpha + \underbrace{\dot{\xi}^2 \cos^2\alpha + \dot{\xi}^2 \sin^2\alpha}_{\dot{\xi}^2}\Big) + \frac{1}{2} m_2 \left(\dot{x}_3^2 + \dot{\xi}^2\right) +$$

$$+ \frac{1}{2} m_3 \dot{x}_3^2 = \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}_3^2 + m_1 \dot{x}_3 \dot{\xi} \cos\alpha + \frac{1}{2}(m_1 + m_2)\dot{\xi}^2$$

$$V = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 + \frac{1}{2} k \xi^2 =$$

Ez mind konstans!

$$= m_1 g d_3 + m_1 g d_1 + m_2 g d_3 + m_2 g d_2 + m_3 g d_3 +$$

$$+ m_1 g \xi \sin\alpha \quad - m_2 g \xi + \frac{1}{2} k \xi^2$$

6.) $\dfrac{\partial L}{\partial x_3} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = (m_1 + m_2 + m_3)\ddot{x}_3 + m_1 \ddot{\xi} \cos\alpha$$

$$\rightarrow \quad (m_1 + m_2 + m_3)\ddot{x}_3 + m \ddot{\xi} \cos\alpha = 0$$

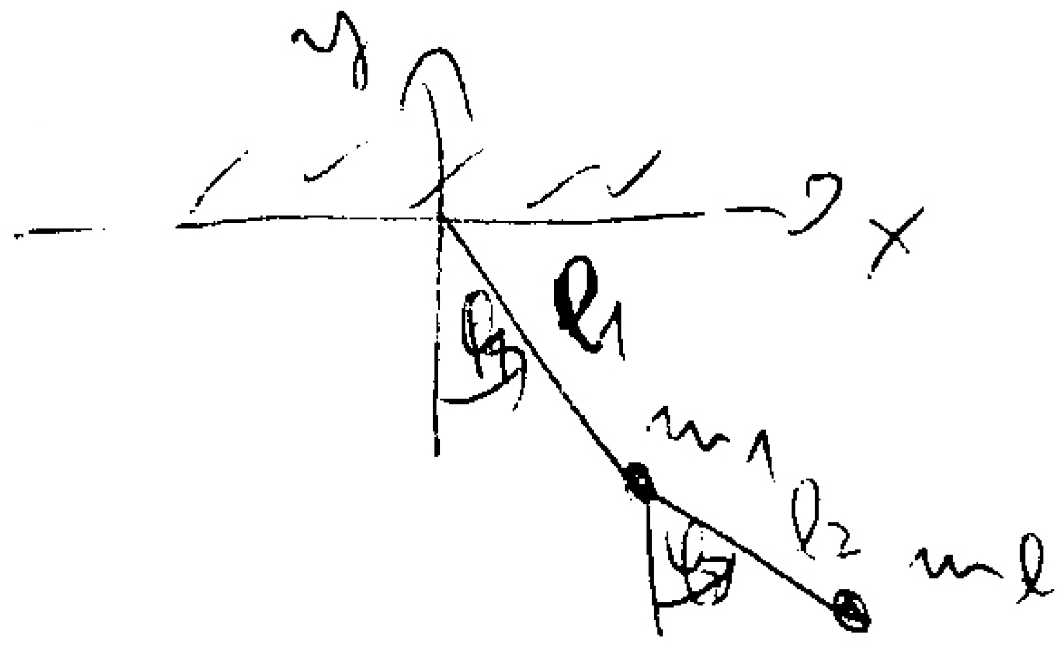
$$\frac{\partial L}{\partial \xi} = -m_1 g \sin\alpha + m_2 g - k\xi$$

$$\frac{\partial L}{\partial \dot{\xi}} = m_1 \dot{x}_3 \cos\alpha + (m_1 + m_2)\dot{\xi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = m_1 \ddot{x}_3 \cos\alpha + (m_1 + m_2)\ddot{\xi}$$

$$\rightarrow \quad -m_1 g \sin\alpha + m_2 g - k\xi = m_1 \ddot{x}_3 \cos\alpha + (m_1 + m_2)\ddot{\xi}$$

3.



1., x_1, y_1, x_2, y_2

2., 2 db fonál

3., φ_1, φ_2

4., $x_1 = l_1 \sin\varphi_1$

$y_1 = -l_1 \cos\varphi_1$

$x_2 = x_1 + l_2 \sin\varphi_2 =$
$= l_1 \sin\varphi_1 + l_2 \sin\varphi_2$

$y_2 = y_1 - l_2 \cos\varphi_2 =$
$= -l_1 \cos\varphi_1 - l_2 \cos\varphi_2$

$\dot{x}_1 = l_1 \cos\varphi_1 \dot{\varphi}_1$

$\dot{y}_1 = l_1 \sin\varphi_1 \dot{\varphi}_1$

$\dot{x}_2 = l_1 \cos\varphi_1 \dot{\varphi}_1 + l_2 \cos\varphi_2 \dot{\varphi}_2$

$\dot{y}_2 = l_1 \sin\varphi_1 \dot{\varphi}_1 + l_2 \sin\varphi_2 \dot{\varphi}_2$

5., $L = T - V$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) =$$

$$= \frac{1}{2} m_1 (l_1^2 \cos^2\varphi_1 \dot{\varphi}_1^2 + l_1^2 \sin^2\varphi_1 \dot{\varphi}_1^2) + \Rightarrow$$

$$+\frac{1}{2}m_2\Big(l_1^2\cos^2\varphi_1\dot\varphi_1^2+2l_1\cos\varphi_1\dot\varphi_1\, l_2\cos\varphi_2\dot\varphi_2+l_2^2\cos^2\varphi_2\dot\varphi_2^2+$$

$$+l_1^2\sin^2\varphi_1\dot\varphi_1^2+2l_1\sin\varphi_1\dot\varphi_1\, l_2\sin\varphi_2\dot\varphi_2+l_2^2\sin^2\varphi_2\dot\varphi_2^2\Big)=$$

$$=\frac{1}{2}m_1 l_1^2\dot\varphi_1^2+\frac{1}{2}m_2\Big[l_1^2\dot\varphi_1^2+l_2^2\dot\varphi_2^2+2l_1l_2\underbrace{(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)}_{\cos(\varphi_2-\varphi_1)}\dot\varphi_1\dot\varphi_2\Big]=$$

$$=\frac{1}{2}(m_1+m_2)l_1^2\dot\varphi_1^2+\frac{1}{2}m_2 l_2^2\dot\varphi_2^2+m_2 l_1 l_2\cos(\varphi_2-\varphi_1)\dot\varphi_1\dot\varphi_2$$

$$V=m_1 g y_1+m_2 g y_2=\quad -m_1 g l_1\cos\varphi_1+m_2 g(-l_1\cos\varphi_1-l_2\cos\varphi_2)=$$

$$=-(m_1+m_2)g l_1\cos\varphi_1-m_2 g l_2\cos\varphi_2$$

E.1 $$\frac{\partial L}{\partial\varphi_1}=m_2 l_1 l_2\dot\varphi_1\dot\varphi_2\cdot\underbrace{\big(-\sin(\varphi_2-\varphi_1)\big)\cdot(-1)}_{\text{„láncszabály"!}}-(m_1+m_2)g l_1\sin\varphi_1$$

$$\frac{\partial}{\partial\varphi_1}(\varphi_2-\varphi_1)=-1$$

$$\frac{\partial L}{\partial\dot\varphi_1}=(m_1+m_2)l_1^2\dot\varphi_1+m_2 l_1 l_2\cos(\varphi_2-\varphi_1)\dot\varphi_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi_1}\right)=(m_1+m_2)l_1^2\ddot\varphi_1+m_2 l_1 l_2\frac{d}{dt}\big(\cos(\varphi_2-\varphi_1)\big)\dot\varphi_2+$$

$$+m_2 l_1 l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2=(m_1+m_2)l_1^2\ddot\varphi_1+$$

$$+m_2 l_1 l_2\big(-\sin(\varphi_2-\varphi_1)\big)\cdot(\dot\varphi_2-\dot\varphi_1)\cdot\dot\varphi_2+m_2 l_1 l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2$$

$$\Rightarrow\ \cancel{m_2 l_1 l_2\dot\varphi_1\dot\varphi_2\sin}(\varphi_2-\varphi_1)-(m_1+m_2)g l_1\sin\varphi_1=$$

$$=(m_1+m_2)l_1^2\ddot\varphi_1-m_2 l_1 l_2\sin(\varphi_2-\varphi_1)\cdot(\dot\varphi_2^2-\cancel{\dot\varphi_1\dot\varphi_2})+m_2 l_1 l_2\cos(\varphi_2-\varphi_1)\ddot\varphi_2$$

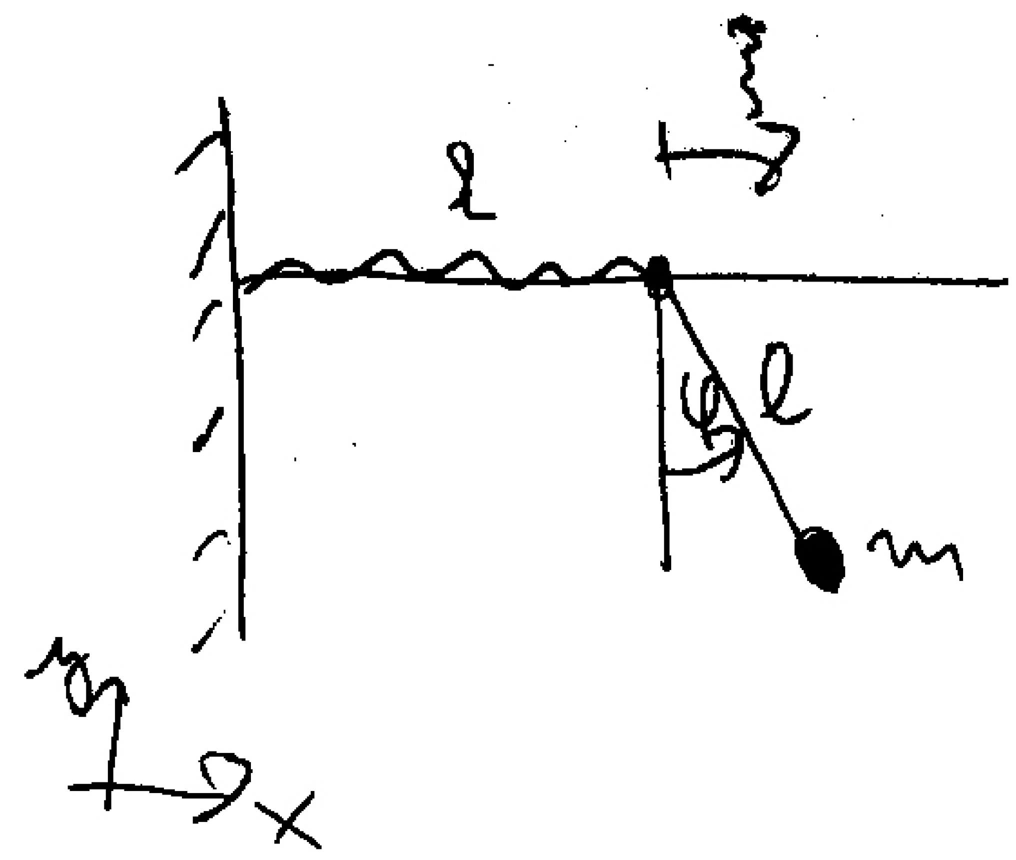
$$\frac{\partial L}{\partial \varphi_2} = m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 (-\sin(\varphi_2 - \varphi_1)) \cdot 1 - m_2 g l_2 \sin \varphi_2$$

$$\frac{\partial L}{\partial \dot{\varphi}_2} = m_2 l_2^2 \dot{\varphi}_2 + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \dot{\varphi}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 \cdot (-\sin(\varphi_2 - \varphi_1)) \cdot (\dot{\varphi}_2 - \dot{\varphi}_1) \cdot \dot{\varphi}_1 +$$
$$+ m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \ddot{\varphi}_1$$

$$\Rightarrow \cancel{- \dot{\varphi}_1 \dot{\varphi}_2 m_2 l_1 l_2 \sin(\varphi_2 - \varphi_1)} - m_2 g l_2 \sin \varphi_2 =$$
$$= m_2 l_2^2 \ddot{\varphi}_2 - m_2 l_1 l_2 \sin(\varphi_2 - \varphi_1) \cdot (\cancel{\dot{\varphi}_1 \dot{\varphi}_2} - \dot{\varphi}_1^2) +$$
$$+ m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \ddot{\varphi}_1$$

4.



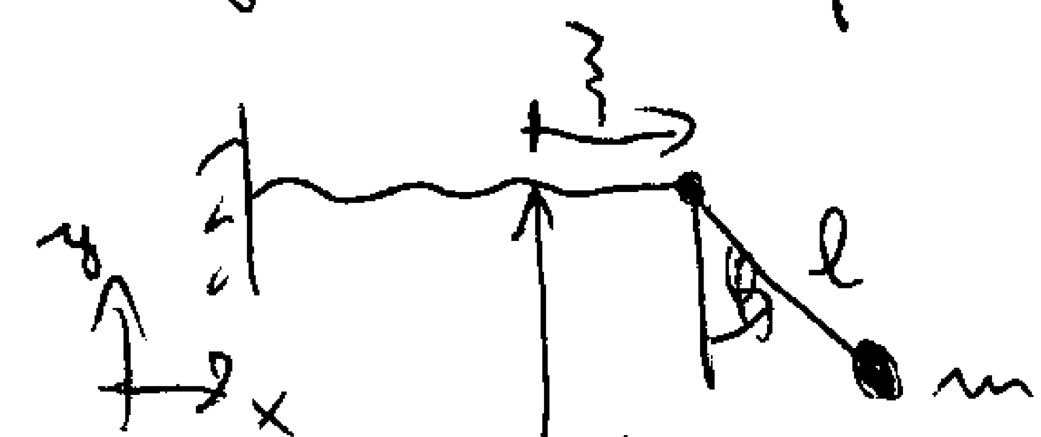
1., x, y (az inga végpontjára)

2., –

3., ξ, φ → a rugó nyugalmi helyzetétől mérve → a rugó végpontjánál mérve

4., $x = c + \xi + l\sin\varphi$

$y = d - l\cos\varphi$



a rugó nyugalmi helyzete, ennek a koordinátái „c" és „d"

$\dot{x} = \dot{\xi} + l\cos\varphi\,\dot{\varphi}$

$\dot{y} = l\sin\varphi\,\dot{\varphi}$

5., $L = T - V$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) =$$

$$= \frac{1}{2}m\left(\dot{\xi}^2 + 2\dot{\xi}\,l\cos\varphi\,\dot{\varphi} + l^2\cos^2\varphi\,\dot{\varphi}^2 + l^2\sin^2\varphi\,\dot{\varphi}^2\right) =$$

$$= \frac{1}{2}m\left(\dot{\xi}^2 + 2l\cos\varphi\,\dot{\varphi}\,\dot{\xi} + l^2\dot{\varphi}^2\right)$$

$$V = mgy + \frac{1}{2}k\xi^2 = mgd - mgl\cos\varphi + \frac{1}{2}k\xi^2$$

6., $$\frac{\partial L}{\partial \xi} = -k\xi$$

$$\frac{\partial L}{\partial \dot{\xi}} = m\dot{\xi} + ml\cos\varphi\,\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = m\ddot{\xi} - ml\sin\varphi\cdot\dot{\varphi}\cdot\dot{\varphi} + ml\cos\varphi\cdot\ddot{\varphi}$$

$$\Rightarrow \quad -k\xi = m\ddot{\xi} - ml\sin\varphi\,\dot{\varphi}^2 + ml\cos\varphi\,\ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = -ml \sin\varphi\, \dot{\varphi}\, \dot{\xi} - mgl \sin\varphi$$

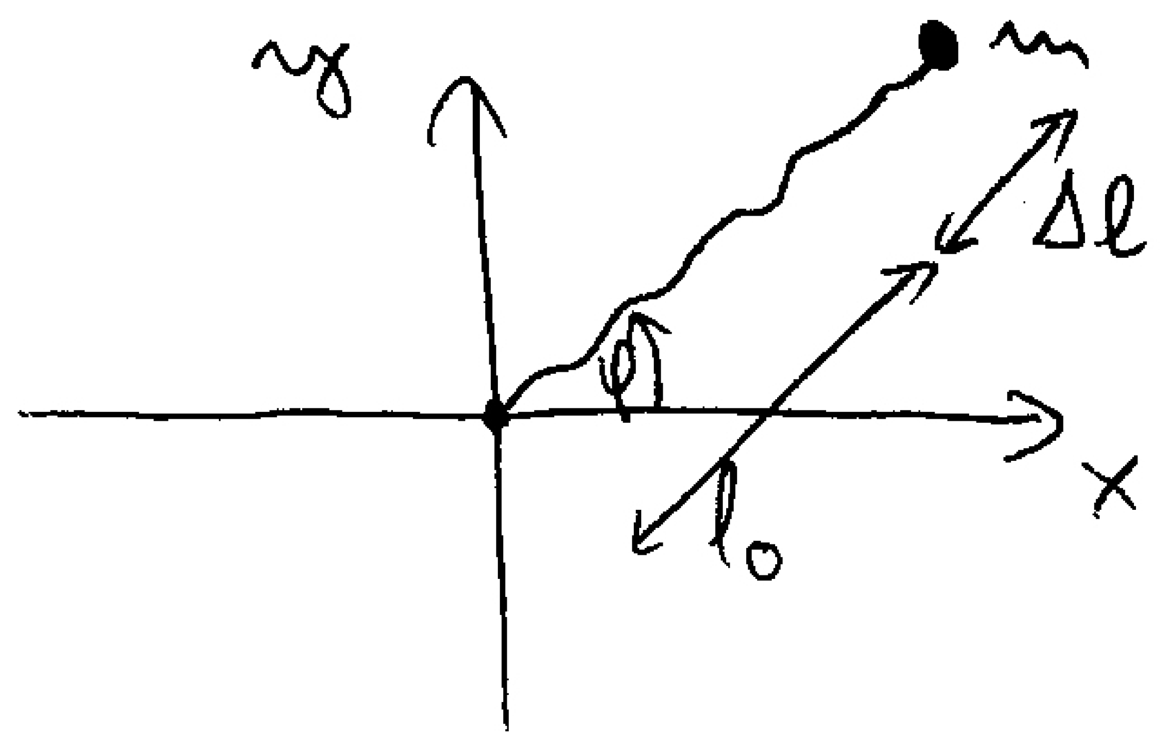
$$\frac{\partial L}{\partial \dot{\varphi}} = ml \cos\varphi\, \dot{\xi} + ml^2 \dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = -ml \sin\varphi \cdot \dot{\varphi} \cdot \dot{\xi} + ml \cos\varphi\, \ddot{\xi} + ml^2 \ddot{\varphi}$$

$$\Rightarrow \quad \cancel{-ml \sin\varphi\, \dot{\varphi}\, \dot{\xi}} - \cancel{m}gl \sin\varphi =$$

$$= \cancel{-ml \sin\varphi\, \dot{\varphi}\, \dot{\xi}} + \cancel{m}l \cos\varphi\, \ddot{\xi} + \cancel{m}l^2 \ddot{\varphi}$$

5.



1. x, y 3. $\varphi, \Delta l$
2. —
4. $x = (l_0 + \Delta l)\cos\varphi$
$y = (l_0 + \Delta l)\sin\varphi$
$\dot{x} = \dot{\Delta l}\cos\varphi - (l_0+\Delta l)\sin\varphi\,\dot{\varphi}$
$\dot{y} = \dot{\Delta l}\sin\varphi + (l_0+\Delta l)\cos\varphi\,\dot{\varphi}$

5. $V = \frac{1}{2}k(\Delta l)^2$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\Big(\dot{\Delta l}^2\cos^2\varphi - \cancel{2\dot{\Delta l}\cos\varphi(l_0+\Delta l)\sin\varphi\,\dot{\varphi}} + (l_0+\Delta l)^2\sin^2\varphi\,\dot{\varphi}^2 + \dot{\Delta l}^2\sin^2\varphi +$$
$$+ \cancel{2\dot{\Delta l}\sin\varphi(l_0+\Delta l)\cos\varphi\,\dot{\varphi}} + (l_0+\Delta l)^2\cos^2\varphi\,\dot{\varphi}^2\Big) =$$
$$= \frac{1}{2}m\left(\dot{\Delta l}^2 + (l_0+\Delta l)^2\dot{\varphi}^2\right) = \frac{1}{2}m\left(v_r^2 + v_\varphi^2\right)$$

így is adódik, ez akkor használható ha a „centrum" áll !

6. $$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m(l_0+\Delta l)^2\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 2m(l_0+\Delta l)\dot{\Delta l}\dot{\varphi} + m(l_0+\Delta l)^2\ddot{\varphi}$$

$$\rightarrow \quad 2\cancel{m(l_0+\Delta l)}\dot{\Delta l}\dot{\varphi} + \cancel{m}(l_0+\Delta l)^{\cancel{2}}\ddot{\varphi} = 0 \quad , \quad 2\dot{\Delta l}\dot{\varphi} + (l_0+\Delta l)\ddot{\varphi} = 0$$

$$\frac{\partial L}{\partial \Delta l} = -k\Delta l + m(l_0+\Delta l)\dot{\varphi}^2$$

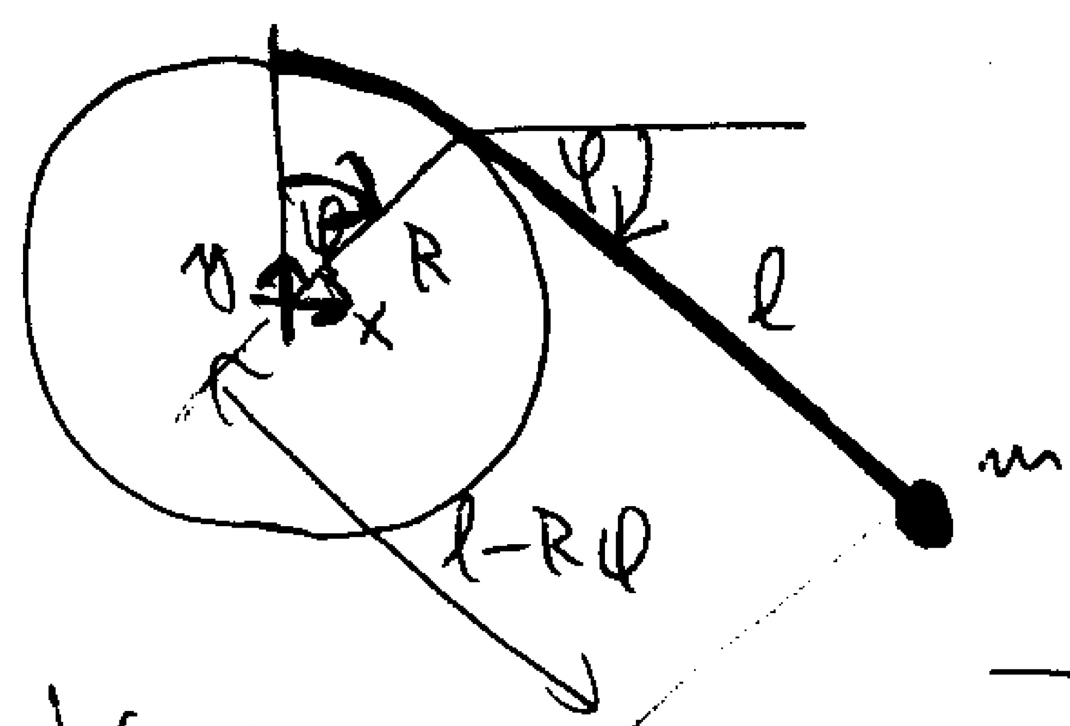
tehát:
$m(l_0+\Delta l)^2\dot{\varphi} \equiv N = \text{állandó}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Delta l}}\right) = m\ddot{\Delta l}$$

$$\rightarrow \quad m\ddot{\Delta l} = -k\Delta l + m(l_0+\Delta l)\dot{\varphi}^2$$

Valójában egyfajta centrális potenciálban történő mozgásról van szó.

6.



1. x, y
2. a kötél érintőirányban „hagyja el" a korongot
3. φ
4. $x = R\sin\varphi + (l-R\varphi)\cos\varphi$

   $y = R\cos\varphi - (l-R\varphi)\sin\varphi$

   ↑ érintési pont helye ↑ a tömegpont helye az érintési ponthoz viszonyítva

   $\dot x = R\cancel{\cos\varphi\,\dot\varphi} - \cancel{R\dot\varphi\cos\varphi} - (l-R\varphi)\sin\varphi\,\dot\varphi = -(l-R\varphi)\sin\varphi\,\dot\varphi$

   $\dot y = \cancel{-R\sin\varphi\,\dot\varphi} + \cancel{R\dot\varphi\sin\varphi} - (l-R\varphi)\cos\varphi\,\dot\varphi = -(l-R\varphi)\cos\varphi\,\dot\varphi$

5. $V = mgy =$

$= mg(R\cos\varphi - (l-R\varphi)\sin\varphi)$

$T = \frac{1}{2}m(\dot x^2 + \dot y^2) =$

$= \frac{1}{2}m((l-R\varphi)^2\sin^2\varphi\,\dot\varphi^2 +$

$+ (l-R\varphi)^2\cos^2\varphi\,\dot\varphi^2) =$

$= \frac{1}{2}m(l-R\varphi)^2\dot\varphi^2$ (*)

6. $\frac{\partial L}{\partial\varphi} = mg(R\sin\varphi - \cancel{R\sin\varphi} + (l-R\varphi)\cos\varphi) = mg(l-R\varphi)\cos\varphi - mR(l-R\varphi)\dot\varphi^2$

$+m(l-R\varphi)\cdot(-R)\dot\varphi^2$

$\frac{\partial L}{\partial\dot\varphi} = m(l-R\varphi)^2\dot\varphi$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) = 2m(l-R\varphi)\cdot(-R\dot\varphi)\cdot\dot\varphi + m(l-R\varphi)^2\ddot\varphi$

$\to \quad \cancel{m}g\cancel{(l-R\varphi)}\cos\varphi = \cancel{-}\cancel{m}\cancel{(l-R\varphi)}R\dot\varphi^2 + \cancel{m}(l-R\varphi)^{\cancel{2}}\ddot\varphi$

$g\cos\varphi = -\cancel{2}R\dot\varphi^2 + (l-R\varphi)\ddot\varphi$

(*) Tehát $v^2 = \dot x^2 + \dot y^2 = v_\varphi^2$. A tanulság az, hogy
($v_r \equiv 0$)

a „centrum" pillanatonként állónak tekinthető ebben

az elrendezésben.

7.

A teljes mechanikai energia:

$$E = T + V = \frac{1}{2} m \underline{v}^2 + V(r) = \frac{1}{2} m (v_r^2 + v_\varphi^2) + V(r) =$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r)$$

Az effektív potenciálban történő mozgás mechanikai energiája:

$$E = T + V_{eff} = \frac{1}{2} m \dot{r}^2 + \frac{N^2}{2mr^2} + V(r) = \frac{1}{2} m \dot{r}^2 +$$

$$+ \frac{(m r^2 \dot{\varphi})^2}{2mr^2} + V(r) = \frac{1}{2} m \dot{r}^2 + \frac{m r^2 \dot{\varphi}^2}{2} + V(r) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r)$$

8.

$$V_{eff}(r) = \frac{N^2}{2mr^2} - \frac{GmM}{r}$$

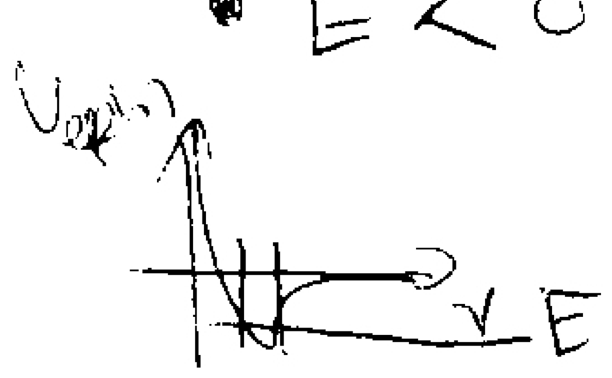
$$V_{eff}(r_{fp}) \overset{!}{=} E$$

$$\rightarrow \quad E r_{fp}^2 + GmM r_{fp} - \frac{N^2}{2m} = 0$$

vagy F, ha E<0

$$r_{fp\,1,2} = \frac{-GmM \pm \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} = -\frac{GmM}{2E} \pm \sqrt{\frac{G^2 m^2 M^2}{4E^2} + \frac{N^2}{2mE}}$$

- $E < 0$: $\quad -\frac{GmM}{2E} > 0, \quad \sqrt{\frac{G^2 m^2 M^2}{4E^2} + \underbrace{\frac{N^2}{2mE}}_{<0}} < -\frac{GmM}{2E}$



$\Longrightarrow$ 2 megoldás van

- $\lim\limits_{E \to 0-} E$. $\quad r_{fp\,2} = \frac{-GmM - \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} \xrightarrow[E \uparrow 0-]{} +\infty$, mindkét ui. ~~[illegible]~~ tag külön-külön is így viselkedik (E<0!)

$$\lim_{E \to 0-} r_{fp1} = \lim_{E \to 0-} \frac{-GmM + \sqrt{G^2 m^2 M^2 + 4E \frac{N^2}{2m}}}{2E} =$$

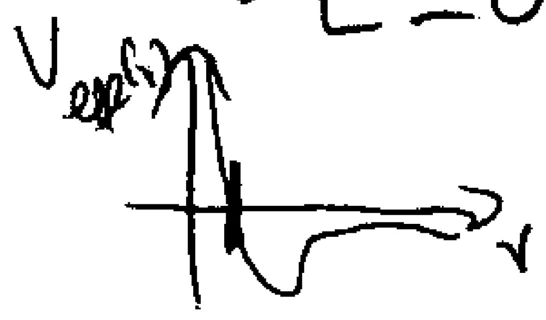
l'Hospital-szabály (a nevező és a számláló is 0-hoz tart)

$$= \lim_{E \to 0-} \frac{0 + \frac{1}{2}\left(G^2 m^2 M^2 + 4E \frac{N^2}{2m}\right)^{-\frac{1}{2}} \cdot \frac{4N^2}{2m}}{2} =$$

$$= \frac{1}{4} \frac{1}{GmM} \frac{4N^2}{2m} = \frac{N^2}{2Gm^2M}$$

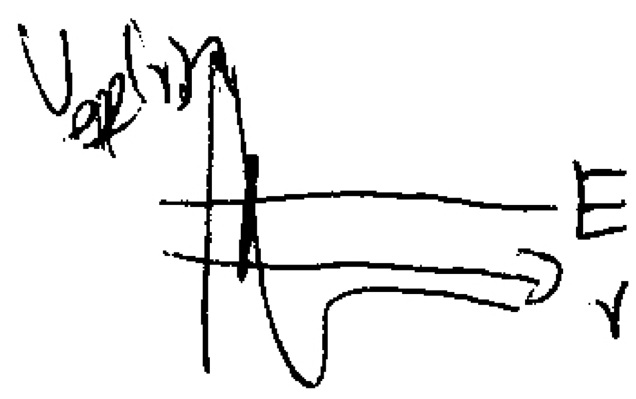
Ez tehát véges marad.

- $E = 0$: $\quad GmMr_{fp} - \frac{N^2}{2m} = 0$



$$r_{fp} = \frac{N^2}{2Gm^2M} \qquad \text{(Megegyezik } r_{fp1} \text{ határértékével } E\to 0\text{-esetben.)}$$

- $E > 0$: $\quad -\frac{GmM}{2E} < 0, \quad \sqrt{\frac{G^2m^2M^2}{4E^2} + \underbrace{\frac{N^2}{2mE}}_{>0}} > \left| -\frac{GmM}{2E} \right|$



$\Rightarrow$ Csak a pozitív előjeles megoldás lesz >0, az valóban csak egy 1 megoldás van.

Pályaalakok:

$E < 0$: ellipszis, r-ben 2 fordulópont: napközel és naptávol
$E = 0$: parabola
$E > 0$: hiperbola

$\left.\begin{array}{l}\text{parabola}\\ \text{hiperbola}\end{array}\right\}$ r-ben 1 fordulópont: csak napközel

(9.)

EMB/Gy2/13

$$V(r) = \alpha r\,, \quad \alpha > 0$$

$$V_{eff}(r) = \frac{N^2}{2mr^2} + \alpha r$$

$$V_{eff}'(r) = -\frac{N^2}{mr^3} + \alpha$$

$$V_{eff}'(r)\Big|_{r=r^*} = -\frac{N^2}{mr^{*3}} + \alpha \overset{!}{=} 0 \quad \Rightarrow \quad r^* = \sqrt[3]{\frac{N^2}{m\alpha}}$$

$$\boxed{\Omega^* = \frac{N}{mr^{*2}} = \frac{N}{m} m^{+2/3} \alpha^{+2/3} N^{-4/3} = \frac{\alpha^{2/3}}{m^{1/3} N^{1/3}}}$$

$$V_{eff}''(r) = 3\frac{N^2}{mr^4}$$

$$V_{eff}''(r)\Big|_{r=r^*} = 3 \cdot \frac{N^2}{m} \cdot \left(\frac{m\alpha}{N^2}\right)^{4/3} = 3N^{-2/3} m^{1/3} \alpha^{4/3}$$

$$V_{eff}''(r)\Big|_{r=r^*} > 0$$

$\Rightarrow$ A körpálya stabil.

$$\omega = \sqrt{\frac{3N^{-2/3} m^{1/3} \alpha^{4/3}}{m}} = \sqrt{3} \cdot \frac{\alpha^{2/3}}{|N|^{1/3} m^{1/3}}$$

$$\frac{\omega}{|\Omega^*|} = \sqrt{3} \notin \mathbb{Q}$$

$\Rightarrow$ A kis rezgések pályája nem záródik.

(10.)

$$V(r) = \alpha r\,, \quad \alpha > 0$$

$$r^* = R \quad \Rightarrow \quad N = ?$$

$$V_{eff}(r) = \frac{N^2}{2mr^2} + \alpha r$$

$$V_{eff}'(r) = -\frac{N^2}{mr^3} + \alpha$$

$$V_{eff}'(r)\Big|_{r=R} = -\frac{N^2}{mR^3} + \alpha \overset{!}{=} 0$$

$$\Rightarrow \quad N = \pm\sqrt{\alpha m R^3}$$

Fontos, két körüljárási irány is van.

11.

$$\varphi(t) = \Omega t \Rightarrow \dot\varphi(t) = \Omega = \text{áll.}$$

$$N = m r^2 \dot\varphi \Rightarrow r(t) = \sqrt{\frac{N}{m\dot\varphi(t)}} = \sqrt{\frac{N}{m\Omega}} = \text{áll.} = r_{eq}$$

$$v = \sqrt{v_r^2 + v_\varphi^2} \underset{\uparrow}{=} |v_\varphi| = r|\dot\varphi| = \sqrt{\frac{N}{m\Omega}} \cdot |\Omega| = \sqrt{\frac{N\Omega}{m}}$$

$$v_r = \dot r = 0$$

$$V_{eff}(r) = \frac{N^2}{2mr^2} + \frac{1}{4}\gamma r^4$$

$$\frac{\partial V_{eff}(r)}{\partial r} = -\frac{N^2}{mr^3} + \gamma r^3$$

$$\left.\frac{\partial V_{eff}(r)}{\partial r}\right|_{r=r_{eq}} = -\frac{N^2}{m r_{eq}^3} + \gamma r_{eq}^3 = 0$$

$$\Rightarrow \gamma = \frac{N^2}{m r_{eq}^6} = \frac{N^2}{m\frac{N^3}{m^3\Omega^3}} = \frac{m^2\Omega^3}{N}$$

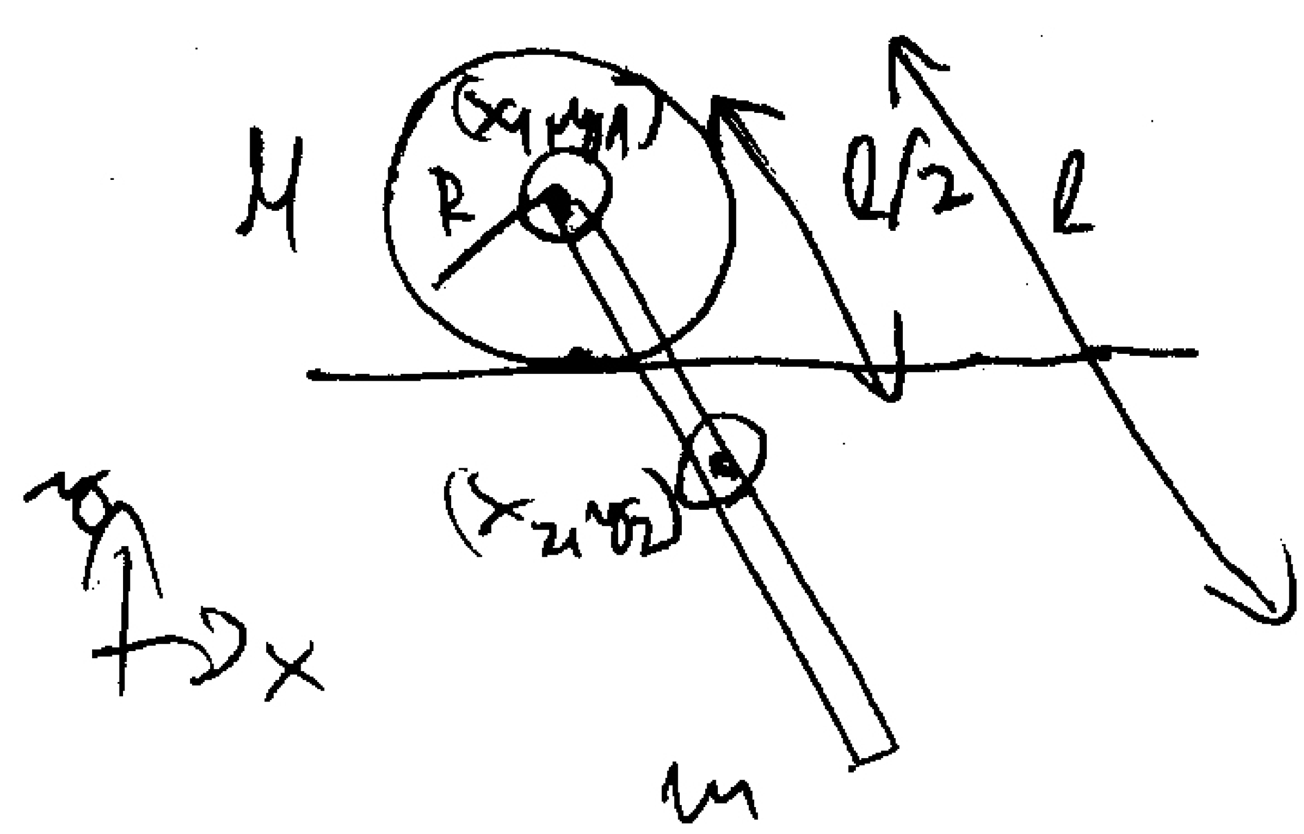
$$\frac{\partial^2 V_{eff}(r)}{\partial r^2} = 3\frac{N^2}{mr^4} + 3\gamma r^2$$

$$\left.\frac{\partial^2 V_{eff}(r)}{\partial r^2}\right|_{r=r_{eq}} = 3\frac{N^2}{m r_{eq}^4} + 3\gamma r_{eq}^2 = 3\frac{\cancel{N^2}}{m\frac{\cancel{N^2}}{m^2\Omega^2}} + 3\cdot\frac{m^2\Omega^3}{\cancel{N}}\cdot\frac{\cancel{N}}{m\Omega} =$$

$$= 6m\Omega^2$$

$$\omega = \sqrt{\frac{1}{m}\left.\frac{\partial^2 V_{eff}(r)}{\partial r^2}\right|_{r=r_{eq}}} = \sqrt{6}|\Omega|$$

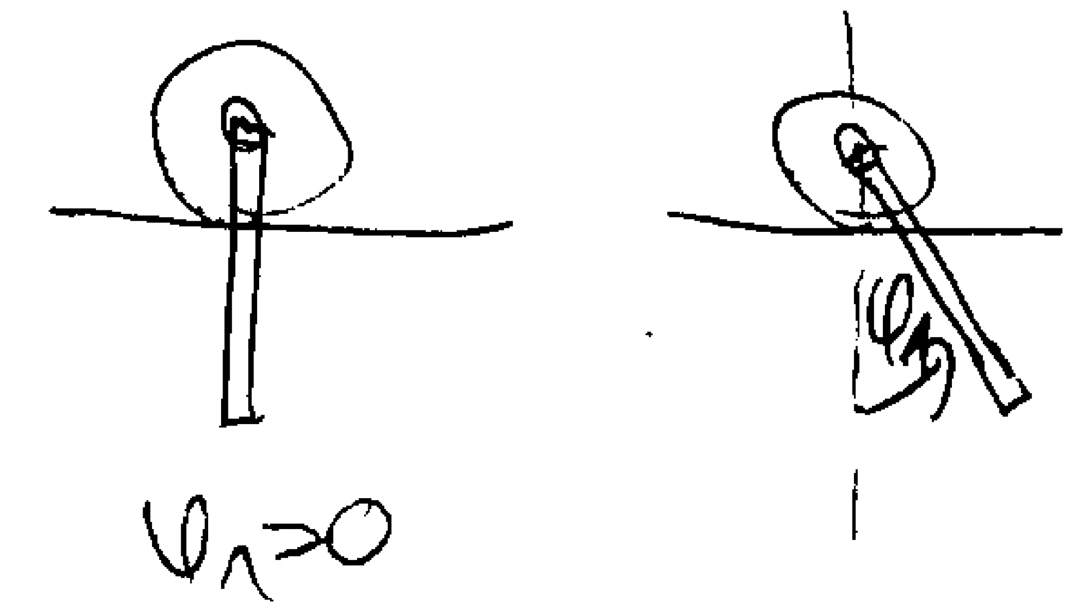
12.



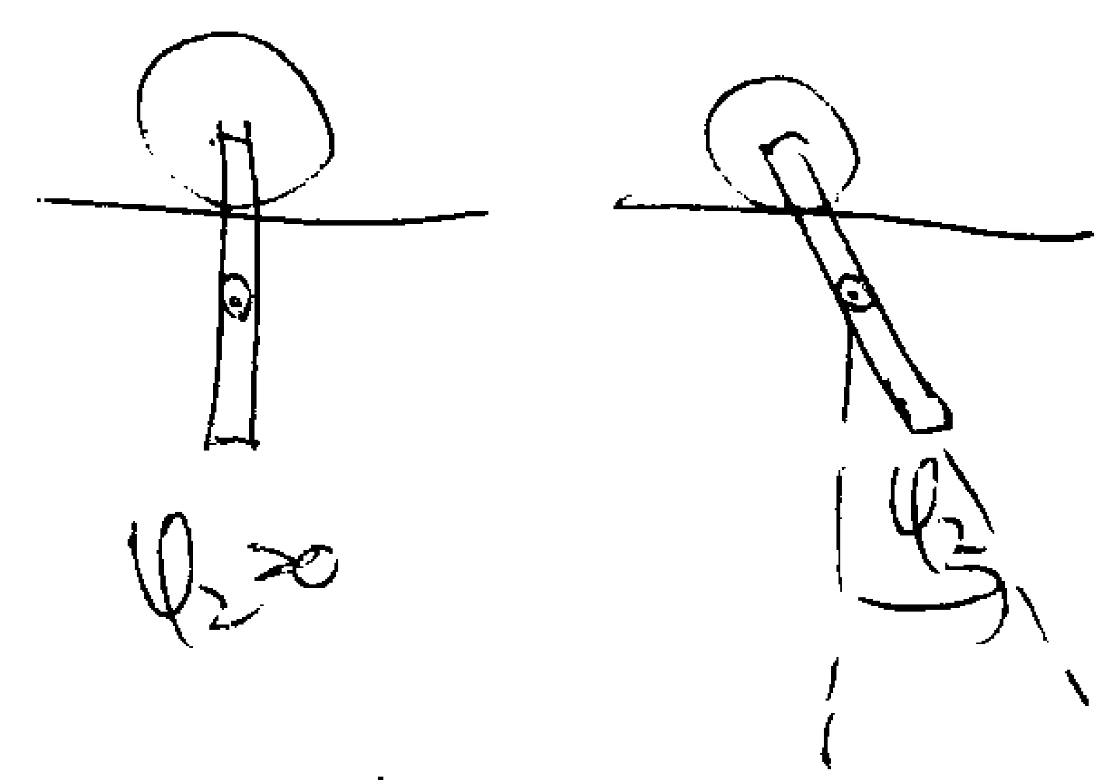
henger — rúd

1., x_1, y_1, φ_1, x_2, y_2, φ_2

legyen $\varphi_1 = 0$, amikor a rúd függőlegesen lefelé mutat:



És legyen „φ_2" is ekkor 0:



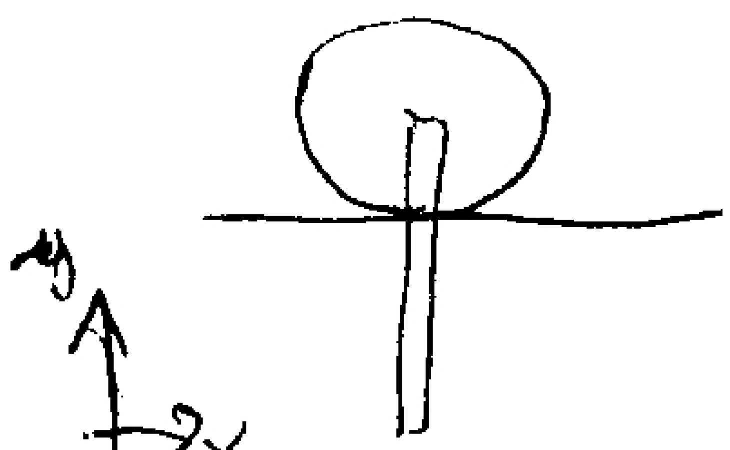
2., $y_1 =$ állandó

• tiszta gördülés

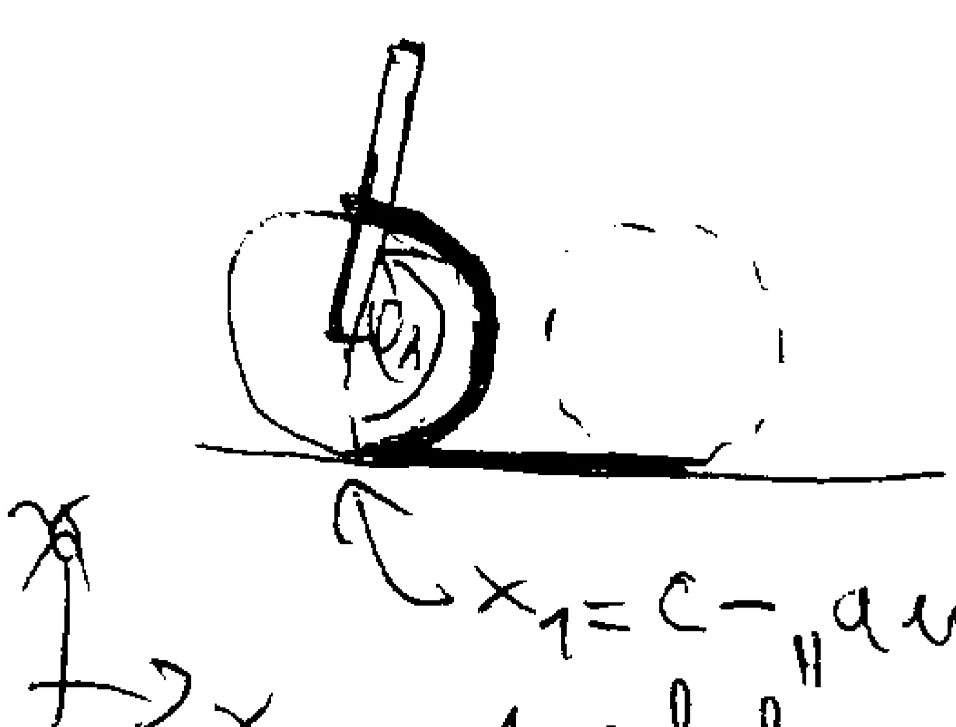
••• a rúd merev rögzítése miatt a rúd helyzetét jellemző mindhárom változó kifejezhető a henger helyzetét jellemző változókkal

3., φ_1

4., $x_1 = -R\varphi_1 + C$ → egy konstans



$\varphi_1 = 0$
$x_1 = C$



$x_1 = C -$ „a vastag vonal hossza" $=$
$= C - R\varphi_1$

$y_1 = d$

$\varphi_1 = \varphi_1$

$x_2 = x_1 + \frac{l}{2}\sin\varphi_1 =$
$= C - R\varphi_1 + \frac{l}{2}\sin\varphi_1$

$y_2 = y_1 - \frac{l}{2}\cos\varphi_1 = d - \frac{l}{2}\cos\varphi_1$

$\varphi_2 = \varphi_1$ (lásd az ábrát az 1., pontban: φ_1 és φ_2 egyállású szögek)

$\dot{x}_1 = -R\dot{\varphi}_1$ $\qquad$ $\dot{x}_2 = -R\dot{\varphi}_1 + \frac{l}{2}\cos\varphi_1\dot{\varphi}_1$

$\dot{y}_1 = 0$ $\qquad$ $\dot{y}_2 = \frac{l}{2}\sin\varphi_1\dot{\varphi}_1$

$\dot{\varphi}_1 = \dot{\varphi}_1$ $\qquad$ $\dot{\varphi}_2 = \dot{\varphi}_1$

5.) $L = T - V$

$$T = \frac{1}{2}M\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}\underbrace{\Theta_{s_1}}_{\Theta_{s_1} = \frac{1}{2}MR^2}\dot{\varphi}_1^2 + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2}\underbrace{\Theta_{s_2}}_{\Theta_{s_2} = \frac{1}{12}ml^2}\dot{\varphi}_2^2 =$$

$$= \frac{1}{2}MR^2\dot{\varphi}_1^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\dot{\varphi}_1^2 + \frac{1}{2}m\left(R^2\dot{\varphi}_1^2 - 2R\dot{\varphi}_1\frac{l}{2}\cos\varphi_1\dot{\varphi}_1 + \right.$$

$$\left. + \frac{l^2}{4}\cos^2\varphi_1\dot{\varphi}_1^2 + \frac{l^2}{4}\sin^2\varphi_1\dot{\varphi}_1^2\right) + \frac{1}{2}\cdot\frac{1}{12}ml^2\dot{\varphi}_1^2 =$$

$$= \frac{3}{4}MR^2\dot{\varphi}_1^2 + \frac{1}{2}mR^2\dot{\varphi}_1^2 - mR\frac{l}{2}\cos\varphi_1\dot{\varphi}_1^2 +$$

$$+ \underbrace{\frac{1}{2}m\frac{l^2}{4}\dot{\varphi}_1^2 + \frac{1}{24}ml^2\dot{\varphi}_1^2}_{\frac{1}{6}ml^2\dot{\varphi}_1^2}$$

$$V = Mgy_1 + mgy_2 = Mgd + mgd - mg\frac{l}{2}\cos\varphi_1$$

6.) $\frac{\partial L}{\partial \varphi_1} = mR\frac{l}{2}\sin\varphi_1\dot{\varphi}_1^2 - mg\frac{l}{2}\sin\varphi_1$

$$\frac{\partial L}{\partial \dot{\varphi}_1} = \frac{3}{2}MR^2\dot{\varphi}_1 + mR^2\dot{\varphi}_1 - mRl\cos\varphi_1\dot{\varphi}_1 + \frac{1}{3}ml^2\dot{\varphi}_1$$

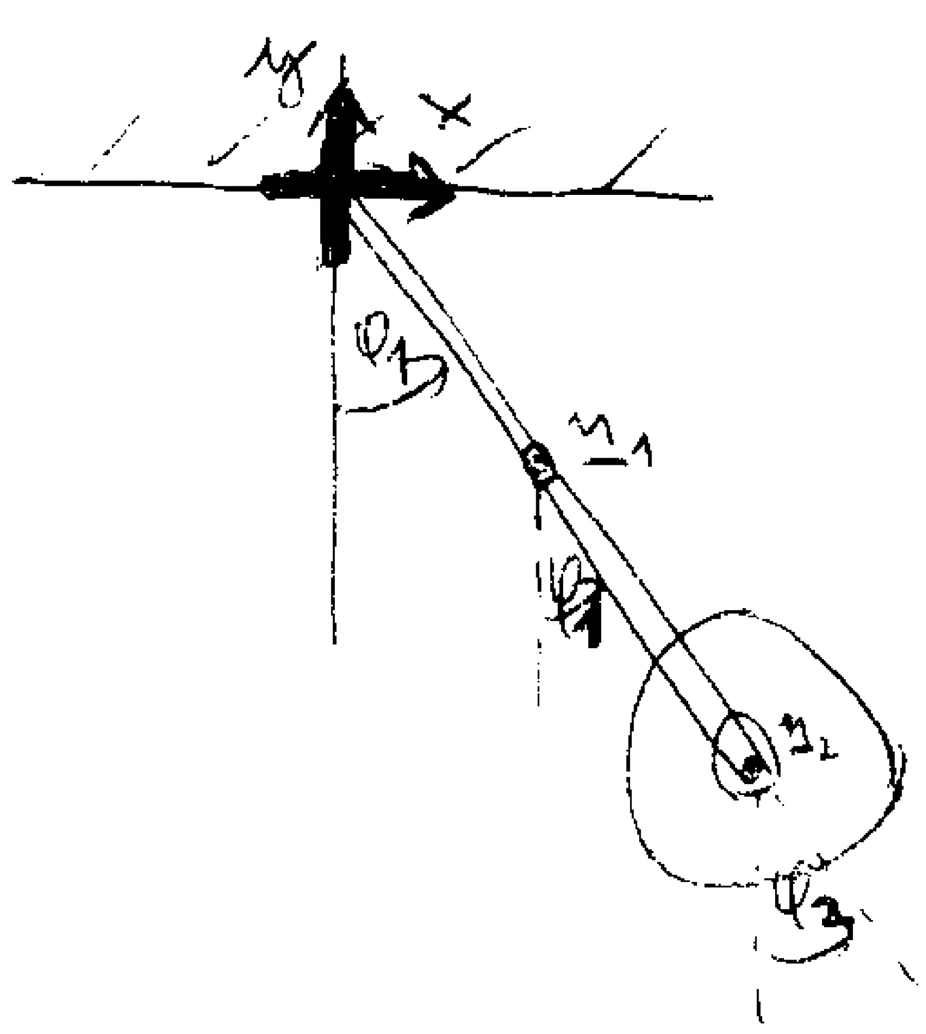
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = \frac{3}{2}MR^2\ddot{\varphi}_1 + mR^2\ddot{\varphi}_1 + mRl\sin\varphi_1\dot{\varphi}_1\cdot\dot{\varphi}_1 - mRl\cos\varphi_1\ddot{\varphi}_1 +$$
$$+ \frac{1}{3}ml^2\ddot{\varphi}_1$$

$$\Rightarrow \cancel{mR\frac{l}{2}\sin\varphi_1\dot{\varphi}_1^2} - mg\frac{l}{2}\sin\varphi_1 =$$

$$= \frac{3}{2}MR^2\ddot{\varphi}_1 + mR^2\ddot{\varphi}_1 + \frac{mRl\sin\varphi_1\dot{\varphi}_1^2}{2} -$$

$$- mRl\cos\varphi_1\ddot{\varphi}_1 + \frac{1}{3}ml^2\ddot{\varphi}_1$$

13.



a)

1. $x_1, y_1, \varphi_1, x_2, y_2, \varphi_2$
2. $\varphi_1 \Rightarrow x_1, y_1, x_2, y_2$, 4 db
3. φ_1, φ_2
4. $x_1 = \frac{l}{2}\sin\varphi_1$

$y_1 = -\frac{l}{2}\cos\varphi_1$

φ_1

$x_2 = l\sin\varphi_1$

$y_2 = -l\cos\varphi_1$

φ_2

$\dot{x}_1 = \frac{l}{2}\cos\varphi_1\dot{\varphi}_1$

$\dot{y}_1 = \frac{l}{2}\sin\varphi_1\dot{\varphi}_1$

$\dot{\varphi}_1$

$\dot{x}_2 = l\cos\varphi_1\dot{\varphi}_1$

$\dot{y}_2 = l\sin\varphi_1\dot{\varphi}_1$

$\dot{\varphi}_2$

$\Theta_{m_1} = \frac{1}{12}ml^2$

$\Theta_{m_2} = \frac{1}{2}MR^2$

b)

+1db: $\varphi_2 = \varphi_1$

csak φ_1

$\varphi_2 = \varphi_1$

$\dot{\varphi}_2 = \dot{\varphi}_1$

5. a)(b)

$$V = mgy_1 + Mgy_2 =$$
$$= -mg\frac{l}{2}\cos\varphi_1 - Mgl\cos\varphi_1 =$$
$$= -\left(\frac{m}{2}+M\right)gl\cos\varphi_1$$

$$T = \frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2\right) + \frac{1}{2}\Theta_{m_1}\dot{\varphi}_1^2 +$$
$$+\frac{1}{2}M\left(\dot{x}_2^2+\dot{y}_2^2\right) + \frac{1}{2}\Theta_{m_2}\dot{\varphi}_2^2 =$$
$$= \frac{1}{2}m\frac{l^2}{4}\dot{\varphi}_1^2 + \frac{1}{24}ml^2\dot{\varphi}_1^2 +$$
$$+\frac{1}{2}Ml^2\dot{\varphi}_1^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\dot{\varphi}_2^2 =$$

$$\dot\varphi_2 = \dot\varphi_1 \;\Downarrow$$

b) $$T = \left[\left(\frac{1}{8}+\frac{1}{24}\right) m\ell^2 + \frac{1}{2}M\ell^2\right]\dot\varphi_1^2 + \frac{1}{4}MR^2\dot\varphi_2^2 =$$

$$= \left[\underbrace{\left(\frac{1}{8}+\frac{1}{24}\right)}_{1/6} m\ell^2 + \frac{1}{2}M\ell^2 + \frac{1}{4}MR^2\right]\dot\varphi_1^2$$

a) $$T = \left[\underbrace{\left(\frac{1}{8}+\frac{1}{24}\right)}_{1/6} m\ell^2 + \frac{1}{2}M\ell^2\right]\dot\varphi_1^2 + \frac{1}{4}MR^2\dot\varphi_2^2$$

6. a) $$\left.\begin{aligned} \frac{\partial L}{\partial \varphi_1} &= -\left(\frac{m}{2}+M\right) g\ell \sin\varphi_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right) &= \left[\frac{1}{3}m\ell^2 + M\ell^2\right]\ddot\varphi_1 \end{aligned}\right\} \to \left(\frac{1}{3}m\ell^2 + M\ell^2\right)\ddot\varphi_1 = -\left(\frac{m}{2}+M\right) g\ell\sin\varphi_1$$

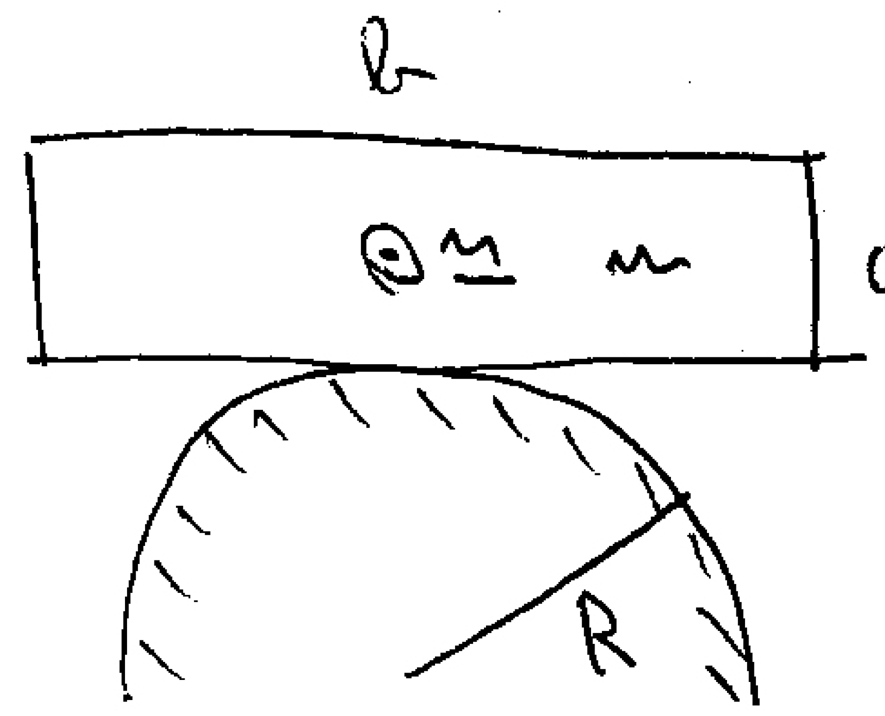
$$\left.\begin{aligned} \frac{\partial L}{\partial \varphi_2} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right) &= \frac{1}{2}MR^2\ddot\varphi_2 \end{aligned}\right\} \to \begin{aligned} \frac{1}{2}MR^2\ddot\varphi_2 &= 0 \\ \ddot\varphi_2 &= 0 \end{aligned}$$

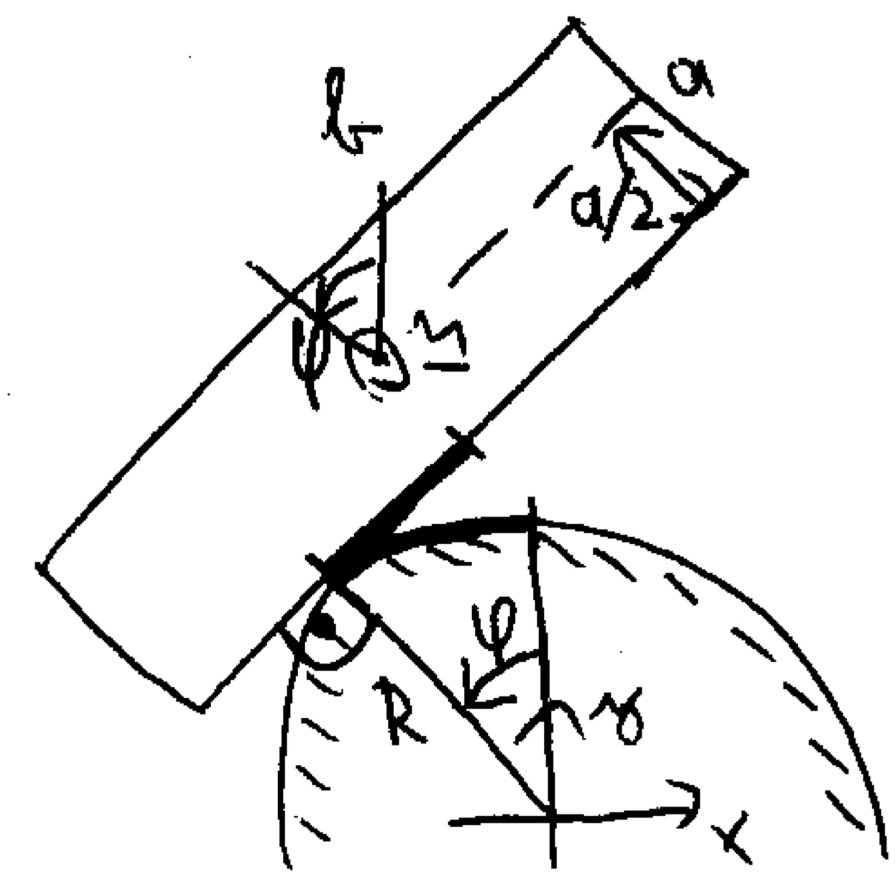
(Semmi sem gyorsítja a korong forgását.)

b) $$\left.\begin{aligned} \frac{\partial L}{\partial \varphi_1} &= -\left(\frac{m}{2}+M\right) g\ell \sin\varphi_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right) &= \left[\frac{1}{3}m\ell^2 + M\ell^2 + \frac{1}{2}MR^2\right]\ddot\varphi_1 \end{aligned}\right\} \to \left(\frac{1}{3}m\ell^2 + M\ell^2 + \frac{1}{2}MR^2\right)\ddot\varphi_1 = -\left(\frac{m}{2}+M\right) g\ell\sin\varphi_1$$

14.

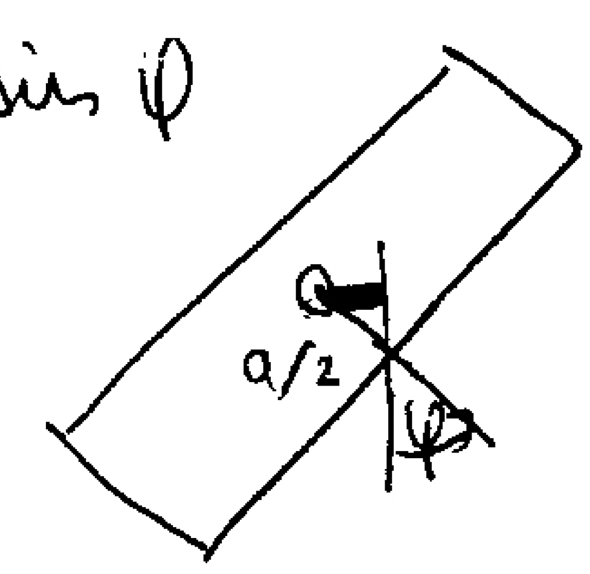
EMB/GY2/20





$\Theta_s = \frac{1}{12} m(a^2+b^2)$

1. x, y, φ
2. tiszta gördülés, „lejtő"
3. φ
4. $x = -R\sin\varphi \quad + R\varphi\cos\varphi - \frac{a}{2}\sin\varphi$



$y = R\cos\varphi + R\varphi\sin\varphi + \frac{a}{2}\cos\varphi$

φ

$\dot{x} = -(\cancel{R} + \frac{a}{2})\cos\varphi\,\dot{\varphi} \cancel{+ R\dot{\varphi}\cos\varphi} - R\varphi\sin\varphi\,\dot{\varphi}$

$\dot{y} = -(\cancel{R} + \frac{a}{2})\sin\varphi\,\dot{\varphi} + \cancel{R\dot{\varphi}\sin\varphi} + R\varphi\cos\varphi\,\dot{\varphi}$

$\dot{\varphi}$

5.

$V = mgy = mg\left[(R+\frac{a}{2})\cos\varphi + R\varphi\sin\varphi\right]$

$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\Theta_s\dot{\varphi}^2 =$

$= \frac{1}{2}m\left[(-\frac{a}{2}\cos\varphi - R\varphi\sin\varphi)^2\dot{\varphi}^2 + (-\frac{a}{2}\sin\varphi + R\varphi\cos\varphi)\dot{\varphi}^2\right] +$

$+ \frac{1}{2}\cdot\frac{1}{12}m(a^2+b^2)\dot{\varphi}^2 = \Rightarrow$

$$= \frac{1}{2} m \left[\frac{a^2}{4}\cos^2\varphi + \cancel{\frac{a}{2} R\varphi \cos\varphi \sin\varphi} + R^2\varphi^2 \sin^2\varphi + \right.$$

$$\left. + \frac{a^2}{4}\sin^2\varphi - \cancel{\frac{a}{2} R\varphi \sin\varphi \cos\varphi} + R^2\varphi^2\cos^2\varphi \right] \dot{\varphi}^2 + \frac{1}{24} m (a^2+b^2)\dot{\varphi}^2 =$$

$$= \frac{1}{2} m \left[\underbrace{\frac{a^2}{4} + R^2\varphi^2}_{\to (*)} \right] \dot{\varphi}^2 + \frac{1}{2} \cdot \frac{1}{12} m (a^2+b^2)\dot{\varphi}^2 =$$

$$= \frac{1}{2} m \left(\frac{a^2}{4} + R^2\varphi^2 + \frac{a^2}{12} + \frac{b^2}{12} \right)\dot{\varphi}^2 = \frac{1}{2} m \left(R^2\varphi^2 + \frac{a^2}{3} + \frac{b^2}{12} \right)\dot{\varphi}^2$$

6. $$\frac{\partial L}{\partial \varphi} = -mg\left[-\left(\cancel{R} + \frac{a}{2}\right)\sin\varphi \cancel{+ R\sin\varphi} + R\varphi\cos\varphi \right] + mR^2\varphi\dot{\varphi}^2 =$$

$$= -mg\left(R\varphi\cos\varphi - \frac{a}{2}\sin\varphi \right) + mR^2\varphi\dot{\varphi}^2$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m\left(R^2\varphi^2 + \frac{a^2}{3} + \frac{b^2}{12} \right)\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 2mR^2\varphi \cdot \dot{\varphi} \cdot \dot{\varphi} + m\left(R^2\varphi^2 + \frac{a^2}{3} + \frac{b^2}{12} \right)\ddot{\varphi}$$

$\to$

$$\cancel{2} mR^2\varphi\dot{\varphi}^2 + m\left(R^2\varphi^2 + \frac{a^2}{3} + \frac{b^2}{12} \right)\ddot{\varphi} =$$

$$= -mg\left(R\varphi\cos\varphi - \frac{a}{2}\sin\varphi \right) + \cancel{mR^2\varphi\dot{\varphi}^2}$$

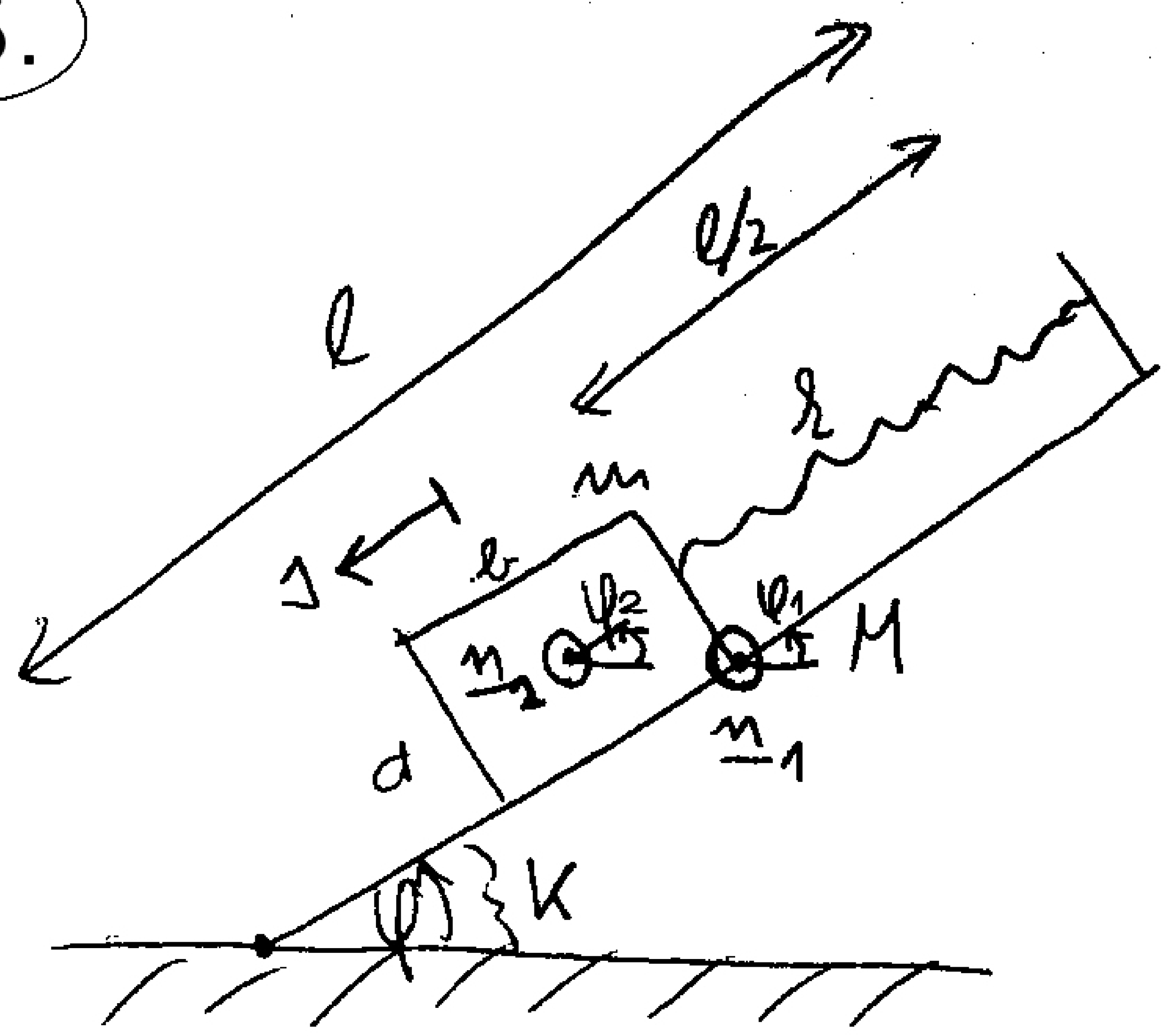
(*) Vegyük észre, hogy ez nem adódik

$$v^2 = v_r^2 + v_\varphi^2$$

(haladó, transzlációs)

alapján, a forgómozgás centrumának nem tekinthetjük az origót!

15.



1. $\underbrace{x_1, y_1, \varphi_1}_{\text{lemez}}, x_2, y_2, \varphi_2$

2. a lemez síkjában 2×, érintkezés, $\varphi_2 = \varphi_1$

3. φ, s

4. $x_1 = \frac{l}{2}\cos\varphi_1 = \frac{l}{2}\cos\varphi$

$y_1 = \frac{l}{2}\sin\varphi_1 = \frac{l}{2}\sin\varphi$

$\varphi_1 = \varphi$

$\varphi_2 = \varphi$

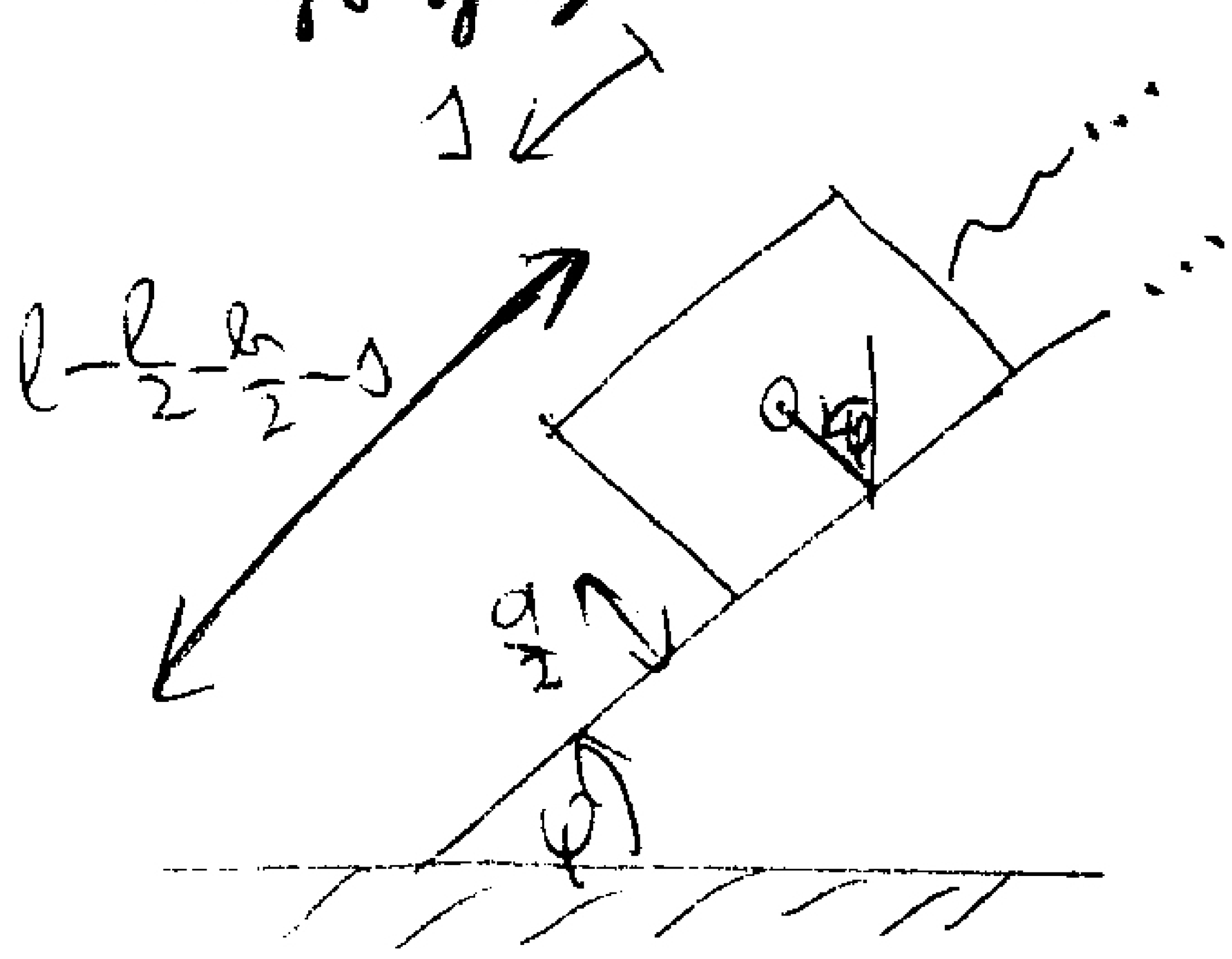
$x_2 = \left(l - \frac{l}{2} - \frac{b}{2} - s\right)\cos\varphi -$

$- \frac{a}{2}\sin\varphi =$

$= \left(\frac{l-b}{2} - s\right)\cos\varphi - \frac{a}{2}\sin\varphi$

$y_2 = \left(\frac{l-b}{2} - s\right)\sin\varphi + \frac{a}{2}\cos\varphi$

(„s" előjeles mennyiség!)



$\dot{x}_1 = -\frac{l}{2}\sin\varphi\,\dot{\varphi}$

$\dot{y}_1 = \frac{l}{2}\cos\varphi\,\dot{\varphi}$

$\dot{\varphi}_1 = \dot{\varphi}$

$\dot{\varphi}_2 = \dot{\varphi}$

$\dot{y}_2 = -\dot{s}\sin\varphi +$

$+ \left(\frac{l-b}{2} - s\right)\cos\varphi\,\dot{\varphi} -$

$- \frac{a}{2}\sin\varphi\,\dot{\varphi} =$

$= -\left(\dot{s} + \frac{a}{2}\dot{\varphi}\right)\sin\varphi +$

$+ \left(\frac{l-b}{2} - s\right)\cos\varphi\,\dot{\varphi}$

$\dot{x}_2 = -\dot{s}\cos\varphi - \left(\frac{l-b}{2} - s\right)\sin\varphi\,\dot{\varphi} -$

$- \frac{a}{2}\cos\varphi\,\dot{\varphi} =$

$= -\left(\dot{s} + \frac{a}{2}\dot{\varphi}\right)\cos\varphi -$

$- \left(\frac{l-b}{2} - s\right)\sin\varphi\,\dot{\varphi}$

$\Theta_{M_1} = \frac{1}{12} M(l^2 + 0) = \frac{1}{12} M l^2, \quad \Theta_{m_2} = \frac{1}{12} m(a^2 + b^2)$

$+\frac{1}{2}k(\varphi-\alpha)^2+\frac{1}{2}ks^2$

5. $V = Mg y_1 + mg y_2 = Mg\frac{l}{2}\sin\varphi + mg\left[\left(\frac{l-b}{2}-s\right)\sin\varphi + \frac{a}{2}\cos\varphi\right] + \frac{1}{2}k(\varphi-\alpha)^2 +$
$+\frac{1}{2}\lambda s^2$

$$T = \frac{1}{2}M(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}\Theta_{s_1}\dot{\varphi}_1^2 + \frac{1}{2}m(\dot{x}_2^2+\dot{y}_2^2) + \frac{1}{2}\Theta_{s_2}\dot{\varphi}_2^2 =$$

$$= \frac{1}{2}M\frac{l^2}{4}\dot{\varphi}^2 + \frac{1}{2}\cdot\frac{1}{12}Ml^2\dot{\varphi}^2 + \frac{1}{2}m\left[\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)^2\cos^2\varphi +\right.$$

$$\cancel{+2\cdot\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)\cos\varphi\cdot\left(\frac{l-b}{2}-s\right)\sin\varphi\dot{\varphi}} + \left(\frac{l-b}{2}-s\right)^2\sin^2\varphi\dot{\varphi}^2 +$$

$$+\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)^2\sin^2\varphi \cancel{-2\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)\sin\varphi\cdot\left(\frac{l-b}{2}-s\right)\cos\varphi\dot{\varphi}} +$$

$$\left.+\left(\frac{l-b}{2}-s\right)^2\cos^2\varphi\dot{\varphi}^2\right] + \frac{1}{2}\cdot\frac{1}{12}m(a^2+b^2)\dot{\varphi}^2 =$$

$$= \frac{1}{6}Ml^2\dot{\varphi}^2 + \frac{1}{2}m\left[\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)^2 + \left(\frac{l-b}{2}-s\right)^2\dot{\varphi}^2\right] +$$

$$+\frac{1}{24}m(a^2+b^2)\dot{\varphi}^2$$

6. $$\frac{\partial L}{\partial\varphi} = -Mg\frac{l}{2}\cos\varphi - mg\left(\frac{l-b}{2}-s\right)\cos\varphi + mg\frac{a}{2}\sin\varphi - k(\varphi-\alpha)$$

$$\frac{\partial L}{\partial\dot{\varphi}} = \frac{1}{3}Ml^2\dot{\varphi} + m\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)\cdot\frac{a}{2} + m\left(\frac{l-b}{2}-s\right)^2\dot{\varphi} + \frac{1}{12}m(a^2+b^2)\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = \frac{1}{3}Ml^2\ddot{\varphi} + m\left(\ddot{s}+\frac{a}{2}\ddot{\varphi}\right)\frac{a}{2} - 2m\left(\frac{l-b}{2}-s\right)\cdot\dot{s}\dot{\varphi} +$$

$$+m\left(\frac{l-b}{2}-s\right)^2\ddot{\varphi} + \frac{1}{12}m(a^2+b^2)\ddot{\varphi}$$

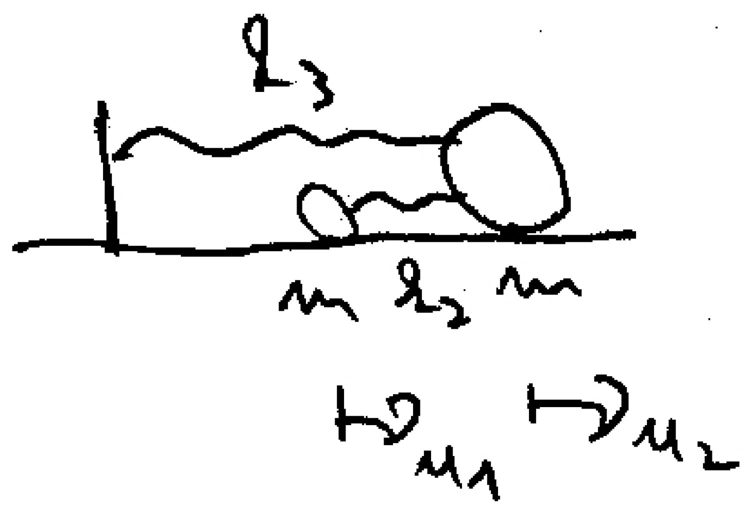
$$\longrightarrow \frac{\partial L}{\partial\varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right)$$

$$\frac{\partial L}{\partial s} = mg\sin\varphi - \lambda s$$

$$\frac{\partial L}{\partial\dot{s}} = m\left(\dot{s}+\frac{a}{2}\dot{\varphi}\right)\cdot 1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{s}}\right) = m\left(\ddot{s}+\frac{a}{2}\ddot{\varphi}\right) \qquad \longrightarrow \frac{\partial L}{\partial s} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{s}}\right)$$

(8)



3. u_1, u_2

5. $V=\frac{1}{2}k_2(u_2-u_1)^2+\frac{1}{2}k_3 u_2^2$

$T=\frac{1}{2}m\dot{u}_1^2+\frac{1}{2}m_2\dot{u}_2^2$

6. $$\left.\begin{aligned}\frac{\partial L}{\partial u_1}&=-k_2(u_2-u_1)\cdot(-1)=-k_2u_1+k_2u_2\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right)&=m\ddot{u}_1\end{aligned}\right\}\to m\ddot{u}_1=-k_2u_1+k_2u_2$$

$$\left.\begin{aligned}\frac{\partial L}{\partial u_2}&=-k_2(u_2-u_1)\cdot 1-k_3u_2\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right)&=m\ddot{u}_2\end{aligned}\right\}\to m\ddot{u}_2=k_2u_1-(k_2+k_3)u_2$$

$$\to \begin{pmatrix}-\frac{k_2}{m} & \frac{k_2}{m}\\ \frac{k_2}{m} & -\frac{k_2+k_3}{m}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}$$

Próbafüggvény: $\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}A_1\\A_2\end{pmatrix}e^{i\omega t} \to \begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\end{pmatrix}=-\omega^2\begin{pmatrix}A_1\\A_2\end{pmatrix}e^{i\omega t}$

$$\to \underbrace{\begin{pmatrix}-\frac{k_2}{m} & \frac{k_2}{m}\\ \frac{k_2}{m} & -\frac{k_2+k_3}{m}\end{pmatrix}}_{\underline{\underline{M}}}\begin{pmatrix}A_1\\A_2\end{pmatrix}\cancel{e^{i\omega t}}=-\omega^2\begin{pmatrix}A_1\\A_2\end{pmatrix}\cancel{e^{i\omega t}}$$

$\begin{pmatrix}A_1\\A_2\end{pmatrix}\not\equiv 0$, ha $\det(\underline{\underline{M}}+\omega^2\underline{\underline{1}})=0$

$$\left(-\frac{k_2}{m}+\omega^2\right)\left(-\frac{k_2+k_3}{m}+\omega^2\right)-\frac{k_2^2}{m^2}\overset{!}{=}0$$

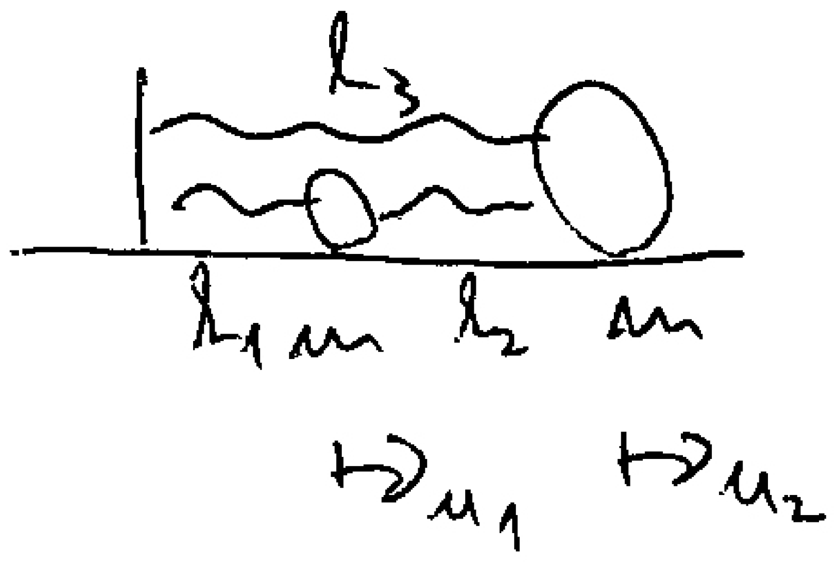
$$\omega^4-\left(\frac{k_2}{m}+\frac{k_2+k_3}{m}\right)\omega^2+\frac{k_2}{m}\cdot\frac{k_2+k_3}{m}-\cancel{\frac{k_2^2}{m^2}}=0$$

$$\omega^4-\frac{2k_2+k_3}{m}\omega^2+\frac{k_2k_3}{m^2}=0$$

$$\to (\omega^2)_{I,II}=\frac{2k_2+k_3}{2m}\pm\sqrt{\left(\frac{2k_2+k_3}{2m}\right)^2-\frac{k_2k_3}{m^2}}$$

Ez ω-ra összesen 4 megoldás.

(9.)



3. $u_1 u_2$

5. $V = \frac{1}{2}k_1 u_1^2 + \frac{1}{2}k_2 (u_2-u_1)^2 + \frac{1}{2}k_3 u_2^2$

$T = \frac{1}{2} m \dot{u}_1^2 + \frac{1}{2} m \dot{u}_2^2$

6. ... $\rightarrow$

$$m\ddot{u}_1 = -(k_1+k_2)u_1 + k_2 u_2$$
$$m\ddot{u}_2 = k_2 u_1 - (k_2+k_3)u_2$$

Próbafüggvény: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} \rightarrow \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$

$$\rightarrow \underbrace{\begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cancel{e^{i\omega t}} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cancel{e^{i\omega t}}$$

$$\det(\underline{\underline{M}} + \omega^2 \underline{\underline{1}}) \overset{!}{=} 0$$

$$\left(-\frac{k_1+k_2}{m}+\omega^2\right)\left(-\frac{k_2+k_3}{m}+\omega^2\right) - \frac{k_2^2}{m^2} = 0$$

$$\omega^4 - \frac{k_1+2k_2+k_3}{m}\omega^2 + \frac{k_1k_2+k_1k_3+k_2k_3}{m^2} = 0$$

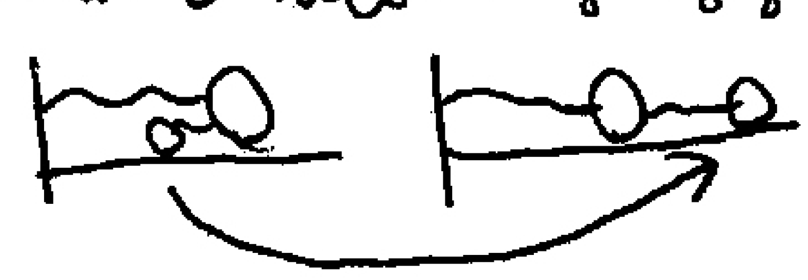
$$\rightarrow (\omega^2)_{I,II} = \frac{k_1+2k_2+k_3}{2m} \pm \sqrt{\left(\frac{k_1+2k_2+k_3}{2m}\right)^2 - \frac{k_1k_2+k_1k_3+k_2k_3}{m^2}}$$

Ez ω-ra 4 megoldás.

$k_3 \rightarrow 0$: Ekkor a nov. 13-i órai feladatot kapjuk vissza a $k_1 = k_A$, $k_2 = k_B$ helyettesítéssel.

$k_1 \rightarrow 0$: Ezen feladatsor (8.) feladata adódik.

Észrevehetjük, hogy a két esetben ugyanolyan alakú sajátfrekvenciákat kapunk a $k_2 = k_B$, $k_1 = k_3$ helyettesítéssel. A sajátfrekvencia nem érzékeny arra, hogy a lánc utolsó golyóját elöl vagy hátra csatoljuk a dobozhoz. $\Rightarrow$



$k_2 \to 0$:

$$(\omega^2)_{I,II} = \frac{k_1+k_3}{2m} \pm \sqrt{\left(\frac{k_1+k_3}{2m}\right)^2 - \frac{k_1k_3}{m^2}} =$$

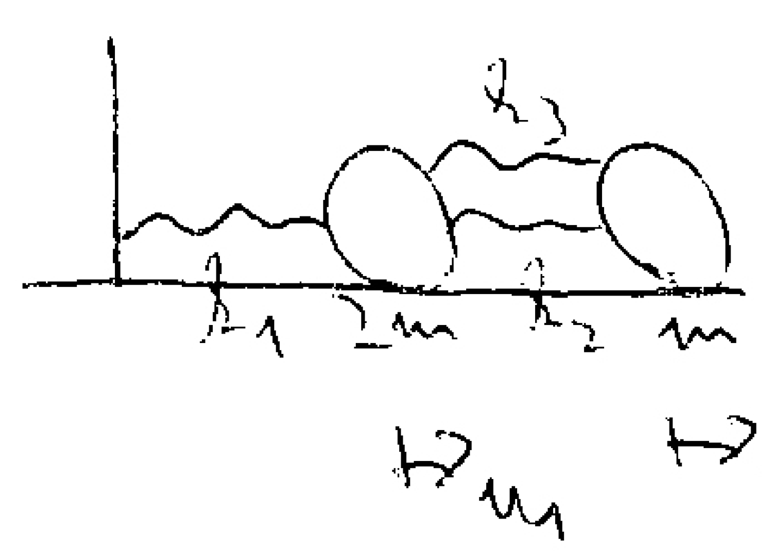
$$= \frac{k_1+k_3}{2m} \pm \sqrt{\frac{k_1^2+2k_1k_3+k_3^2}{4m^2} - \frac{4k_1k_3}{4m^2}} =$$

$$= \frac{k_1+k_3}{2m} \pm \sqrt{\frac{(k_1-k_3)^2}{4m^2}} = \frac{k_1+k_3}{2m} \pm \frac{|k_1-k_3|}{2m} = \begin{cases} \frac{k_1}{m} \\ \frac{k_3}{m} \end{cases}$$

(Ez a két megoldás van $k_1 > k_3$-ra és $k_1 < k_3$-ra is.)

Tehát az egyes rugók „saját" frekvenciája jelenik meg. Ez nem meglepő, mivel $k_2 = 0$ esetén két független alrendszer van: 

10.



3. u_1, u_2

5. $V = \frac{1}{2}k_1u_1^2 + \frac{1}{2}k_2(u_2-u_1)^2 + \frac{1}{2}k_3(u_2-u_1)^2 =$

$= \frac{1}{2}k_1u_1^2 + \frac{1}{2}\underbrace{(k_2+k_3)}_{=:k_{eff}}(u_2-u_1)^2$

Ez a tag éppen olyan, mintha a két golyó között 1 db, k_{eff} (effektív) direkciós erejű rugó lenne. Tehát „párhuzamosan kapcsolt" rugók direkciós állandói összeadódnak.

$T = \frac{1}{2}\cdot 2m\dot{u}_1^2 + \frac{1}{2}m\dot{u}_2^2$

6. $\frac{\partial L}{\partial u_1} = -k_1u_1 - k_{eff}(u_2-u_1)\cdot(-1) =$

$= -k_1u_1 + k_{eff}(u_2-u_1) =$

$= -(k_1+k_{eff})u_1 + k_{eff}u_2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) = 2m\ddot{u}_1$

$\frac{\partial L}{\partial u_2} = -k_{eff}(u_2-u_1)\cdot 1 =$

$= k_{eff}u_1 - k_{eff}u_2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right) = m\ddot{u}_2$

$\Rightarrow 2m\ddot{u}_1 = -(k_1+k_{eff})u_1 + k_{eff}u_2$

$\Rightarrow m\ddot{u}_2 = k_{eff}u_1 - k_{eff}u_2$

10. (folytatás)

$$\begin{pmatrix} -\frac{k_1+k_{eff}}{2m} & \frac{k_{eff}}{2m} \\ \frac{k_{eff}}{m} & -\frac{k_{eff}}{m} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix}$$

Próbafüggvény: $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} \rightarrow \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$

$$\rightarrow \underbrace{\begin{pmatrix} -\frac{k_1+k_{eff}}{2m} & \frac{k_{eff}}{2m} \\ \frac{k_{eff}}{m} & -\frac{k_{eff}}{m} \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cancel{e^{i\omega t}} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cancel{e^{i\omega t}}$$

$$\det\left(\underline{\underline{M}} + \omega^2 \underline{\underline{1}}\right) \stackrel{!}{=} 0$$

$$\left(-\frac{k_1+k_{eff}}{2m} + \omega^2\right)\left(-\frac{k_{eff}}{m} + \omega^2\right) - \frac{k_{eff}^2}{2m^2} = 0$$

$$\omega^4 - \frac{k_1+3k_{eff}}{2m}\omega^2 + \frac{k_1 k_{eff}}{2m^2} = 0$$

$$\rightarrow \left(\omega^2\right)_{I,II} = \frac{k_1+3k_{eff}}{2m} \pm \sqrt{\left(\frac{k_1+3k_{eff}}{2m}\right)^2 - \frac{k_1 k_{eff}}{2m^2}}$$

ω-ra ez 4 megoldást jelent.